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REPORT ON RESEARCHES ON THE CHEMICAL AND MINERALOGICAL COMPOSITION OF METEORITES, WITH ESPECIAL REFERENCE TO THEIR MINOR CONSTITUENTS.

BY

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A REPORT ON RESEARCHES ON THE CHEMICAL AND MINERALOGICAL COMPOSITION OF METEORITES, WITH ESPECIAL REFERENCE TO THEIR MINOR CONSTITUENTS.

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I. INTRODUCTION AND SCOPE OF INVESTIGATION.

In June, 1909, in view of the current speculations regarding earth history, the writer published a paper on the composition of stony meteorites compared with that of terrestrial igneous rocks.[a] In the preparation of this paper he was impressed with the comparatively small number of satisfactory chemical analyses available, but 99 being found which were considered sufficiently complete and accurate for his purpose. A second fact was the apparent similarity in, and simplicity of, meteoric composition, there being shown scarcely any of those elements which recent rock analyses have found to be common constituents, though in small quantities, of terrestrial rocks. These facts, coupled with the occasional reported occurrences of such elements as platinum, gold, lead, zinc, etc., and the high degree of perfection reached by modern analytical chemistry, suggested to him the advisability of undertaking a systematic investigation of the chemical nature of both stone and iron meteorites, with particular reference to the occurrence of such elements as had been reported as doubtful or found only in traces. On mentioning the matter to Prof. Morley, he was encouraged to make application for financial assistance from the J. Lawrence Smith fund of the National Academy of Sciences. This was promptly granted, and a preliminary report of progress was published in 1913.[b] An application for further assistance being granted, the work has been continued down to approximately the present date, the analytical work, as before, being placed in the hands of Dr. J. E. Whitfield, of Booth, Garrett & Blair, in Philadelphia.

As is well known, and was stated in the preliminary report, the nongaseous elements characteristic of meteorites, the presence of which has been established by quantitative methods beyond controversy, are silicon, aluminum, iron, chromium, manganese, nickel, cobalt, magnesium, calcium, sodium, potassium, sulphur, phosphorus, and carbon. In addition there have been reported, usually under such conditions as to need authentication or at least corroboration, antimony, arsenic, copper, gold, lead, palladium, platinum, tin, titanium, tungsten, uranium, vanadium, and zinc. The preliminary investigation above referred to was made mainly for the purpose of fixing the presence or absence of these last named in amounts sufficient for determination by a skillful analyst, though the possible occurrence of other elemental constituents of terrestrial rocks was not ignored, and during the final researches on feldspathic types great care was taken in searching for barium, strontium, and zirconium. Wherever possible samples of the same meteorites in which an element had been doubtfully reported were analyzed. In other cases meteorites were taken which had not before been subjected to analysis, or the analyses of which were unsatisfactory for one reason or another. Whenever possible, too, an amount of material was taken sufficient to warrant a representative selection; as a rule 50 grams and upwards were thus utilized. In a few instances, particularly

[a] Amer. Journ. Sci., vol. 27, June, 1909, pp. 469–474.
[b] On the Minor Constituents of Meteorites, Amer. Journ. Sci., vol. 35, May, 1913, pp. 509–525.

in the case of the older falls, the absurdly high price demanded by holders of the material necessitated a lower limit, which, however, was rarely less than 10 grams. In all cases the meteorite was made the subject of careful microscopic study, and the purpose ever held in view of not merely ascertaining the presence or absence of any constituent but of relegating the same, as well, to its proper source.

II. ELEMENTS DOUBTFULLY REPORTED OR OF UNUSUAL OCCURRENCE.

Before discussing the results it will be well to repeat what was given in the preliminary paper regarding the previously reported occurrences of the unusual elements.

Arsenic.—The first determination of arsenic in meteorites of which I have record is that of Karl Rumler who, in 1840, reported[a] getting distinct arsenical reactions from the olivine-like mineral occurring in both the Atacama, Bolivia, and the Krasnojarsk, Siberia, pallasites. It is difficult to detect possible sources of error in Rumler's method as given. The fact, however, that no one has since been able to corroborate his work would suggest some possible impurity in his reagents. Silliman and Hunt also reported traces of arsenic (and copper) in the iron of Cambria, N. Y.[b] The only other reported occurrence of the element known to me is that of Fischer and Duflos in the Braunau iron.[c] This determination can to-day scarcely be considered satisfactory. The solution remaining after the precipitation of the copper was evaporated, the dry residue mixed with soda and heated before the blowpipe; result, a garlic odor. In stating the analysis, copper, manganese, arsenic, lime, magnesium, silicon, carbon, chromium, and sulphur are all thrown together as amounting to 2.072 per cent.

Antimony.—Traces of this metal were reported by Trottarelli in the stone of Collescipoli. I have not seen the original paper, but an abstract by Max Bauer[d] gives, among other constituents, lead, antimony, tin, and lithia, as occurring in traces, palladium to the amount of 0.7745 per cent, and soda (Na_2O) to the unheard of amount of 10.386 per cent (!). I have therefore a natural feeling of skepticism regarding the results as a whole. (See new analyses, p. 14.)

Copper.—Copper in amounts from traces up to weighable quantities has been reported by such authorities as Rammelsberg, Rose, J. L. Smith, and many others, and should be removed from the doubtful list.

Gold.—Gold, so far as I am aware, was first plausibly suggested as a meteoric constituent by A. Liversidge,[e] who thought to find it in an iron from Boogaldi, New South Wales. Notwithstanding the fact that the work of Prof. Liversidge seems to have been performed with proper care, there exists a lingering doubt in the minds of many as to the actual occurrence of this element as an original constituent of the iron. It is to be noted, however, that more recent investigations by J. C. H. Mingaye are confirmatory.[f]

Lead.—Trottarelli, whose analysis is above referred to, reported traces of lead in the Collescipoli stone. R. P. Greg also reported[g] native lead lining the cavities in an iron from the Tarapaca desert of Chile. J. L. Smith, however, concluded from his own examination[h] that the metal was altogether foreign to the stone when it fell.

Lithia.—Lithia was reported by Story Maskelyne[i] to the amount of 0.016 per cent in the enstatite and in traces in the augitic constituent of the Busti stone. J. L. Smith likewise reported[j] traces of lithia in the stones of Waconda, Kans., and Bishopville, S. C. Others report it determined by spectroscopic methods.

Platinum, palladium, and iridium.—Platinum, palladium, and iridium come in for occasional reference as meteoric constituents, but almost invariably in amounts too small to weigh,

[a] Pogg. Ann. Phys. Chem., vol. 49, 1840, p. 591.
[b] Amer. Journ. Sci., vol. 2, 1846, p. 376.
[c] Pogg. Ann. Phys. Chem., vol. 72, 1847, p. 479.
[d] Neues Jahrb. für Min., etc., 1891, vol. 2, p. 238.
[e] Journ. Proc. Roy. Soc. of. N. S. Wales, vol. 36, 1902.
[f] Records Geol. Surv. N. S. Wales, vol. 7, 1904, p. 306.
[g] London, Edinburgh & Dublin Philos. Mag., vol. 10, 1855, p. 12. Also Amer. Journ. Sci., vol. 23, 1857, p. 118.
[h] Amer. Journ. Sci., vol. 49, 1870, p. 305.
[i] Philos. Trans. Roy. Soc., vol. 160, 1870, pp. 206–7.
[j] Amer. Journ. Sci., vol. 13, 1877, p. 212, and vol. 38, 1869, p. 226.

and often in analyses made under such conditions as to give rise to a feeling of doubt as to their correctness. Trottarelli's reported finding of palladium has already received attention. J. M. Davison [a] obtained from 608.6 grams of the Coahuila iron 0.014 gram of platinum; from 464 grams of the Toluca iron a few crystals of potassium platinic chloride were obtained which showed a reddish color and probably contained iridium. Tassin [b] doubtfully reported the acid soluble portion of the Persimmon Creek iron as containing traces of platinum too small to weigh. Mallet's work on the Canon Diablo iron is confirmatory, however, of its occasional occurrence (see p. 10).

Tin.—Tin to the amount of 0.17 per cent SnO_2 was reported [c] by Stromeyer and Walmstedt as long ago as 1825 as occurring in the olivine of a pallasite. Unfortunately some doubt exists as to whether this was the pallasite of Krasnojarsk or Steinbach. Rammelsberg in 1884 [d] reported finding 0.08 per cent Sn in the metallic portion of the Klein-Wenden aerolite, and he also tabulates [e] 0.57 per cent Sn in the analysis of the Nashville (?) Tenn., iron. Jackson [f] thought to have found 0.063 per cent Sn in an iron from Dakota, while C. A. Joy reported [g] 0.44 per cent SnO_2 in the mineral portion of the Atacama pallasite, and Mallet reported [h] 0.002 per cent to 0.003 per cent Sn in the iron from Staunton, Virginia. Still more recently traces of the metal have been reported in the Barraba and Cowra, New South Wales, irons, by J. C. H. Mingaye.[i] Coming from such a source the statement might well be considered as conclusive (see further p. 18). Numerous other occurrences of like small amounts are mentioned in the literature, the copper and tin being frequently undifferentiated. Although not so stated, the inference may be drawn, with the possible exception of that found in the olivine above noted, that tin, if present at all, occurs mainly if not wholly in the metallic portion.

Titanium.—Rammelsberg [j] found 0.16 per cent titanic oxide (TiO_2) in the insoluble residue from the Juvinas stone. This is the first reported occurrence. Davison reported [k] traces of titanium in the stone from Estacado, Tex. Everhart [l] found 0.09 per cent in that of Pickens County, Ga.; and Stokes [m] found 0.08 per cent in the stone of Allegan, Mich. These occurrences are all dwarfed by Tschermak's [n] determination of 2.39 per cent in the stone of Angra dos Reis. The amounts are, however, mostly very small, and knowing the difficulties in the way of determination, it seems unquestionable that it is a common and widespread constituent.

Vanadium.—Apjohn reported [o] finding unmistakable evidences of vanadium in the Limerick stone, but in amount too small for quantitative determination. The occasional presence of the element seems now confirmed beyond doubt.

Zinc.—E. Pfeiffer, in the report [p] of his analysis of the Parnallee stone, included traces of copper, tin and zinc. J. L. Smith [q] found in the schreibersite from the Tazewell iron a trace of zinc. I find no other recorded occurrence of this element.

III. DETAILED CHEMICAL AND MINERALOGICAL DETERMINATIONS.

In the following pages are given in considerable detail the results of my investigations, the chemical analyses upon which most reliance is placed being mainly those of Dr. J. E. Whitfield, as before stated, though in the final tabular statement I have brought together the work of such other analysts as seems sufficiently detailed or given in such forms as to merit attention. The results given in the first paper are here repeated. The order of arrangement is alphabetical, under the three heads Irons, Stony-irons, and Stones.

(1) *Iron.*—Canon Diablo, Ariz. A coarse octahedral iron with numerous interlaminations of schreibersite and inclusions of graphite and troilite, the latter sometimes an inch or

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[a] Amer. Journ. Sci., vol. 7, 1899, p. 4.
[b] Proc. U. S. Nat. Mus., vol. 27, 1904, p. 959.
[c] See Rose, Beschreibung u. Eintheilung der Meteoriten, etc., 1864, p. 77.
[d] Pogg. Ann. Phys. Chem., vol. 62, 1844, p. 449.
[e] Die Chemischer Natur Meteoriten, 1870, p. 146.
[f] Amer. Journ. Sci., vol. 36, 1863, p. 260.
[g] Amer. Journ. Sci., vol. 37, 1864, p. 245.
[h] Amer. Journ. Sci., vol. 2, 1871, p. 13.
[i] Records Geol. Surv. N. S. Wales, vol. 7, pt. 4, 1904, p. 305.
[j] Pogg. Ann. Phys. Chem., vol. 73, 1848, p. 585.
[k] Amer. Journ. Sci., vol. 22, 1906, p. 59.
[l] Science, vol. 30, 1909, p. 772.
[m] Proc. Wash. Acad. Sci., vol. 2, July, 1900, p. 48.
[n] Tsch. Min. Pet. Mittheil., vol. 28, 1909, p. 110.
[o] Journ. Chem. Soc. London, vol. 27, 1874, p. 104.
[p] Sitz. k. Akad. Wiss. Wien, vol. 47, 1863, p. 461.
[q] Amer. Journ. Sci., vol. 19, 1855, p. 155.

more in diameter. In the selection of samples for analyses these were avoided as far as possible. Carbon is present in form of microscopic diamonds and also as graphite. It has been the subject of numerous analyses, yielding variable results owing to its coarse crystallization. Whitfield's analysis is given in column I below. In column II is given the average of three analyses by Moissan, Booth, Garrett & Blair, and Tassin.

Constituents.	I	II
	Per cent.	*Per cent.*
Silicon	Trace.	0.032
Sulphur	0.009	.007
Phosphorus	.261	.159
Manganese	None.	None.
Copper	.015	Trace.
Nickel	7.335	5.828
Cobalt	.510	.044
Combined carbon	.105	.465
Graphitic carbon	.028	
Iron oxides	2.520	....
Iron protochloride	.097	....
Iron	89.167	93.425
Total	100.047	99.96

In process of analysis schreibersite to the amount of 1.832 per cent separated out. This afforded the following composition:

	Per cent.
Iron	55.04
Nickel	29.58
Phosphorus	15.38

Platinum was looked for in two portions of fifty grams each, but none was found. Neither is its presence recorded in previous analyses by this same firm, by Derby,[a] Tassin or Moissan.[b] J. W. Mallet, working on a residue from the solution of 25 pounds of the iron in dilute hydrochloric acid obtained results representing 3.63 grams of platinum and 14.95 grams of iridium per metric ton of the original iron, "with probably a trace of rhodium." He suggests, and this is in accordance with our own results, that the platinoid metals are not uniformly distributed in the iron.[c] This may account for the failure to find it on the part of others. The quantity of material worked upon is undoubtedly an important factor, however.

(2) *Iron.*—Casas Grandes, Mexico. Medium octahedrite. Previously analyzed and described by Tassin.[d] The results of Dr. Whitfield's analyses are given in Column I below. In Column II are given Mr. Tassin's results as previously obtained.

Constituents.	I	II
	Per cent.	*Per cent.*
Silicon	0.010	....
Iron	90.470	95.13
Nickel	7.742	4.38
Cobalt	.604	.27
Copper	.012	Trace.
Phosphorus	.166	.24
Sulphur	.029	None.
Combined carbon	.145	Traces.
Graphitic carbon	.032	
Iron oxides	.794	....
Total	100.004	100.02

Whitfield's analyses were made on 50-gram samples, free from evident inclusions of troilite. The following elements were especially looked for but not found; antimony, arsenic, tin, lead, palladium, platinum, titanium, tungsten, vanadium, uranium, chromium, manganese, molybdenum, and zinc.

[a] Amer. Journ. Sci., vol. 49, 1895, pp. 101-110.

[b] See contributions to the Study of the Canyon Diablo Meteorite, by G. P. Merrill and W. Tassin, Smithsonian Misc. Coll., Quar. Issue, vol. 50, pt. 2, 1907, p. 209.

[c] Proc. Acad. Nat. Sci. Phila., Dec., 1905, p. 913. Footnote on p. 862.

[d] Proc. U. S. Nat. Mus., vol. 25, 1902, pp. 69-74.

(3) *Iron.*—Mount Joy, Pa. Coarse octahedrite; brecciated. Previous analysis by Eakins[a] yielded:

	Per cent.
Iron	93. 80
Nickel	4. 81
Cobalt	. 51
Copper	. 005
Phosphorus	. 190
Sulphur	. 01
	99. 325

Further tests for the minor constituents by Dr. Whitfield on a 50-gram sample yielded:

	Per cent.
Chromium	0. 006
Manganese	. 075
Copper	. 008
Chlorine	. 255
Platinum	Trace.

No vanadium, molybdenum, tungsten, gold, silver, lead, or tin, nor in fact any other element in amounts large enough to be determined by wet analysis, were found.

(4) *Iron.*—Perryville, Mo. Described by Merrill[b] as belonging to Brezina's group of finest octahedrites (Off). As the entire iron was in possession of the National Museum, and it had not before been described, all the necessary material was sacrificed for a very detailed analysis, with the results tabulated below:

	Per cent.
Iron	89. 015
Nickel	9. 660
Cobalt	. 545
Copper	. 025
Manganese	None.
Phosphorus	. 365
Sulphur	. 002
Silicon	. 003
Carbon	. 015
Iridium	Traces.
Palladium	
Platinum	
Ruthenium	
	99. 63

The amount of the rarer elements found in different samples of this iron was quite variable, but always small. From one portion of 25 grams was obtained 0.004 gram of platinum and from another portion of 100 grams weight but 0.0002 gram. The precipitates of ammonium platinic chloride were in all cases faintly orange, indicating the presence of palladium but in amounts too small for determination. In a 100-gram sample of the iron were found 0.014 gram of ruthenium and 0.028 gram of iridium, while another portion of equal weight yielded but 0.0009 gram of ruthenium and 0.0011 gram of iridium.

So far as I am aware this is the first recorded occurrence of ruthenium in a meteoric iron. The probable presence of iridium in the Toluca and Coahuila irons was recognized by Davison, as already noted, but is not elsewhere recorded. No chromium, vanadium, molybdenum, or titanium were found.

The mineral schreibersite, constituting 2.61 per cent of the iron, was isolated and analyzed with the following results:

	Per cent.
Phosphorus	14. 00
Iron	51. 10
Nickel	34. 13
Cobalt	. 30
	99. 53

[a] Amer. Journ. Sci., vol. 44, 1892, p. 416.

[b] Proc. U. S. Nat. Mus., vol. 43, No. 1943, 1913.

The high percentage of nickel shown by this analysis is comparable with that of the schreibersite from the Magura iron. Even greater amounts have been reported as yielded by the irons of Seeläsgen, Germany (36.17 per cent), and Cranbourne, Australia (38.24 per cent).

(5) *Stony-iron (Pallasite)*.—Mount Vernon, Ky. A coarse pallasite, consisting of large blebs of olivine in a mesh of metal. Described by Tassin.[a] No complete (bulk) analyses made owing to the coarse nature of the stone. The nickel alloy yielded Tassin as follows:

	Per cent.
Iron	82. 520
Nickel	14. 044
Cobalt	. 949
Copper	. 104
Sulphur	. 288
Silica	. 808
Aluminum	. 410
Carbon	. 465
Phosphorus	. 390
Chlorine	Trace.
	99. 978

He also gave analyses of the included tænite, schreibersite, troilite, chromite, and olivine separately, but found no constituents of unusual occurrence.

A 50-gram sample, badly oxidized, submitted to Dr. Whitfield for tests for minor elements, yielded:

	Per cent.
Chromium	0. 300
Copper	. 016
Nickel	2. 960
Cobalt	. 090
Manganese	. 151
Vanadium	Trace.

No trace of molybdenum, tungsten, antimony, tin, lead, zinc, gold, silver, or platinum was found.

(6) *Stony-iron (Pallasite)*.—Krasnojarsk, Siberia. A well-known historic meteorite, stated by previous workers to contain arsenic and tin. The material being too coarse for bulk or mass analysis, the metal and olivine were examined independently. The metallic portion yielded:

	Per cent.
Fe	89. 90
Ni	9. 52
Co	. 60
P	. 085
	100. 105

The silicate portion (olivine) yielded:

	Per cent.
SiO_2	37. 22
FeO	15. 21
Al_2O_3	. 46
MgO	47. 07
	99. 96

with no traces of arsenic, chromium, manganese, sulphur, or tin, in either portion.

(7) *Meteoric stone, Chladnite*.—Bishopville, S. C. This unique stone fell on March 25, 1843, and was first described by Shepard in 1846.[b] It has since been the subject of numerous writings (see Wülfing, pp. 29–31), and in several instances subjected to partial analyses. The widely variant results obtained are due as much to imperfect sampling as to incorrect determinations. The Museum collection possessing several grams of fragmental material, it was deemed advisable to sacrifice them to the searching methods of modern analytical chemistry. The stone was described by Shepard as consisting in large part of a

[a] Proc. U. S. Nat. Mus., vol. 28, 1905, p. 213.

[b] Amer. Journ. Sci., vol. 2, 1846, p. 379.

light gray material, regarded by him as a persilicate of magnesia, and to which he gave the name *chladnite* in honor of the chemist Chladni. Researches in 1864 by J. Lawrence Smith showed the mineral to be identical with enstatite. The stone was described in 1854 by W. Sartorious von Waltershausen, who thought to show that the siliceous portion was made up of 95.011 per cent chladnite and 4.985 per cent labradorite. Rammelsberg in 1863 stated as a result of his examinations that it contained no feldspar. Later microscopic investigations by Wadsworth [a] and Tschermak [b] show the stone to be a crystalline granular mass of enstatite plagioclase feldspar, and an iron sulphide identified as pyrrhotite. Wadsworth includes also augite and metallic iron. The probable presence of oldhamite, confirmed by the present researches, was suggested but not proven by Maskelyne.[c] Below are given the results of Dr. Whitfield's analyses, made with especial reference to the presence or absence of the mineral oldhamite, and the elements barium, strontium, and zirconium. The commonly quoted analyses of J. Lawrence Smith, it should be noted, were not of the stone in mass, but of the selected white pyroxenic constituent, the chladnite of Shepard.

	Per cent.
Silica (SiO_2)	57. 034
Alumina (Al_2O_3)	1. 706
Ferric oxide (Fe_2O_3)	1. 406
Manganous oxide (MnO)	. 189
Lime (CaO)	2. 016
Magnesia (MgO)	33. 506
Cobalt oxide (CoO)	Trace.
Nickel oxide (NiO)	. 538
Soda (Na_2O)	1. 027
Potash (K_2O)	. 089
Ignition (H_2O)	1. 995
Iron (Fe)	. 181
Nickel (Ni)	. 039
Sulphur (S)	. 297
	100. 023
Minus O for S	. 147
	99. 876

An amount of calcium equivalent to 0.67 per cent calcium sulphide was liberated by boiling the finely pulverized stone for two hours in distilled water. Inspection of the stone in mass shows, in addition, occasional granules of an iron sulphide (troilite or pyrrhotite) which were not included in the portion analyzed. No traces of barium, strontium, or zirconium could be detected. This is worthy of note in view of the feldspathic nature of the stone. The amount of material utilized in the analyses was not as large as could have been desired.

(8) *Meteoric stone, Chondrite (Cc).*—Collescipoli, Italy. This stone, which fell on February 3, 1890, is of a gray color, somewhat friable, of a common chondritic type (*Cc*), composed essentially of olivine and enstatite with the usual sprinkling of metal and metallic sulphide. It presents no unusual features macro- or microscopically, and attention was given it here only on account of the extraordinary array of the rarer elements reported in Trottarelli's analyses.[d] It is unfortunate that although this fall is supposed to have comprised some 4 or 5 kilograms, but 1,802 grams are known to-day and hence the prices quoted by dealers ($0.70 to $1.27 per gram) are so high as to place it out of reach for exhaustive investigation. Fortunately a few grams were found which, on account of their minutely fragmental condition, could be purchased at prices justifying sacrifice. Below are given the results obtained by Whitfield:

	Per cent.
Metallic portion	18. 60
Silicate portion	81. 40
	100. 00

[a] Lithological Studies, p. 200, 1884.

[b] Sitz. k. Akad. Wiss. Wien, vol. 88, pt. 1, 1883, p. 363.

[c] Proc. Royal Soc. Edinburgh, vol. 18, 1870, p. 146.

[d] Gazetta Chimica Italiana, vol. 20, 1890, pp. 611-615.

The metallic portion yielded:

	Per cent.
Iron (Fe)	91.60
Nickel (Ni)	8.00
Cobalt (Co)	.49
	100.09

The silicate portion yielded:

	Per cent.
Silica (SiO_2)	42.50
Alumina (Al_2O_3)	7.90
Iron protoxide (FeO)	19.50
Lime (CaO)	2.20
Magnesia (MgO)	26.00
Potash (K_2O)	.32
Soda (Na_2O)	1.80
	100.22

Careful search was made for other elements, and in particular, for the reasons noted, for arsenic, antimony, copper, and platinum. None were found.

In order to compare these results with those of Trottarelli, the analyses were recalculated and the results given in column I below; in column II are given those of Trottarelli. I will not attempt to account for the discrepancy.

Constituents.	I	II
	Per cent.	*Per cent.*
Silica (SiO_2)	34.59	31.057
Alumina (Al_2O_3)	6.43	.9304
Ferrous oxide (FeO)	15.87	
Magnesia (MgO)	21.17	.0186
Lime (CaO)	1.79	.1169
Potash (K_2O)	.26	Traces.
Soda (Na_2O)	1.46	10.386
Iron (Fe)	17.04	40.983
Nickel (Ni)	1.49	1.544
Cobalt (Co)	.09	Traces.
Palladium (Pd)	None.	.7745
Manganese (Mn)	None.	1.0060
Chromium (Cr)	None.	.5616
Sulphur (S)	None.	7.679
Ignition (H_2O)	None.	2.100
Total	100.19	[a] 97.157

[a] With traces of lead, antimony, tin, lithia, sulphuric anhydride, and chlorine.

(9) *Meteoric stone, Chondrite* (*Cc*).—Cullison, Kans. Described by Merrill.[a] Thin sections showed it to be of the normal chondritic type containing olivine, enstatite, monoclinic pyroxene, plagioclase feldspar, with the usual sprinkling of metal and metallic sulphides. The separation of the component parts by an electromagnet and treatment with iodine resulted as follows:

	Per cent.
Troilite	6.07
Metal	19.40
Silicate	74.50
Schreibersite	.10
	100.07

The metallic portion yielded:

	Per cent.
Silicon	0.129
Sulphur	Trace.
Phosphorus	.071
Nickel	9.207
Cobalt	.507
Copper	.040
Chromium	.160
Carbon	.088

[a] Proc. U. S. Nat. Mus., vol. 44, 1913, pp. 325-330.

	Per cent.
Manganese	.080
Iron	89.700
	99.982

with no traces of tungsten, vanadium, or molybdenum.

The silicate portion yielded:

	Per cent.
Silica (SiO_2)	47.36
Alumina (Al_2O_3)	5.67
Ferric oxide (Fe_2O_3)	.10
Ferrous oxide (FeO)	11.25
Lime (CaO)	.84
Magnesia (MgO)	31.72
Manganous oxide (MnO)	.36
Soda (Na_2O)	2.42
Potash (K_2O)	.23
Titanic oxide (TiO_2)	None.
	99.95

Combining the metallic and nonmetallic portions and recalculating with the usual assumption that the mineral called troilite is the monosulphide FeS, and that the schreibersite conforms to the formula Fe_2NiP, the following figures are obtained representative of the composition of the stone as a whole:

	Per cent.
Silica (SiO_2)	35.30
Alumina (Al_2O_3)	4.24
Ferric oxide (Fe_2O_3)	.75
Ferrous oxide (FeO)	8.38
Lime (CaO)	.62
Magnesia (MgO)	23.631
Manganous oxide (MnO)	.268
Soda (Na_2O)	1.804
Potash (K_2O)	.171
Sulphur (S)	2.184
Phosphorus (P)	.0138
Nickel (Ni)	1.80
Cobalt (Co)	.098
Copper (Cu)	.008
Chromium (Cr)	.029
Carbon (C)	.017
Manganese (Mn)	.015
Iron (Fe)	21.270
	100.5988

None of the rarer elements, other than those noted, were found.

(10) *Meteoric stone, Chondrite.*—Elm Creek, Kans. This stone was plowed up in a field some time in May, 1906. Nothing is known of its fall. It was considerably oxidized on the outside, indicating that it had lain some time in the soil. It is described by Howard[a] as of a dark gray, nearly black color, thickly studded on a polished surface with metallic points and indistinct chondrules, which break, in part, with the groundmass. The silicate portion, as shown by the microscope, consists essentially of olivine and enstatite with a polysynthetically twinned monoclinic pyroxene. It had never before been subjected to chemical analysis, and was therefore open to critical investigation. Dr. Whitfield found:

	Per cent.
Silicates	93.18
Metal	6.82
	100.00

[a] Amer. Journ. Sci., vol. 28, 1907, p. 380.

The silicate portion yielded:

	Per cent.
Silica (SiO_2)	36.76
Alumina (Al_2O_3)	3.10
Ferric oxide (Fe_2O_3)	13.23
Ferrous oxide (FeO)	14.22
Chromic oxide (Cr_2O_3)	.35
Lime (CaO)	1.62
Magnesia (MgO)	25.66
Water (H_2O)	5.10
	100.04

No barium, strontium, zirconium, or other rare elements could be detected.

The metallic portion yielded:

	Per cent.
Iron (Fe)	87.13
Nickel (Ni)	11.30
Cobalt (Co)	1.42
Manganese (Mn)	.15
Copper (Cu)	None.
	100.00

The amount of metal available (1.35 grams) was not sufficient for an exhaustive examination for the rarer elements.

(11) *Meteoric stone, Chondrite (Ci).*—Fisher, Polk County, Minn. This, the first and only meteoric stone thus far reported from Minnesota, is supposed to be the representative of a fall which took place on the 9th of April, 1894. It was made the subject of an investigation by Prof. N. H. Winchell,[a] which was, however, not completed. The matter was subsequently taken up by the present writer, and a detailed account of it published in the Proceedings of the U. S. National Museum.[b] The stone is described as consisting of a confused aggregate of irregular crystalline granules of olivine and pyroxene interspersed with numerous imperfectly outlined chondrules of the same mineral, throughout which are occasional interstitial areas of a colorless, pellucid, isotropic material referred to maskelynite. The pyroxenes are wholly of the enstatite type and devoid of twinned structure. The analyses yielded results as below:

	Per cent.
Metallic constituents	11.44
Silicate constituents	88.56
	100.00

The silicate portion yielded:

	Per cent.
Silica (SiO_2)	43.70
Alumina (Al_2O_3)	4.96
Ferrous oxide (FeO)	18.27
Manganous oxide (MnO)	.38
Nickel oxide (NiO)	.23
Lime (CaO)	2.19
Magnesia (MgO)	29.38
Chromite ($FeOCr_2O_3$)	.80
	99.91

The metallic portion, freed from the last trace of siliceous matter, yielded:

	Per cent.
Iron (Fe)	85.00
Nickel (Ni)	14.15
Cobalt (Co)	.74
Copper (Cu)	Trace.
	99.89

[a] American Geologist, vol. 14, 1894, p. 389; vol. 17, 1896, p. 173; vol. 20, 1897, p. 316.
[b] Vol. 18, 1915, pp. 503-506.

A recalculation of these results gives the bulk or mass composition of the stone as follows:

	Per cent.
Silica (SiO_2)	38.70
Alumina (Al_2O_3)	4.39
Ferrous oxide (FeO)	16.406
Manganous oxide (MnO)	.336
Nickel oxide (NiO)	.204
Lime (CaO)	1.939
Magnesia (MgO)	26.018
Chromic oxide (Cr_2O_3)	.482
Metallic iron (Fe)	9.724
Metallic nickel (Ni)	1.608
Metallic cobalt (Co)	.084
	99.891

No traces could be discovered of barium, strontium, zirconium, or potassium.

(12) *Meteoric stone, Chondrite.* Holbrook, Ariz. Described by Merrill.[a] A gray chondritic stone, very fresh, having fallen on July 19, 1912. Contains very little metallic iron, but is correspondingly rich in sulphide. Analyses yielded:

	Per cent.
Schreibersite	0.11
Troilite	7.56
Metal	4.85
Silicates	87.48
	100.00

The metallic portion yielded:

	Per cent.
Nickel	8.68
Cobalt	.64
Copper	.29
Iron	90.50
	100.11

The silicate portion yielded:

	Per cent.
Silica (SiO_2)	41.93
Alumina (Al_2O_3)	4.30
Ferrous oxide (FeO)	21.85
Lime (CaO)	2.40
Soda (Na_2O)	Trace.
Magnesia (MgO)	29.11
Manganous oxide (MnO)	.25
Nickel oxide (NiO)	.08
	99.92

Specific gravity at 22.6° C., 3.48.

None of the rarer elements under consideration were found, even in traces. The sulphide occurs in such forms as to be readily separated mechanically, and yielded on analysis:

	Per cent.
Iron	63.62
Sulphur	36.50
Nickel, cobalt, and copper	None.
	100.12

This shows the mineral to be troilite, though its specific gravity (4.61) is low. It is, however, wholly unattracted by the magnet, and apparently there is no question as to its true nature. Its occurrence in this form is interesting in a stone so low in the metallic constituent.

(13) *Meteoric stone, Carbonaceous Chondrite (Cc).*—Indarch, Elizabethpol, Russia. This interesting stone fell, according to Meunier,[b] on the 9th of April, 1891. Although made the subject of numerous brief papers, it seems never to have been previously analyzed, and was, there-

[a] Smithsonian Misc. Coll. vol. 60, 1912, No. 2149.

[b] Comptes Rendus, Acad. Sci., Paris, vol. 125, 1897, p. 894.

fore, taken up in connection with the present work and a detailed description of it has been given in the Proceedings of the National Museum.[a] The stone is there described as of a dark greenish gray color, firm and compact, admitting of a polish and thickly studded with small, dark, almost black chondrules and nodular masses of metal and troilite. A microscopic examination shows it to consist of a dense black irresolvable ground, throughout which are scattered iron and iron sulphide, together with abundant sharp splinters of pyroxene and numerous more or less fragmentary chondrules of the same mineral. No olivine, feldspar, or other silicate mineral was determined. The presence of carbonic acid, as shown by the analysis, suggested the mineral breunnerite, but this could not be determined absolutely owing to the obscuring effect of the abundant graphite. The sulphide of calcium, oldhamite, was detected both by chemical means and by the microscope. The analyses yielded results as follows:

Metallic portion separated by mercuric chloride solution:

	Per cent.
Iron (Fe)	90.44
Nickel (Ni)	8.26
Cobalt (Co)	.18
Phosphorus (P)	.08
Manganese (Mn)	1.04
	100.00

Silicate portion, free as possible from the metal, sulphides, and graphite, yielded:

	Per cent.
Silica (SiO_2)	47.970
Alumina (Al_2O_3)	2.647
Ferrous oxide (FeO)	19.283
Phosphoric acid (P_2O_5)	.699
Manganous oxide (MnO)	.175
Nickel oxide (NiO)	.739
Cobalt oxide (CoO)	.067
Lime (CaO)	1.559
Magnesia (MgO)	22.736
Carbonic acid (CO_2)	.363
Soda (Na_2O)	Trace.
Potash (K_2O)	None.
Water (H_2O)	3.762
	100.00

A recalculation of these analyses gives the following, showing the composition of the stone as a whole:

	Per cent.
Silica (SiO_2)	35.699
Alumina (Al_2O_3)	1.969
Ferrous oxide (FeO)	25.790
Manganous oxide (MnO)	.130
Nickel oxide (NiO)	.549
Cobalt oxide (CoO)	.049
Lime (CaO)	1.160
Magnesia (MgO)	16.920
Carbonic acid (CO_2)	.271
Phosphoric acid (P_2O_5)	.520
Water (H_2O)	2.799
Iron (Fe)	10.400
Nickel (Ni)	.949
Cobalt (Co)	.020
Phosphorus (P)	.092
Manganese (Mn)	.119
Carbon (graphite) (C)	.310
Sulphur (S)	5.100
	102.846
Minus O for S	2.54
	100.306

No barium, strontium, or zirconium could be detected.

[a] Proc. U. S. Nat. Mus., vol. 49, 1915, p. 109.

The mineral composition so far as determined by analysis and microscopic examination is:

	Per cent.
Silicate (enstatite)	74.42
Metal	11.50
Troilite	13.296
Oldhamite	.596
Graphite	.31
	100.122

Specific gravity, 3.42.

(14) *Meteoric stone, Eukrite (Eu).*—Juvinas, France. This stone has been widely circulated and is represented in all the collections of importance throughout the world. As a result it has been made the subject of numerous memoirs and briefer notices, Wülfing recording forty-seven titles in his catalogue. I find, however, no recorded analysis later than that quoted below, which dates back to 1848. In view of this, and the additional fact that it is a feldspathic stone, it seemed worth the while to give it consideration here with especial reference to the possible occurrence of barium, strontium and zirconium. Wadsworth, in his review of the mineralogical determinations made by Rammelsberg, Tschermak, and others, states that the stone consists of anorthite and augite with small amounts of pyrrhotite and nickel-iron. Rammelsberg noted chromite and ilmenite, while Fouqué and Lévy detected also enstatite.

In Column I below are given the results obtained by Whitfield and in II those of Rammelsberg.

Constituents.	I	II
	Per cent.	*Per cent.*
Silica (SiO_2)	47.99	49.23
Titanic oxide (TiO_2)	.57	.10
Alumina (Al_2O_3)	13.50	12.55
Ferric oxide (Fe_2O_3)	.22	1.21
Iron (Fe)	Traces.	.16
Nickel oxide (NiO)	.11	....
Cobalt oxide (CoO)	Trace.	....
Ferrous oxide (FeO)	18.63	20.33
Lime (CaO)	10.60	10.23
Magnesia (MgO)	7.20	6.44
Barium oxide (BaO)	None.	....
Strontium oxide (SrO)	None.	....
Zirconium oxide (ZrO)	None.	....
Potash (K_2O)	None.	.12
Soda (Na_2O)	.55	.63
Chromic oxide (Cr_2O_3)	Trace.	.24
Phosphoric acid (P_2O_5)	None.	.28
Sulphur (S)	.054	.09
Sulphuric anhydride (SO_3)	.02	....
Total	99.444	101.61

The amount of metal was so small as to make it practically impossible to determine the proportional amounts of nickel, cobalt, and iron. The SO_3 and a part of the Fe_2O_3 were doubtless derived from the iron sulphide through oxidation, and have so been considered in the final tabulation. Tests were made for oldhamite by boiling the powdered material in water, but no calcium could be detected. The absence of chromium in Whitfield's analysis is doubtless due to the sporadic occurrence of the mineral chromite and the small size of the sample submitted, but 9 grams being available.

(15) *Meteoric stone, Black Chondrite (Cs).*—McKinney, Collin County, Tex. Referred to by Brezina[a] and relegated to his Cs type, characterized by colorless chondrules firmly imbedded in a dark gray to black ground. The mineral composition and structure are given, but no analyses. Dr. Whitfield found the stone to consist of:

	Per cent.
Troilite	6.26
Schreibersite	.58
Metal	5.70
Chromite	.11
Silicate minerals	87.35
	100.00

[a] Ann. k. k. Hofmus., Band 10, Heft 3 u. 4, 1895 (96), p. 252.

The silicate portion yielded:

	Per cent.
Silica (SiO_2)	43.30
Alumina (Al_2O_3)	15.18
Ferrous oxide (FeO)	8.45
Lime (CaO)	1.88
Magnesia (MgO)	30.48
Manganous oxide (MnO)	.25
Nickel oxide (NiO)	.51
	100.05

The metallic portion yielded:

	Per cent.
Iron (by difference)	85.84
Cobalt	.92
Copper	.08
Nickel	13.16
	100.00

Barium, strontium and zirconium, in addition to the other rarer elements, were looked for but no traces discovered.

(16) *Meteoric stone, Gray Chondrite* (*Cg*).—Monroe, Cabarrus County, N. C. This stone has been briefly described by several writers and subjected to at least one previous chemical analysis. The first examination was made by C. U. Shepard [a] who relied for his mineralogical determinations upon the results of chemical analyses, thin sections at that date not being available. Wadsworth mentioned it briefly in his Lithological Studies, but added little excepting to justly remark that "judging from its general character Shepard's analysis is incorrect and it is hoped a new one may be made." Wülfing in his catalogue places the stone in Brezina's group, Cga, that is, with stones consisting of a tuff-like mass with variously colored chondrules firmly embedded in the ground. It remains to be stated that the chondrules are in part of olivine and in part of pyroxene, both varieties occurring in porphyritic or in barred or radiating forms. Two varieties of pyroxene are recognizable, the one obscurely twinned and monoclinic—the klinoenstatite variety—and the other occurring frequently in large pellucid forms and orthorhombic in crystallization. Metallic iron and iron sulphides with their oxidation products complete the list of recognizable minerals. The wide discrepancies between the analyses of Shepard and Whitfield can be accounted for on the supposition that the material utilized by the first named was not representative and his methods imperfect.

Shepard's analysis:

	Per cent.
Silica (SiO_2)	56.168
Ferrous oxide (FeO)	18.108
Magnesia (MgO)	10.406
Alumina (Al_2O_3)	1.797
Nickel-iron with traces of chromium	6.326
Magnetic pyrite	3.807
Lime, soda, potash, and loss	3.394
	100.006

Whitfield's results as follows:

	Per cent.		Per cent.
Silica (SiO_2)	36.71	Silicates..	82.60
Alumina (Al_2O_3)	3.59		
Chromic oxide (Cr_2O_3)	Trace.		
Ferrous oxide (FeO)	14.80		
Manganous oxide (MnO)	.23		
Nickel and cobalt oxides (NiOCoO)	.46		
Lime (CaO)	2.27		
Magnesia (MgO)	24.54		
Iron (Fe)	12.58	Metal.....	13.54
Nickel (Ni)	.87		
Cobalt (Co)	.09		

[a] Amer. Journ. Sci., vol. 10, 1850, p. 127.

			Per cent.
Iron (Fe)	2.39	Troilite...	3.80
Sulphur (S)	1.41		
	99.94		99.94

No trace of barium, strontium, lithium, soda, potash, zirconium, or copper, could be discovered.

(17) *Meteoric stone, Chondrite (Ci).*—Ness County, Kans. Described by H. L. Ward.[a] No analysis given. The stone was somewhat decomposed through weathering, but yielded approximately 15 per cent of nickeliferous iron which showed:

	Per cent.
Copper	0.30
Nickel	7.00
Cobalt	.20
Iron	92.04
	99.54

A bulk analysis in which all the combined and oxidized iron was determined as ferric oxide, yielded:

	Per cent.
Silica (SiO_2)	38.340
Ferric oxide (Fe_2O_3)	8.551
Alumina (Al_2O_3)	8.259
Chromic oxide (Cr_2O_3)	.587
Lime (CaO)	1.180
Magnesia (MgO)	24.040
Metal (FeNi)	15.000
Loss on ignition	3.500
	99.457

Recalculating the first analysis in order to include the components of the metallic portions, and thus obtain the composition of the stone as a whole, we have:

	Per cent.
Silica (SiO_2)	38.340
Ferric oxide (Fe_2O_3)	8.551
Alumina (Al_2O_3)	8.259
Chromic oxide (Cr_2O_3)	.587
Lime (CaO)	1.180
Magnesia (MgO)	24.040
Loss on ignition	3.500
Iron (Fe)	13.860
Nickel (Ni)	1.050
Cobalt (Co)	.030
Copper (Cu)	.050
	99.447

None of the rarer elements were found. The high ignition is due largely to the hydrous sesquioxide of iron formed through weathering.

(18) *Meteoric stone, Chondrite (Cc).*—Selma, Ala. Described by Merrill[b] but no analyses given. Examination of thin sections showed the presence of the usual sulphide particles together with olivine, enstatite, and a monoclinic pyroxene.

A bulk analysis as given in my preliminary report was so unsatisfactory, particularly with reference to the iron and alkali content, that additional material was selected and new analyses made with results as given below. No metallic iron could be detected, though whether or not this was due to oxidation, as seems probable from the high content of ferric oxide and water, could not be determined.

[a] Amer. Journ. Sci., vol. 7, 1899, p. 233.

[b] Proc. U. S. Nat. Mus., vol. 32, 1907, p. 59.

	Per cent.
Silica (SiO_2)	31.06
Alumina (Al_2O_3)	4.30
Phosphoric acid (P_2O_5)	.25
Chromic oxide (Cr_2O_3)	.41
Ferric oxide (Fe_2O_3)	18.15
Ferrous oxide (FeO)	13.07
Manganous oxide (MnO)	.26
Nickel oxide (NiO)	1.45
Cobalt oxide (CoO)	.15
Lime (CaO)	2.13
Magnesia (MgO)	21.21
Soda (Na_2O)	3.96
Potash (K_2O)	.07
Vanadium oxide (V_2O_3)	Trace.
Water (H_2O)	3.07
Troilite (S)	.19
Troilite (Fe)	.32
	100.05

No traces of other constituents than those mentioned.

(19) *Meteoric stone, Eukrite.*—Stannern, Moravia. This stone, which fell on the 22d of May, 1808, has become, on account of its wide distribution in public and private collections throughout the world, one of the best known of meteorites. It naturally follows that it has been repeatedly the subject of investigation and publication. Wülfing's catalogue gives 74 independent publications between the date of fall and 1894, two of which included chemical analyses. Of these only that of Rammelsberg, given in Column II below, needs consideration. The latest analysis, by Whitfield, is given in column I. This was made with especial reference to the possible presence of barium, strontium, or zirconium, it being a feldspathic stone.

Constituents.	I	II
	Per cent.	*Per cent.*
Silica (SiO_2)	47.94	48.30
Titanic oxide (TiO_2)	.41	
Alumina (Al_2O_3)	11.19	12.65
Ferric oxide (Fe_2O_3)	1.20	
Iron (Fe)	Trace.	
Nickel oxide (NiO)	.25	
Cobalt (Co)	Trace.	
Ferrous oxide (FeO)	18.97	19.32
Lime (CaO)	10.36	11.27
Magnesia (MgO)	7.14	6.87
Manganous oxide (MnO)		.81
Barium oxide (BaO)	None.	
Strontium oxide (SrO)	None.	
Zirconium oxide (ZrO)	None.	
Potash (K_2O)	.13	.23
Soda (Na_2O)	.75	.62
Chromic oxide (Cr_2O_3)	.35	.54
Iron (Fe)	.55	
Sulphur (S)	.31	Trace.
Phosphoric acid (P_2O_5)	.14	
Ignition (H_2O)	.30	
Total	99.99	100.61

(20) The three meteorites mentioned below were subject to partial analyses only.

Ballinoo, Australia.—Iron. Finest octahedrite. This iron, described by Sjöström,[a] so closely resembles in its physical and chemical properties that of Perryville, Mo., that it was thought advisable to test it for the rarer or minor constituents not reported in its published analysis. A 30-gram fragment submitted to Dr. Whitfield showed neither platinum nor iridium, but on the other hand did show unmistakable traces of palladium and ruthenium.

Glorieta Mountain, N. Mex.—Iron. Medium octahedrite. This iron, described by G. F. Kunz, with analyses by L. G. Eakins[b] was stated to carry 0.03 per cent zinc. Dr. Whitfield, however, working on another portion, failed to find a trace of the metal.

[a] Sitz. k. Preus. Akad. Wiss. Berlin, 1908, p. 21.

[b] Ann. N. Y. Acad. Sci., vol. 3, 1885, p. 334.

Misshof, Courland, Russia.—Stone, Cc. Johanson's analysis shows 0.18 per cent Cu and 0.156 per cent SnO_2.[a] Two fragments, one of 7 and one of 28 grams, were submitted to Dr. Whitfield, who reports 0.01 per cent Cu in the first and 0.008 per cent in the second, but no traces of tin in either.

IV. DISCUSSION OF RESULTS.

Gold and the platinoid elements.—It will be noted that our work has failed to substantiate the reported occurrence of gold in meteorites, either iron or the stony varieties, while the occurrence of platinum, palladium, iridium, and ruthenium in the irons appears proven beyond a doubt. That these last are not more frequently reported is probably due to the careful, detailed work involved in their determination, and perhaps also to their very irregular distribution, noted on page 5. The close relationship existing between the gold and platinum metals would render their association in the meteorites not surprising were it not that the mineral associations of the two are in terrestrial rocks so unlike, gold being rarely if ever reported from rocks as basic as the peridotites. I do not find, however, that the terrestrial peridotites have as yet been subjected to the careful scrutiny necessary to decide this absolutely.

While, however, Dr. Whitfield's analyses fail to bring to light a trace of gold, it should be noted that in addition to Liversidge's determinations, J. C. H. Mingaye,[b] in a very thorough analysis of the Mount Dyrring, N. S. Wales, pallasite found traces of gold, together with platinum, iridium, and palladium. The same authority also reported platinum and iridium and traces of tin in the Barraba iron.

Phosphorus.—The phosphorus shown by analyses to occur in meteorites is usually relegated to the schreibersite of the metallic portion or to apatite of the silicate portion. So far as the writer at present recalls, the presence, in a meteorite, of the mineral apatite has been satisfactorily demonstrated by optical means in but a single instance, though Shepard claimed to have found in the Richmond, Va., stone particles of such size that he was able to isolate them for qualitative tests.[c] These results, so far as I am aware, have never been corroborated. Berwerth [d] identified the mineral in granular, short, prismatic, and skeleton forms in the silicate secretions of the Kodaikanal iron.[e] Numerous other supposed instances of its occurrence have been reported, based mainly upon the presence of a soluble phosphate in the silicate or nonmetallic portions. That the phosphatic mineral is not schreibersite is conclusively shown by its solubility, and the presence of lime in the same soluble portions is naturally suggestive of its combination as apatite, the form in which it exists in corresponding igneous rocks. In my own work I have repeatedly obtained reactions for phosphorus by digesting for a short time the pulverized stony material in an acid solution far too weak to attack the schreibersite. This is the case with the Alfianello, Bluff, Dhurmsala, Estherville, Farmington, Felix, Indarch, Quenggouk, and several other stones which have been examined.

Investigations made to settle the question of its occurrence in some other form of combination than that of apatite have yielded unexpected results which may be briefly mentioned here, though elaborated elsewhere.[f] It may be recalled that in the writer's description of the meteorite from Rich Mountain, N. C.,[g] he mentioned the occurrence of a doubtful mineral occurring in plates of irregular outline, faintly gray or almost completely colorless, showing very faint, short, sharp cleavage lines with weak polarization colors, and which were optically biaxial. This he referred to the monticellite-like mineral described by Tschermak,[h] though confessing to a feeling of doubt as to its true nature. Since Tschermak's writing the mineral has been ob-

[a] Arb. des Natur. Vereins zu Riga, Neue Folge, Siebentes Heft, 1891, p. 51.

[b] Records Geol. Surv., N. S. Wales, vol. 7, pt. 4, 1904, p. 305.

[c] Amer. Journ. Sci., vol. 45, 1843, p. 102.

[d] Tsch. Min. Pet. Mittheil., vol. 25, 1906, p. 188.

[e] Tschermak in his paper on the Angra dos Reis meteorite describes a singly refracting, optically negative, colorless mineral concerning which he remarks, "Man kann wohl als sicher annehmen, dass diese Körnchen dem Apatite angehören." This occurrence was overlooked in my paper on the monticellite-like mineral in meteorites elsewhere referred to.

[f] See "On the Monticellite-like Mineral in Meteorites," Proc. Nat. Acad. Sci., vol. 1, 1915.

[g] Proc. U. S. Nat. Mus., vol. 32, 1907, p. 243.

[h] Sitz. k. Akad. Wiss. Wien, vol. 88, 1883, p. 355.

served by numerous other investigators, the more recent notices being those of Lacroix in the stone of St. Christophe la Chartreuse,[a] and Borgström in the stone of St. Michel, Finland.[b] In connection with the present investigations it was determined if possible to settle the identity of this mineral. With this in view, slides of both the Rich Mountain and Alfianello stones were submitted to Dr. F. E. Wright, of the Geophysical Laboratory, who reported the refractive indices of the mineral in question as $\alpha = 1.623 \pm 0.002$, $\gamma = 1.627 \pm 0.002$, birefringence weak, less than 0.005, and interference colors not exceeding gray white of the first order. The additional data still left the exact mineral species undetermined, though the refractive indices suggested that if a known mineral it is allied to the phosphate francolite. With this in mind, several slides, embracing those of Alfianello and Rich Mountain, were uncovered, and the mineral submitted to microchemical tests, which proved conclusively its phosphatic nature. The objections to considering the mineral francolite are, that so far as known among terrestrial rocks this mineral is of secondary origin, and a product of aqueous deposition, thus suggesting conditions which are not supposed to prevail among meteorites.

Farrington,[c] it will be recalled, thought to have found native phosphorus in the meteorite of Saline County, Kans. Notwithstanding the care exercised in his determinations, one can but feel that the observations need corroboration before acceptance. The stone had lain some three years in the ground when found and the examinations were not made for a year or so later. Under these conditions, when the nature of phosphorus is considered, it seems well-nigh impossible that material so susceptible to change could have remained unaltered.

Silica or silicon is not infrequently reported in analyses of meteoric iron in amounts rarely exceeding 0.2 of 1 per cent. The condition under which the element exists is in some cases at least problematical. Hunt and Silliman, in describing the iron of Lockport (Cambria), N. Y.,[d] refer to a reddish brown residue obtained by them as being "either silica with a trace of carbon or silicon," which last, they add, "Prof. Shepard has already shown to exist in the Oswego iron." Prof. Shepard, however, in his paper simply tabulates his results as "Silicon 0.20 per cent" and does not commit himself as to the condition under which the element may exist. Prof. Mallet, in his analysis of the Staunton, Va., iron, gives 0.067 per cent, 0.061 and 0.056 per cent SiO_2, but adds by way of explanation, "some of it (i. e., the Si) seems to have in reality existed as a silicide of iron."[e] Cohen, in his Meteoritenkunde (p. 55), refers the Ca, Mg, Al, K, and N very properly to the silicate minerals, and adds, "Das gleiche gilt wohl auch in der Regel für Silicium; doch führte Winkler in metallischen Theil von Rittersgrün gefundene Kieselsäure auf Silicium zurück, welches mit Eisen verbunden war, und nahm das Vorhandensein eines Siliciumeisen von der Formel Fe_2Si an, dessen Menge er für das Nickeleisen zu 0.329 Procent berechnete." Tassin in 1907 announced verbally in an informal communication before the National Academy of Sciences "the discovery of elemental silicon" in the meteoric iron of Casas Grandes, Mexico, incidentally claiming it as "the first announcement of the occurrence of this element in nature." With reference to these reports it may be stated that an examination of the insoluble residues from all of these irons reveals the presence of minute particles of quartz, sometimes shreds of glass and sundry silicates.[f] It seems most probable, therefore, that the small percentage of this constituent found had existed either as free quartz (SiO_2) or as a silicide of iron. Until the element shall be actually isolated it is unsafe to claim its existence in other form than that of combination with other elements.

[a] Bull. Soc. sci. nat. Ouest, 2d ser., vol. 6, 1906, p. 81.

[b] Bull. Com. Geol. Finlande, No. 34, 1912

[c] Amer. Journ. Sci., vol. 15, 1903, p. 71.

[d] Amer. Journ. Sci., vol. 2, 1846, p. 374.

[e] Amer. Journ. Sci., vol. 2, 1871, p. 14.

[f] See Cohen & Weinschenk on the Toluca, Mexico, meteoric iron, Meteoreisen-Studien, Ann. k. k. Hof.-Mus., 1891, p. 140. It should be added, however, that personally I regard the preterrestrial origin of these particles as open to serious doubt. In residues from a quantity of shavings from the Casas Grandes iron and from a 10-gram piece showing a portion of the original surface, though carefully cleansed, I found easily recognizable, clear, glassy quartz, both in form of crystals and angular fragments, shreds of colorless glass and also undetermined silicate minerals. Two other determinations on pieces cut from a depth of 2 centimeters below the surface yielded no such results, the residues being clean graphitic particles and schreibersite flakes. A few minute, colorless, isotropic particles, too small to manipulate, were crushed under the microscope between glass slides and were found to scratch and bite into the glass with all the energy of the diamond.

Sulphur.—The sulphur in meteorites is unquestionably combined in large part with iron, though the form of combination, whether as troilite or pyrrhotite or some intermediate compound, is recognized as still open to discussion. The work of Dr. E. T. Allen, of the Geophysical Laboratory, has shown that in the presence of an excessive amount of iron apparently only the mono-sulphide is possible of formation. That it sometimes occurs in this form in stony meteorites poor in iron has been shown by the writer [a] and also by Ramsay and Borgström.[b] The subject is thoroughly reviewed up to date of publication by Cohen in his Meteoritenkunde (vol. 1, 1894, pp. 182–209), and it is evident that further work is needed before the question can be considered fully decided. A portion of the sulphur, and one that heretofore has been almost wholly ignored if not overlooked, is that combined with calcium in the form of—

Oldhamite.—The presence of this sulphide was first noted in 1870 by Maskelyne in the meteorite of Busti. [c] Its probable occurrence was also suggested in that of Bishopville. Since that time the mineral has been determined both chemically and microscopically by Borgström in the meteorite of Hvittis, and by Lacroix and the present writer in that of Indarch. Its probable presence as indicated by a soluble calcium-bearing mineral (oldhamite or its alteration product gypsum) has been also shown by the present writer in the stones of Morristown [d] and Cullison, and in that of Allegan by Tassin. [e] These results rendering it probable that, as suggested by Maskelyne, the mineral was more commonly distributed than the published description would lead one to suppose, quantities of a gram or so from each of the stones listed below were finely pulverized and boiled for an hour in distilled water, the solutions being then tested for calcium by the ordinary ammonium oxalate method. The residues from this boiling were in some cases boiled also in very dilute hydrochloric acid and the solutions tested for phosphoric acid, with a view of deciding if the phosphorus shown in the bulk analyses was from the schreibersite, which would be insoluble under these conditions, or existed in the form of apatite or other soluble mineral. The results of the calcium tests were as follows:

Alfianello	Positive reaction.
Allegan [f]	Positive reaction.
Beaver Creek	Negative reaction.
Bishopville	Positive reaction.
Cullison	Positive reaction.
Dhurmsala	Positive reaction.
Dores dos Campos	Positive reaction.
Estherville	Positive reaction.
Farmington	Faint positive reaction.
Fayette	Positive reaction.
Felix	Positive reaction.
Forest City	Positive reaction.
Hessle	Faint positive reaction.
Holbrook	Positive reaction.
Homestead	Positive reaction.
Knyahinya	Negative reaction.
L'Aigle	Positive reaction.
Mocs	Positive reaction.
Monroe	Negative reaction.
New Concord	Doubtful reaction.
Parnallee	Faint positive reaction.
Pultusk	Positive reaction.
Quenggouk	Positive reaction.
Stannern	Negative reaction.
Tennasilm	Positive reaction.

[a] A recent meteorite fall near Holbrook, Ariz., Smithsonian Misc. Coll., vol. 60, No. 9, 1912, p. 4.
[b] Bull. Com. Geol. Finlande, No. 12, 1912.
[c] Phil. Trans. Roy. Soc. London, vol. 160, 1870, pp. 189–214.
[d] Amer. Journ. Sci., vol. 4, 1896, p. 149.
[e] Proc. U. S. Nat. Mus., vol. 34, 1908, p. 433.
[f] Quantitative tests on this stone by Whitfield show but 0.064 per cent of this constituent, as against upwards of 6 per cent as found by Tassin (Proc. U. S. N. M., vol. 34, 1908, p. 433). Although not so stated, it seems probable that the latter used an acid solution and decomposed in part the phosphatic mineral and iron sulphide. This would account for the present writer's inability to detect the mineral in the thin section.

There being a question, which is suggested by Maskelyne's description of the Busti stone, as to the sporadic occurrence of the calcium sulphide, three individuals from the Pultusk fall were selected and tested, two of which yielded distinct traces of calcium in the water solution, while the third showed not the slightest trace. The results then are apparently to the effect that oldhamite, or its alteration product, is a fairly common constituent of meteorites, but that it is by no means uniformly distributed throughout the mass of the stone. The cause of its being overlooked is doubtless due in part to the small size of the granules, to their breaking away in the process of cutting the section, or to the obscure form of its alteration products. The most careful examination by the writer has failed to reveal it in distinct crystalline form in any of the cases listed above.

Tin.—The occurrence of this metal has for a long time been regarded as open to question by the writer, notwithstanding the apparent care and skill under which the various analyses had been made. The skepticism was based in part upon the conditions under which the metal occurs in terrestrial rocks, where, as is well known, it is limited almost wholly to acidic types; in but two exceptions has it been found to occur in rocks of intermediate (andesitic) type. Genetically then it is fair to assume there is some connection. Among the common mineral associations of terrestrial tin, in the form in which it usually occurs (cassiterite) are, further, several very characteristic species such as fluorite, tourmaline, wolframite, topaz, etc., which are utterly unknown in meteorites. It is of course possible that the metal, if present, is in the form of the sulphide (stannite) or as an alloy with iron, but none of the recorded analyses of meteoric sulphides show a trace of the element, nor do analyses of terrestrial irons, as those of Ovifak, Greenland, or the various terrestrial nickel-irons as josephinite, awaruite, etc.[a]

Other elements reported.—Concerning the occurrence and distribution of some of the other less abundant elements, there is still a lingering doubt. The reported occurrence of titanium, nickel, cobalt, and chromium in the silicate portions, freed from metal, may reasonably be construed as indicating their combination in silicate compounds, particularly the pyroxenes, as in terrestrial rocks. Dr. Whitfield in his analyses has aimed at deciding this by exercising particular caution in separating the metallic from the nonmetallic portions. The analyses of the latter, it will be noted, still show small amounts of nickel and cobalt. It may be recalled in this connection that Tschermak[b] reported 2.39 per cent TiO_2 in the meteorite of Angra dos Reis, all of which he relegated to the augitic constituent.

V. RÉSUMÉ.

To sum up in brief the results of this investigation: So far as the minor elements are concerned, we have not merely failed to confirm but in most instances have thrown grave doubts on previous determinations of antimony, arsenic, gold, lead, tin, tungsten, uranium, and zinc. The occasional presence of platinum is apparently confirmed beyond question, and in two instances of vanadium.[c] Palladium, ruthenium, and iridium have also been found in traces. It is very probable that further investigations on the iron meteorites would yield confirmatory results. The presence of platinum was to be expected from the analogy with the terrestrial sources of this metal. Vanadium and titanium were also not unexpected in view of their widespread occurrence in terrestrial peridotites, as shown by Hillebrand's investigations.[d]

The apparent universal absence of barium and strontium may perhaps be accounted for by the paucity of the meteorites examined in feldspathic minerals. It is unfortunate that the National Museum collections are very poor in feldspathic types, and the prices per gram asked by dealers, and even other museums and collectors, are practically prohibitive.[e]

[a] These analyses are brought together in convenient form and discussed on pp. 313-15, 2nd. ed. of Clarke's Data of Geochemistry.

[b] Tsch. Min. Pet. Mittheil, vol. 28, 1909, pp. 110-114.

[c] Traces of vanadium are also reported by H. C. White (Records Geol. Surv. N. S. Wales, vol. 7, 1904, p. 312) in the meteorite of Mount Browne.

[d] Bull. 167 U. S. Geol. Surv., pp. 49-55.

[e] Ward's Values of Meteorites quotes prices of the feldspar-bearing Eukrites and Howardites varying from $1 to $4 per gram. The Juvinas and Stannern among the Eukrites alone drop to prices from 50 cents to $2 a gram.

Table of selected analyses of stony meteorites.

Name	Date	Analyst	Totals	Remarks	Total known weight of fall.
[illegible]	1896	H. [illegible]	100.01	No Ba or Sr. Carries oldhamite........	Grams. 14,100
[illegible]	[illegible]	J. Mal[illegible]	99.98		54,866
[illegible]	[illegible]	Hardy and Tschermak	100.02		1,700-2,000
[illegible]	[illegible]	G. T. Prior	100.17	No Ba or Sr	2,000+
[illegible]	[illegible]	W. F. Hillebrand	99.9[illegible]		14,000
[illegible]	[illegible]	J. E. Whitfield	100.177	No Ba, Sr, or Zr. Carries oldhamite.	6,100
[illegible] Finland	[illegible]	W. Ram[illegible]	100.04		400,000
[illegible]	[illegible]	J. E. Whitfield	99.79		146,000
[illegible] Mo.	[illegible]	J. Penfield	100.00		2,058+
[illegible]	[illegible]	J. E. Whitfield	100.19	No As, Sb, [illegible] Pt, or other minor constituent found	4,500
[illegible]	[illegible]	do.	100.53	None of the rarer constituents other than those mentioned.	10,100
[illegible]	1911	G. T. Prior	100.15	No Ba, Sr, or Zr found..	10,000
[illegible] Kan.	1906	J. E. Whitfield	98.779	do..........	7,075
[illegible] Tex.	[illegible]	J. M. Davison	100.27	Less O for S, 0.48	290,000
[illegible] La.	1900	L. G. Eakins	100.00		84,000
[illegible]	1900	J. [illegible]	99.79	....	2,049
[illegible] Minn.	1896	J. E. Whitfield	99.751	No Ba, Sr, Zr, or K found.	9,900
Forest City, Iowa	1890	L. G. Eakins	99.99		122,037
[illegible]	1901	W. Tassin	99.45		6,000
[illegible] Sweden	[illegible]	A. Nordenskjöld	99.32		22,895
Holbrook, Ariz.	1912	J. E. Whitfield	99.81	No traces of other minor constituents.	218,310
[illegible] Finland	1901	L. H. Borgström	100.28		14,040
[illegible]	[illegible]	J. E. Whitfield	100.071	No traces of other constituents than those noted. Carries oldhamite.	27,000
[illegible] Serbia	1889	M. Losanitsch	99.205		33,830
[illegible]	[illegible]	H. S. Washington	98.56		29,543
[illegible] France	[illegible]	J. E. Whitfield	99.42	No Ba, Sr, Zr, and but 0.008 per cent metal.	50,041
[illegible]	[illegible]	M. E. Pisani	100.77		80,000
[illegible]	[illegible]	— Kovalo	99.79		7,088
[illegible]	[illegible]	C. Rammelsberg	100.05		3,250
[illegible]	[illegible]	[illegible]	100.22		16,[illegible]00
[illegible]	[illegible]	[illegible]	99.99		1,802
[illegible]	[illegible]	[illegible]	100.405		20,000
[illegible]	[illegible]	H. W. Nichols	99.95		5,640
[illegible]	[illegible]	[illegible]	99.96		11,000
[illegible]	[illegible]	J. E. Whitfield	100.004	No traces found of Ba, Sr, Zr, or other minor constituents	100,000
[illegible]	[illegible]	H. [illegible]	100.74		2,136
[illegible]	[illegible]	[illegible]	99.728		5,800
[illegible]	[illegible]	[illegible]	[illegible]		245,000
[illegible]	[illegible]	[illegible]	99.46		16,000
[illegible]	[illegible]	[illegible]	99.91	No constituents other than those tabulated could be detected	8,800
[illegible]	[illegible]	H. [illegible]	100.00	No Ba, Sr, Zr, [illegible] Sn, Sb, [illegible]	25,250
[illegible]	[illegible]	J. L. Smith	100.00		150,000
[illegible]	[illegible]	E. H. von Baumhauer	100.00		1,[illegible]
[illegible]	[illegible]	[illegible]	[illegible]		2,058
[illegible]	[illegible]	J. L. Smith	99		140
[illegible]	[illegible]	[illegible]	100.10		5,500
[illegible]	[illegible]	[illegible]	101.[illegible]	Contains oldhamite	13,783
[illegible]	[illegible]	L. H. Borgström	100.26		7,000
[illegible]	[illegible]	H. W. Nichols	[illegible]		20,000
[illegible]	[illegible]	[illegible]	[illegible]		875
[illegible]	[illegible]	[illegible]	100.00		8,815
[illegible]	[illegible]	J. E. Whitfield	100.10	Contains traces of [illegible] but no other minor constituents than those mentioned.	140,000
[illegible]	[illegible]	L. H. Borgström	[illegible]		18,000
[illegible]	[illegible]	G. Lindström	99.84		34,000
[illegible]	1868	[illegible]	[illegible]	No Ba, Sr, or Zr. No oldhamite	32,000
[illegible]	[illegible]	G. [illegible]	99.916		28,[illegible]
[illegible]	[illegible]	L. G. Eakins	100.16		2,400
[illegible] Holland	[illegible]	E. H. von Baumhauer	99.21		9,700
[illegible]	1880	G. T. Prior	100.10		Unknown

It may be added that there must invariably be more or less variation in the proportional amounts of the different elements as found by analysis, owing to the difficulties in sampling, without sacrificing what is felt to be too large a quantity of material, a rock in which metallic iron is so prominent a constituent. This might account for the failure on the part of some to detect platinum and allied elements, or such minerals as oldhamite and the problematic phosphate. An analysis of the stony forms, accompanied, as it always should be, by an examination in thin section, leaves, however, little excuse for lapses of this nature. The reported occurrences in the older analyses of elements not found in later investigations may very naturally be ascribed, in part, to impure chemicals.

VI. TABLE OF ANALYSES AND DISCUSSION.

In the following table the writer has brought together the more satisfactory complete analyses of stony meteorites made during the progress of this investigation, as well as such made by others as seem up to the modern standard. In this work of compilation, analyses have been ruled out in most cases in which the totals fell 1 per cent and more short of 100 per cent, and also those that footed up approximately to 100 per cent, but in which certain elements which were obviously present were not mentioned. It is, of course, possible that in all cases full justice in the selections has not been done to the other workers, but the error, if there is one, is of omission rather than commission.

The purpose of tabulation in this form is to render the analyses comparable with those of terrestrial rocks. In the work of preparation it has been necessary to recalculate in part several of them. Those constituents the statement of which required most frequent attention have been the ferrous sulphide which has been tabulated as Fe and S; the ferric oxide, which is mainly secondary, and has been recalculated as ferrous, and the chromite which has been recalculated as chromic oxide (Cr_2O_3) and ferrous oxide (FeO), a proceeding which is recognized as not absolutely correct, since the meteoric chromites almost invariably contain several per cent of alumina and appreciable quantities of magnesia. The phosphorus has been allowed to stand as given, either as P or P_2O_5, though the probabilities are that it belongs in all cases to the silicate portion, and to be consistent should be tabulated as P_2O_5. The recalculation and tabulation of the iron in the sulphide as Fe is also open to question, and hence in such cases the percentage amounts are inclosed in brackets. In all these cases the sulphide is assumed to be in the form of FeS. Where an element has been recalculated as an oxide or the reverse, allowance for the gain or loss in the total percentages has been made in the customary manner.

It is to be noted that one of the most common and widely disseminated of the minor meteoric constituents—chlorine—has been ignored in this investigation, as in that of the majority of workers on stony meteorites. This is due in part to the ready oxidation of the mineral lawrencite, in which it mainly occurs, and in part to its seemingly trifling amount. Its presence in other form of combination than with iron—and perhaps nickel—has yet to be satisfactorily shown.[a] As will be seen by reference to the table, I have found but five recorded determinations from which to calculate averages.

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[a] See Cohen's Meteoritenkunde, pp. 55 and 227.

The above analyses, it will be noted, cover, to a fair extent, the entire range in composition of the stony meteorites. Although the number is small, it will nevertheless be not without interest to average them, as was done in my paper of 1909,[a] and by Farrington at a later date.[b] In making this average I have adhered to the plan first adopted of considering only the actual determinations of any particular constituent. Further, it has seemed advisable to rule out a few cases in which the percentage amounts of any constituent were so high as to be considered anomalous, as that of 2.39 per cent TiO_2 in the stone of Angra dos Reis, or 6.33 per cent Cr_2O_3 in that of Long Island.

This elimination is of course open to serious objection, and the question may well be raised as to the desirability of omitting entirely from the calculations the analyses in which such anomalies occur. The only answer that can be given is that in so doing the total number would be so reduced as to make any average of little interest. But it must be borne in mind that the value of any average that can be made to-day lies not in its showing the actual average composition, but rather in showing what has been done and inferentially what remains to be done.

The figures given in column I are therefore averages of 53 analyses with the 15 exceptions noted. In column II is shown for comparison the average composition of terrestrial igneous rocks, and in III that of the lithosphere, as given by Clarke.[c] In columns IV and V are repeated the meteoric averages given in my previous paper and that of Dr. Farrington.

Average composition of stony meteorites compared with terrestrial rocks.

Constituents.	I	II	III	IV	V
	Per cent.	*Per cent.*	*Per cent.*	*Per cent.*	*Per cent.*
SiO_2	38.68	59.93	59.85	38.732	39.12
TiO_2	.18[1]	.74	.73	......	.02
SnO_2	None.	......	......	......	.02
ZrO_2	None.	.03	.03	......	......
Al_2O_3	2.88	14.97	14.87	2.733	2.62
Fe_2O_3	......	2.58	2.63	.835	.38
Cr_2O_3	.47[2]	.05	.05		.41
V_2O_3	Trace.	.02	.02	......	......
Fe	11.98	......	......	11.536	11.46
Ni	1.15[3]	......	......	1.312	1.15
Co	.07[4]	......	......		.05
FeO	14.58	3.42	3.35	16.435	16.13
NiO	.48[5]	.03	.03	......	.21
CoO	.06[6]	......	......	......	......
CaO	2.42	4.78	4.81	1.758	2.31
BaO	None.	.11	.10	......	......
MgO	22.67	3.85	3.77	22.884	22.42
MnO	.29[7]	.10	.09	.556	.18
SrO	None.	.04	.04	......	......
Na_2O	.87[8]	3.40	3.29	.943	.81
K_2O	.21[9]	2.99	3.02	.328	.20
Li_2O	Trace.	.01	.01	......	......
H_2O (Ign.)	.75[10]	1.94	2.05	......	.20
P_2O_5	.26[9]	.26	.25	......	.12
S	1.80[11]	.11	.10	1.839	1.98
Cu	.014[12]	......	......	......	......
C	.15[13]	......	.03	......	.06
Cl	.08[14]	.06	.06	......	......
F	?	.10	.10	......	......
CO_2	?	.48	.70	......	......
SO_3	......	......	.02	......	......
Ni,Mn	......	......	......	......	.02
Cu,Sn					
Totals	100.044	100.00	100.00	99.891	99.87

1 Average of 46 determinations.
2 Average of 42 determinations.
3 Average of 50 determinations.
4 Average of 41 determinations.
5 Average of 19 determinations.
6 Average of 6 determinations.
7 Average of 33 determinations.
8 Average of 49 determinations.
9 Average of 44 determinations.
10 Average of 15 determinations.
11 Average of 51 determinations.
12 Average of 16 determinations.
13 Average of 8 determinations.
14 Average of 5 determinations.

[a] Amer. Journ. Sci., vol. 27, p. 469.
[b] Publ. 151 Field Mus. Nat. Hist., Geol. Series, vol. 3, No. 9, 1911.
[c] Bull. 491, U. S. Geol. Surv., 1911, p. 32.

It is, I think, commonly recognized by all who have given the matter thought, that greater uniformity in both method and statement of meteorite analyses is desirable. That the prevailing practice of separation into soluble and insoluble portions is also desirable is, I believe, without question, but the results of these processes can be accepted as little more than suggestive and not final, unless accompanied by a careful and detailed microscopic examination.

The solution obtained by digestion in distilled water will show with a fair degree of safety, the presence, or absence, of calcium sulphide and ferrous chloride as well as ferrous decomposition products, and the metallic and silicate portions be left practically untouched. The moment an acid is added, however, care needs be exercised in the interpretation of results since one has no means of telling how complete may have been the solution of the "soluble" constituents or the extent to which the "insoluble" may have been attacked. To illustrate: It was found, when working on the olivine of the Admire pallasite, that the material when not reduced to an impalpable powder was not completely dissolved even through repeated digestions over the steam bath in standard hydrochloric acid and sodium carbonate solutions, but left, without previous evaporation to dryness, a skeleton residue of white and completely isotropic silica, amounting to some 13 per cent of the original material. As showing the approximate proportional amounts of the olivines and pyroxenes the method is, however, unquestionably useful, particularly when accompanied by the microscopic examination. The late work of Borgström, [a] Prior, [b] and Cohen [c] furnishes an excellent example of what may be accomplished, and certainly the carefully detailed analyses of Mingaye [d] leave little to be desired. The practice, within the prescribed limits, of stating what elements were looked for and not found, as well as what were found, is worthy of commendation and should be followed. Such extremely careful and detailed work as has in times past been done by Fletcher [e] is quite beyond the possibility of general practice, and with the now almost universal use of the microscope is perhaps not essential.

[a] Der Meteorit von St. Michel, Bull. Com. Geologique de Finland, No. 34, 1912.

[b] The Meteoric Stones of Baroti, etc., Min. Mag., Dec., 1913.

[c] On the Meteoric Stone which Fell at the Mission of St. Marks, etc. Ann. South African Museum, vol. 5, 1905.

[d] Notes on, and analyses of, the Mount Dyrring, Barraba, and Cowra Meteorites. Records Geol. Surv. N. S. Wales, vol. 7, 1904.

[e] Min. Mag., vol. 10, No. 48, 1894, and vol. 13, No. 59, 1901.

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COMPLETE CLASSIFICATION OF THE TRIAD SYSTEMS ON FIFTEEN ELEMENTS.

BY

H. S. WHITE, F. N. COLE, AND LOUISE D. CUMMINGS.

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PART 1.

TRIAD SYSTEMS ON 15 ELEMENTS WHOSE GROUP IS OF ORDER HIGHER THAN UNITY.

By H. S. White.

§ 1. INTRODUCTION.

The problem which first led to the study of triad systems (*Tripel-systeme*) was proposed in the first place for 15 elements, Kirkman's "Fifteen schoolgirls problem." [1] In various journals, during the past 60 years, are described many different ways of constructing triad systems on 15 things. There was not known, prior to 1912, any short and decisive method for comparing systems apparently different; accordingly duplicates were produced, and up to 1912 only 10 noncongruent systems had been found. A triad system, like any other arrangement of elements, may have its appearance changed while its structure is unaltered by a permutation among the elements—a substitution. When one system can be derived from another by a substitution, the two are called *congruent* or *equivalent;* otherwise they are *incongruent* or *nonequivalent.* If there are substitutions which transform a system into itself (usually permuting its triads among themselves), all such substitutions together are called *the group of that triad system.*

The number of elements in a triad system must be of the form $6n+1$ or $6n+3$, where n is an integer.[2] Such numbers are 3, 7, 9, 13, 15, 19, etc. For any number of elements under 15 the exact number of nonequivalent triad systems has been known for some time, namely, one system for each of the numbers 3, 7, 9, while for 13 there are two systems. These two systems on 13 elements have different groups, of orders 6 and 39, respectively. One might anticipate that for numbers above 13 the same thing might happen, so that the group would serve as a distinctive mark or characteristic for its system. Miss Cummings has shown, however, that for 15 the case is different; that sometimes the same group belongs to two or more incongruent systems.[3]

Another test has been employed by E. H. Moore to prove the nonequivalence of two systems.[4] If among the 35 triads in 15 elements there occur 7 triads on any 7 elements, exclusively, the larger system, which Moore denotes by $\triangle_{15}$, is said to contain the smaller, a $\triangle_7$. Then if one $\triangle_{15}$ does contain a $\triangle_7$, while another does not contain any $\triangle_7$, the two are obviously incongruent. But this is of course not a conclusive test for equivalence, since Miss Cummings has found 23 incongruent $\triangle_{15}$'s, each of which does contain a $\triangle_7$.

Probably it has been the lack of convenient and reliable tests for equivalence or nonequivalence that has deterred investigators from the task of finding how many essentially distinct triad systems are possible in 15 elements. But now we have available two distinct methods of comparison, both of which have given reliable results, positive as well as negative, in all cases where they have been tried, although their value as positive tests of equivalence still lacks a priori demonstration. One of these, the *method of trains,* uses a given triad system

[1] T. P. Kirkman: On a problem in combinations. Cambridge and Dublin Mathematical Journal, vol. 2 (1847), p. 191. See also Note on an unanswered prize question, ibidem, vol. 5 (1850), p. 255.

[2] Netto: *Substitutionentheorie*, p. 220, § 192.

[3] L. D. Cummings: *Note on the groups for triple systems.* Bulletin of the American Mathematical Society, vol. 19 (1913), p. 355.

[4] E. Hastings Moore: *Concerning triple systems.* Mathematische Annalen, vol. 43 (1893), p. 271.

as a transformer of all possible combinations of the elements in threes, and so assembles these combinations into self-evolvent aggregates or trains, analogous to the "path-curves" of a point transformation.[1] The entire set of such trains, abstractly considered, belonging to any triad system on 15 elements is characteristic for that system and for all systems equivalent to it. A second method, that of *sequences* and *indices*, is described and exemplified by Miss Cummings in her dissertation.[2] These two methods have given, so far, accordant results, and both lead to the easy discovery of the substitutions which transform two equivalent systems into each other.

With these two direct and simple methods available for testing triad systems it is reasonable to attempt the complete enumeration of those on 15 elements. In this part, however, I shall undertake an easier task, that of constructing all triad systems on 15 elements whose groups are of order higher than unity; all which have groups containing at least one operation different from the identity. In this research the group, and indeed the particular substitution, will furnish the starting point.

§2. SUBSTITUTIONS ADMISSIBLE IN THE GROUP OF A $\triangle_{15}$.

Not every type of substitution can occur in the group of a triad system. Consider in particular a system $\triangle_{15}$ whose elements are denoted by letters, and S a substitution of its group. Represent S, as is uniquely possible, by a product of mutually exclusive cycles:

$$S\equiv(A_1\,A_2\;.\;.\;.\;.\;A_a)\;(B_1\,B_2\;.\;.\;.\;.\;B_b)\;(C_1\,C_2\;.\;.\;.\;.\;C_c)\;.\;.\;.\;.\;.\;.\;.\;.$$

It is required by the definition of a triad system that every pair of elements shall occur in some triad, and that no pair occur twice. The pair A_1B_1 must occur in some triad, as $A_1B_1C_1$. As a special case, the third element C_1 may be in one of the cycles (A) or (B). We note that in S the order c of cycle (C) must be a factor of the L. C. M. of a and b; otherwise the pair A_1B_1, would occur in two or more triads. Let $a=m\beta$, $b=m\alpha$, where α and β are relatively prime; then must

$$m\alpha\beta=\gamma c\ (\gamma \text{ an integer}).$$

For similar reasons

a is a factor of the L. C. M. (b, c), $m\beta$ divides $m\alpha c$;

b is a factor of the L. C. M. (a, c), $m\alpha$ divides $m\beta c$.

Hence, as α is prime to β, c is a multiple of the product $\alpha\beta$, $c=\mu\alpha\beta$, or

$$\gamma\ c=\gamma\mu\alpha\beta=m\alpha\beta,$$

therefore $m=\gamma\mu$. Now we find, more exactly, that

$$a(=\gamma\mu\beta) \text{ divides the L. C. M. } (\gamma\mu\alpha,\ \mu\alpha\beta)$$

which is $\mu\alpha$. L. C. M. (γ, β).

Therefore

$$\gamma\ \beta \text{ divides } \alpha.\ \text{L. C. M. } (\gamma, \beta)$$

and

$$\gamma\ \alpha \text{ divides } \beta.\ \text{L. C. M. } (\gamma, \alpha)$$

Hence γ is prime to both α and β. We have accordingly for the three orders of cycles (A), (B), and (C),

$$a=\mu\beta\gamma,\ b=\mu\gamma\alpha,\ c=\mu\alpha\beta;$$

and we have proved this theorem:

> If in the group of a $\triangle_{15}$ there occurs a substitution containing two mutually exclusive cycles of orders $a=\mu\beta\gamma$, $b=\mu\gamma\alpha$ respectively, then that substitution contains also a cycle (possibly coincident with one of the first two) of order $c=\mu\alpha\beta$; α being prime to β, and γ some common factor of a and b but prime to α and β.

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[1] H. S. White: Triple systems as transformations, and their paths among triads. Transactions of the American Mathematical Society, vol. 14 (1913), pp. 6–13.

[2] L. D. Cummings: On a method of comparison for triple systems. Transactions of the American Mathematical Society, vol. 15 (1914), pp. 311–327.

Particular cases under this theorem are:

$a=1$, giving cycles of orders $\mu\beta$, $\mu\gamma$, $\mu\beta\gamma$;

$a=\beta=1$, giving cycles of orders $\mu\gamma$, $\mu\gamma$, μ;

$a=\beta=\gamma=1$, giving cycles of orders μ, μ, μ;

$\mu=1$, giving cycles of orders $\beta\gamma$, γa, $a\beta$.

When two of the numbers are equal, they may refer to the same cycle, under conditions which it is not necessary to examine here.

For our present purpose, the important deduction from this theorem is the simplest case, namely, that if $a=1$ and $b=1$, then must $c=1$. In other words, if there are two letters (or elements) A and B invariant under the substitution S of the group, the triad containing these two must contain a third, C, also unaltered by the substitution S. An immediate extension of this corollary gives the following rule:

> All the elements not altered by a substitution in the group of a triad system (cycles of period unity) must constitute a complete triad system contained in the principal system, a subordinate system. Hence, for a $\triangle_{15}$, the number of cycles of period unity in any operation in its group can only be 0, 1, 3, 7, or 15.

The numbers 9, 13 are excluded, since a subordinate system can not contain half as many elements as the principal systems.

If the substitution S has cycles of different periods both higher than unity, $a<b$, then a power S^a of that operator will have invariant a additional elements. If S^a still contains cycles of different periods, another power can be found to diminish the number of different periods; and ultimately some power of S will be found with 0, 1, 3, or 7 cycles of period 1, and having the rest of its cycles of equal prime period.

Every triad system on 15 elements, whose group is not the identity, is invariant under at least one substitution of one (at least) of the following seven types:

1. (5) (5) (5)
2. (3) (3) (3) (3) (3)
3. (1) (7) (7)
4. (1) (2) (2) (2) (2) (2) (2) (2)
5. (1) (1) (1) (3) (3) (3) (3)
6. (1) (1) (1) (2) (2) (2) (2) (2) (2)
7. (1) (1) (1) (1) (1) (1) (1) (2) (2) (2) (2),

where the digit in any parenthesis indicates the period of a cycle.

These seven types of substitution offer a natural means of classifying triad systems on 15 elements, provided they admit groups of substitutions. Under each type I shall construct all possible invariant triad systems, omitting obviously equivalent repetitions. In one case, type 4, no system can exist; all the others have actual systems. It will still happen that certain systems occur in two or more classes, their groups containing substitutions of two or more of these seven types. Further reduction is undertaken by Miss Cummings (Part 2), who furnishes the proof of nonequivalence of the net residue, 44 systems. Of these 44, 24 were known previously, more than half of them discovered by Miss Cummings. It will be noted that the 20 new $\triangle_{15}$'s contain no $\triangle_7$; they are not of the "odd-and-even" structure; they may be called "headless," while any $\triangle_7$ contained in an earlier known $\triangle_{15}$ is termed its head. One headless system only, discovered by Heffter, has been known heretofore. The discussion of groupless systems, and their enumeration, is deferred to Parts 3 and 4.

§3. CLASS I, INCLUDING THE KIRKMAN AND HEFFTER SYSTEMS.

For each substitution the work of constructing systems of triads must be special; no general conclusions are to be developed. Three requirements guide us: (1) Every pair of elements shall occur; (2) no pair shall occur twice; and (3) the triads shall be grouped in sets conjugate under the particular substitution (operator).

Denote by S_1 the operator of type 1, and take for its three cycles of five, respectively, English and Greek letters and Arabic numerals.

$$S_1 \equiv (a\ b\ c\ d\ e)\ (\alpha\ \beta\ \gamma\ \delta\ \epsilon)\ (1\ 2\ 3\ 4\ 5).$$

Triads of a system may contain elements from one cycle only, or from two, or from three. Further subdivision gives us 10 classes. Indicate the numbers of triads in these several classes as follows:

$5u_1$ of type abc, $5u_2$ of type $\alpha\beta\gamma$, $5u_3$ of type 123,
$5v_1$ of type $ab\alpha$, $5v_2$ of type $\alpha\beta 1$, $5v_3$ of type $12a$,
$5w_1$ of type $ab1$, $5w_2$ of type $\alpha\beta a$, $5w_3$ of type 12α,
$5t$ of type $a\alpha 1$.

As all possible pairs must occur, we have the diophantine equations to satisfy:

$$15u_i + 5v_i + 5w_i = 10 \quad (i = 1, 2, 3),$$
$$10v_i + 10w_{i+1} + 5t = 25 \quad (i = 1, 2, \text{ or } 3 \equiv 0).$$

The solutions are

	1st.	2d.	3d.
$t = 1,\ u_1 = u_2 = u_3 = 0,$			
$v_1 = v_2 = v_3 =$	2,	1,	0,
$w_1 = w_2 = w_3 =$	0,	1,	2.

Of these the third solution may be dropped, since it is related to the above arrangement of elements in the operator S_1 exactly as the first solution is to the operator S^2_1 (or S_1^{-1}). The first and second solution yield two kinds of systems, as follows:

First kind.—$t = 1,\ u_i = 0,\ v_i = 2,\ w_i = 0 \quad (i = 1, 2, 3)$.

The triads enumerated by $5t = 5$ shall be these: $a\alpha 1$, $b\beta 2$, $c\gamma 3$, $d\delta 4$, $e\epsilon 5$. With the pair ab must occur δ; for if the triad were $ab\gamma$, then must follow in cyclic order $bc\delta$, ----, $ea\beta$. There remains, therefore, no letter of the second cycle to join with the pair ac, since we have already the pairs $a\alpha$, $c\gamma$, and $a\gamma$, $a\beta$, $c\delta$, $c\epsilon$. The same reasoning excludes the combination $ab\epsilon$. This leaves only the possibility $ab\delta$. Under S_1 there follow from $ab\delta$ the pairs $c\epsilon$, $a\gamma$, and there is left for the pair ac only the letter β, hence $ac\beta$. Similar reasons apply to triads from the second and third cycles and from third and first cycles. The entire system is accordingly determined uniquely, and is given by the following seven triads and the conjugates derived from them by the operator S_1:

System I1: $a\alpha 1$; $ab\delta$, $ac\beta$; $\alpha\beta 4$, $\alpha\gamma 2$; $12d$, $13b$.

Second kind.—$t = 1,\ u_i = 0,\ v_i = w_i = 1 \quad (i = 1, 2, 3)$.

After the five triads, $a\alpha 1$ and its conjugates under S_1, the apparent possibilities are $ab\gamma$ and $ab\delta$.

Assume first a triad $ab\gamma$. Its conjugates contain the pairs αe, αd, and βa, βc, so that there remains a possibility of the triad $\alpha\beta c$. But also there are found the pairs γc, γb; whence the pair $\alpha\gamma$ can join with no letter from the first cycle, but must be completed by either 2, 4, or 5. We examine these successively.

(1) TRIAD $\alpha\gamma 2$.

Conjugate triads will involve the pairs 1ϵ, 1β, 2α, 2γ, 3β, 3δ; and already we had 1α, 2β, 3γ. Therefore the pair 13 can not be completed from the Greek cycle, nor can its four conjugate pairs. But $w_3 = 1$, and for the pair 12 we have available the letter δ only. Necessarily one triad is 12δ. We must complete

13 by either b, d, or e,
ac by either 2, 4, or 5.

The hypothesis of $13b$ would result in excluding ac from 2, 5, and 4. The hypothesis $13e$ leaves open only the alternative $ac2$, having $eb1$ as a conjugate, which is inconsistent with $13e$. Dismissing, therefore, $13b$ and $13e$, we have $13d$, requiring $ac4$. The remainder of the system follows uniquely from these.

System I2: $a\alpha 1$, $ab\gamma$, $ac4$, $a\beta c$, $a\gamma 2$, 12δ, $13d$, and their conjugates under the operator S_1.

(2) Triad $\alpha\gamma 4$.

By similar scrutiny it is found that triad $\alpha\gamma 4$ excludes the pair 12 from completion by the Greek cycle, but allows the requisite triad 13δ and its conjugates. To be tested are now $12c$, $12d$, and $12e$. Of these, $12c$ leads to $ac5$ and so to $ce2$ inconsistent with $12c$; and the third alternative $12e$ would exclude the pair ac from the Arabic cycle altogether, and is hence inadmissible. There remains $12d$, which allows further $ac2$. Complete the system thus, uniquely,

$$a\alpha 1,\ ab\gamma,\ ac2,\ 12d,\ 13\delta,\ \alpha\beta c,\ \alpha\gamma 4.$$

This system differs from I2, above, only in the interchange of the Greek cycle and the English. Omit it therefore as a duplicate.

(3) Triad $\alpha\gamma 5$.

By considerations like the above we find that $\alpha\gamma 5$ and $a\alpha 1$ with their conjugates will exclude 12 from completion in the Greek cycle, and will necessitate the occurrence of 13ϵ, which in turn requires $\alpha\gamma 5$. Next, 12 must be joined with either c, d, or e. The first and last of these lead to inconsistent triads, and we have remaining $12d$, and therefore $ac2$. The sole admissible system here is therefore that given by these seven:

$$a\alpha 1,\ ab\gamma,\ ac2,\ \alpha\gamma 5,\ \alpha\beta c,\ 13\epsilon,\ 12d.$$

But this triad system is related to the operator S_1^2,

$$S_1^2 \equiv (\alpha\ \gamma\ \epsilon\ \beta\ \delta)\ (1\ 3\ 5\ 2\ 4)\ (a\ c\ e\ b\ d),$$

in exactly the same way as system I2, above, is related to the operator S_1,

$$S_1 \equiv (a\ b\ c\ d\ e)\ (\alpha\ \beta\ \gamma\ \delta\ \epsilon)\ (1\ 2\ 3\ 4\ 5).$$

We may therefore omit it as a duplicate.

This exhausts the possibilities under the first assumption, i. e., that the second triad was $ab\gamma$. Test now the other alternative: Assume $ab\delta$. By trials similar to the foregoing, we construct the following five systems, exhausting the possibilities:

(a) $a\alpha 1$, $ab\delta$, $\alpha\gamma b$, $ac4$, $\alpha\beta 3$, $13d$, 12γ.
(b) $a\alpha 1$, $ab\delta$, $\alpha\gamma b$, $ac5$, $\alpha\beta 3$, $13e$, 12γ.
(c) $a\alpha 1$, $ab\delta$, $\alpha\gamma b$, $ac4$, $\alpha\beta 5$, $13d$, 12ϵ.
(d) $a\alpha 1$, $ab\delta$, $\alpha\gamma b$, $ac5$, $\alpha\beta 5$, $13e$, 12ϵ.
System I3: $a\alpha 1$, $ab\delta$, $\alpha\gamma b$, $ac2$, $\alpha\beta 4$, 13β, $12d$.

Four of these systems are equivalent to the two already found. Notice first that two of them, (d) and (c), reduce to (a) and (b), respectively, by the reversal of order in each cycle, i. e., by using for operator S_1^4 in place of S_1. Then (a) is seen to become system I2 by exchange of English and Arabic cycles. To show that (b) is congruent to system I2, replace operator S_1 by S_1^3 in a changed order of cycles, thus

$$S_1 \equiv (a\ b\ c\ d\ e)\ (\alpha\ \beta\ \gamma\ \delta\ \epsilon)\ (1\ 2\ 3\ 4\ 5)$$
$$S_1^3 \equiv (\alpha\ \delta\ \beta\ \epsilon\ \gamma)\ (1\ 4\ 2\ 5\ 3)\ (a\ d\ b\ e\ c)$$

The same substitution will change (b) into I2.

The net result of this section is therefore the construction of three systems, I1, I2, and I3. The first and third of these are the well-known systems of Heffter and Kirkman, respectively, the second hitherto unknown.

§4. CLASS II. EIGHT SYSTEMS INVARIANT UNDER A SUBSTITUTION OF THE TYPE $(3)^5$.

A first separation into two divisions is found when we distinguish three kinds of triads. Denote the elements of the five cycles by the letters a, b, c, d, e, attaching to each in their order the subscripts 1, 2, 3:

$$S \equiv (a_1\ a_2\ a_3)\ (b_1\ b_2\ b_3)\ (c_1\ c_2\ c_3)\ (d_1\ d_2\ d_3)\ (e_1\ e_2\ e_3).$$

Denote any three different letters from among these five by k, l, m, leaving the subscripts undetermined. Then obviously every triad that can occur is of one of the three types represented by

kkk (Denote their number by u),
kkl (Denote their number by $3v$),
klm (Denote their number by $3w$).

In a given triad system, every possible pair of elements occurs once. There are in the entire 105,

15 pairs of type kk,

90 pairs of type kl, making in combination 35 triads.

Compare these with the numbers of each found in triads of each of the three types above. We find the necessary relations:

$$3u+3v=15,$$
$$6v+9w=90,$$
$$u+3v+3w=35$$

We can have, therefore, the two kinds of systems, divisions 1 and 2:

	u	v	w
Division 1.........	2	3	8
Division 2.........	5	0	10

Division 1.—As u is 2, assume triads $a_1a_2a_3$ and $b_1b_2b_3$. Now subdivide further, and let k represent either an a or a b, and l, m represent any two of the letters c, d, e, subscripts being disregarded. Since $3v=9$, let us distinguish—

Of type abm, $9=3.3$ triads,
Of type kll, $3x$ triads,
Of type klm, $3y$ triads,
Of type lmm, $3z$ triads,
Of type cde, $3t$ triads.

These numbers have to satisfy the following conditions:

Triads, $3x+3y+3z+3t+2+9=35$;
Pairs kl, $18+6x+6y=54$,
Pairs ll, $3x+3z=9$,
Pairs lm, $3y+6z+9t=27$.

These conditions admit three solutions, which will be taken up in order.

	x	y	z	t
Family a...........	3	3	0	2
Family b...........	2	4	1	1
Family c...........	1	5	2	0

FAMILY 1a.

The only possible schedule, prior to the assignment of subscripts, is this:

aaa, bbb; acc, add, aee;
abc, abd, abe;
bcd, bce, bde;
cdc, cde.

For the hypothesis *acc, add*, and *bee*, for example, would require a_1 to unite with two e's in the one triad left for it after the three where it occurs with b's; whereas ee is found already with b in *bee*, whence the impossibility.

With entire generality we fix, in the above schedule, the subscripts in eight typical triads:

$a_1\,a_2\,a_3$; $b_1\,b_2\,b_3$; $a_1\,c_2\,c_3$, $a_1\,d_2\,d_3$, $a_1\,e_2\,e_3$;
$a_1\,b_1\,c_1$, $a_1\,b_2\,d_1$, $a_1\,b_3\,e_1$.

Six trials now suffice, to reduce the possible ways of supplying subscripts for the remaining triads to the following two, supplementary to the above:

System II, 1_1: $c_1\,d_1\,b_3$, $c_1\,d_2\,e_3$, $c_1\,d_3\,e_2$;
$b_1\,c_3\,e_3$, $b_1\,d_1\,e_1$.

System II, 1_2: $c_1\,d_2\,b_2$, $c_1\,d_1\,e_1$, $c_1\,d_3\,e_2$;
$b_1\,c_2\,e_1$, $b_1\,d_2\,e_3$.

FAMILY 1b.

Prior to assigning subscripts, the only possible schedule is this:

aaa, bbb; ace, add; cee; cde;
abe, abe, abc;
bcd, bcd; ade, bde.

(The other apparent possibility, *aee, bdd, cee*, leads to the impossibility of five times three pairs *ee*.)

Subscripts may be fixed arbitrarily, in order, in the following triads (subject, of course, to the cyclic permutation of 1, 2, 3):

$a_1a_2a_3$, $b_1b_2b_3$; $a_1b_1e_1$, $a_1b_2e_3$, $a_1b_3e_1$,
$a_1e_2d_1$; $a_1c_2c_3$, $a_1d_2d_3$.

The remaining five sets are found by trial to admit two arrangements only. With the above we may unite either of these:

System II, 1_3: $b_1c_3d_1$, $b_1c_1d_2$, $b_1c_3d_3$, $c_1d_3e_3$, $c_1e_1e_2$.

System II, 1_4: $b_1c_3d_3$, $b_1c_1d_1$, $b_1c_3d_2$, $c_1d_2c_1$, $c_1e_2e_3$.

FAMILY 1c.

The association of letters in triads, disregarding subscripts, may follow two schedules. The first possible schedule is this:

aaa, bbb;
acc, cdd, cee;
abc, abd, abe;
ade, ade, bde, bcd, bce.

In the first six that follow *aaa* and *bbb* we may dispose of subscripts by fixing $a_1b_1c_1$, then $a_1c_2c_3$, $c_1d_2d_3$, $c_1e_2e_3$. So far letters d and e are exchangeable, also subscripts 2 and 3. This observation reduces to five the number of essentially different ways of affixing subscripts to the next two triads, *abd* and *abe*, viz:

(1) $a_1d_1b_2$, (2) $a_1d_1b_2$, (3) $a_1d_2b_2$, (4) $a_1d_2b_3$, (5) $a_1d_2b_2$,
$a_1e_1b_3$; $a_1e_2b_3$; $a_1e_2b_3$; $a_1e_3b_2$; $a_1e_3b_3$.

Of these five only the first and the fifth can be completed to full systems. They give each a unique system.

System II, 1_5: $a_1a_2a_3$, $b_1b_2b_3$;
$a_1c_2c_3$, $c_1d_2d_3$, $c_1e_2e_3$;
$a_1b_1c_1$,
$a_1d_1b_2$, $a_1e_1b_3$;
$d_1e_1b_1$, $d_1e_2a_3$, $d_1e_3a_2$;
$c_1d_1b_3$, $c_1e_1b_2$.

System II, 1_6: $a_1a_2a_3$, $b_1b_2b_3$;
$a_1c_2c_3$, $c_1d_2d_3$, $c_1e_2e_3$;
$a_1b_1c_1$,
$a_1d_2b_2$, $a_1c_3b_3$;
$d_1e_1a_1$, $d_1e_2b_3$, $d_1e_3a_2$;
$c_1d_1b_2$, $c_1e_1b_3$.

The second of the two possible schedules diverges from the first in its fifth triad and is the following, with seven triads void of subscripts:

$a_1a_2a_3$, $b_1b_2b_3$; $a_1c_2c_3$, $c_1d_2d_3$, $d_1e_2e_3$;
$b_1c_1d_1$; adb, adb, ade;
ceb, ceb, cea; abe.

On trial only two ways are found for affixing subscripts to the triads still left blank, and these differ only by the interchange of 2 and 3. Hence we have finally in this family 1c only the one additional system:

System II, 1_7: $a_1a_2a_3$, $b_1b_2b_3$, $a_1c_2c_3$, $c_1d_2d_3$, $d_1e_2e_3$, $b_1c_1d_1$
$a_1d_1b_3$, $a_1d_3b_1$, $a_1d_2e_2$;
$c_1e_1b_3$, $c_1e_2b_2$, $c_1e_3a_1$;
$a_1b_2e_1$.

Division 2.—In the second principal division of this class there are five triads formed from single cycles of three letters, and these are necessarily

$a_1a_2a_3$, $b_1b_2b_3$, $c_1c_2c_3$, $d_1d_2d_3$, $e_1e_2e_3$.

The ten sets of three triads which are to complete the system contain, as we saw, each three different letters; and no combination of three letters can be repeated, as is shown readily by writing the diophantine equations of condition. There are only 10 combinations possible, so that each will occur once, i. e., in one set of three triads. If we attend first to the triads containing a, remembering that in each of the five cycles the subscripts may be permuted cyclically, it is found that there are but two distinct kinds of sets. To characterize them, we may best use Cole's term, *interlacing;* either there is an interlacing in the triads containing symbols a or there is none. In the first case the four symbols concerned may be given index or subscript 1, $b_1c_1d_1e_1$, and the four triads may be:

$a_1b_1c_1$, $a_1d_1e_1$, $a_2b_1d_1$ $a_2c_1e_1$.

In the second case we find, by permutations of letters and of the five cycles independently, that we may denote four triads by

$a_1b_1c_1$, $a_1d_1e_1$, $a_2b_1d_1$, $a_3c_1e_1$.

Each of these is completed, in two equivalent ways, to a full system.

System II, 2_1: $a_1b_1c_1$, $a_1d_1e_1$, $a_2b_1d_1$, $a_2c_1e_1$, $a_3b_1e_1$, $a_3c_1d_1$,
$b_1c_2d_3$, $b_1c_3e_2$, $b_1d_2e_3$, $c_1d_3e_2$.

System II, 2_2: $a_1b_1c_1$, $a_1d_1e_1$, $a_2b_1d_1$, $a_2b_3e_1$, $a_3c_1e_1$, $a_3c_2d_1$,
$b_1c_2d_3$, $b_1c_3e_1$, $b_1d_2e_3$, $c_1d_1e_3$.

This last system is found to have no interlacings whatever, and so is evidently the exceptional system in Cole's enumeration, the headless cyclical system of Heffter.

The preceding system, however, No. II, 2_1, is found by quite obvious indications to be equivalent to System II, 1_1 and the one is transformed into the other by the following substitution:

II, 1_1: $a_1a_3a_2\ b_1b_3b_2\ c_1c_2c_3\ d_1d_2d_3\ e_1e_2e_3$.

II, 2_1: $b_1b_2b_3\ a_1a_2a_3\ c_1d_2e_3\ e_1c_2d_3\ d_1e_2c_3$.

Further, the two systems II, 1_3, II, 1_4 are equivalent by the interchange of letters $(a_3\ a_2)\ (b_1\ b_2)\ (c_3\ c_2)\ (d_3\ d_2)\ (e_1\ e_3)$. In conclusion, therefore, this class II contains not more than seven triad systems that are essentially distinct, Nos. 1_1, 1_2, 1_3, 1_5, 1_6, 1_7, 2_2.

§5. TRIAD SYSTEMS INVARIANT UNDER AN OPERATION OF TYPE (1)(7)(7).

There are three distinct systems, and no more, which admit a substitution of period 7. The proof is almost intuitional, no long analysis being needed. Indicate the $1+7+7$ letters, and the substitution, thus:

$$S \equiv (A)\ (a\ b\ c\ d\ e\ f\ g)\ (1\ 2\ 3\ 4\ 5\ 6\ 7).$$

By diophantine equations of condition it is found that there must be seven triads—a system $\triangle_7$, constituted within the one cycle, e. g., the cycle of letters $(a\ b\ c\ d\ e\ f\ g)$; seven triads like $Aa1$, and 21 triads composed of one letter and two digits.

Take S to be such a power of the cyclic operation that the included $\triangle_7$ consists of the triad abd and its six conjugates. Of the 21 now remaining to be determined, three must contain the letter a. Indicate on a circle in their order of sequence the seven digits at equal intervals. Since a and 1 are already together in the triad $Aa1$, we have now to connect in pairs, by three chords, all six digits 2, 3, . . ., 7. No two of these chords can be of equal length, since then the rotation effected by the substitution S would produce a repetition of a pair, contrary to the definition of a triad system. Trial shows at once that chords 23 and 25 would lead necessarily to equality of at least two chords, hence these are excluded; while 24, 26, and 27 lead to one solution each, as represented in figure 1.

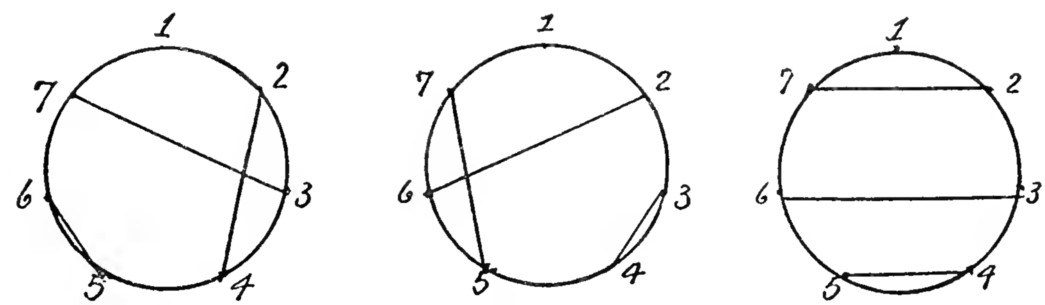


Fig. 1.

Accordingly the three possible systems are given by the following, with the triads conjugate to them under the operation S.

System III, 1: $Aa1$, abd, $a24$, $a37$, $a56$.

System III, 2: $Aa1$, abd, $a26$, $a34$, $a57$.

System III, 3: $Aa1$, abd, $a27$, $a36$, $a45$.

The first of these is evidently the one ordinarily constructed from the $\triangle_7$ by the method for passing from n to $2n+1$; viz. by substitution of corresponding elements from the second cycle in triads of the first, there are formed from abd, for example, three others: $a24$, $1b4$, $12d$. The 21 of this structure, the original 7, and the 7 like $Aa1$ make up the complete set. It is, prima facie, the Kirkman system (No. IIIA of Miss Cummings's dissertation).

§6. NO TRIAD SYSTEM CAN EXIST THAT ADMITS A SUBSTITUTION OF TYPE $(1)(2)^7$.

If a triad system can be invariant under an operation of the type $(1)\ (2)^7$, denote the element in the unit cycle by A, and in each dual cycle denote one element by a letter, the other by a digit:

$$S \equiv (A)\ (a\ 1)\ (b\ 2)\ (c\ 3)\ (d\ 4)\ (e\ 5)\ (f\ 6)\ (g\ 7).$$

Every pair $a1$ from one duad must occur in some one triad and can be associated with no third element save A; hence seven triads are like $Aa1$. The remaining 28 triads may be arranged in two pairs of classes, each pair equal in number by the symmetry of the operator S in letters and digits, thus:

x triads like abc, x triads like 123;
y triads like $ab1$, y triads like $a12$.

Since the nature of the system calls for 21 pairs of letters, as ab, and 21 pairs of digits, while there must be 42 mixed pairs such as $a2$, we have the equations of condition:

$$3x+y=21,\ 4y=42,$$

insoluble in integers. Hence no triad system of this type can exist.

§ 7. SUBSTITUTIONS OF TYPE $(1)^3(3)^4$ AND THEIR INVARIANT TRIAD SYSTEMS.

Denote by A, B, C three elements not affected by a certain substitution, and by $(a_1\ a_2\ a_3)$, $(b_1\ b_2\ b_3)$, $(c_1\ c_2\ c_3)$, $(d_1\ d_2\ d_3)$, four cycles of period 3 in that substitution.

$$S\equiv(A)\ (B)\ (C)\ (a_1\ a_2\ a_3)\ (b_1\ b_2\ b_3)\ (c_1\ c_2\ c_3)\ (d_1\ d_2\ d_3).$$

Any triad system invariant under S must contain the triad ABC; and equations of condition show that of triads and sets like $a_1a_2a_3$, b_1b_2c, there are respectively either 1, 3 or 4, 0. It will be shown that this gives in letters, irrespective of their subscripts, five possible schedules. There are 18 triads like Aab, falling into three sets of twice three. By the subscripts of these latter each of the first five schedules is made a source of five subclasses.

The equations of condition show further that there must be either two or four sets of three triads like abc, formed from three different cycles in S. Hence there are at least three different pairs of letters, as ab, that can not occur more than twice (i. e., 2×3 times) in conjunction with letters A, B, or C. The complementary pairs are excluded thereby also; e. g., a set of triads Aab would imply another set Acd, since all possible pairs of elements occur in a system $\triangle_{15}$. Where four triads of period 1 occur, like $a_1a_2a_3$, and therefore four sets like abc, no pair ab can occur with two letters from the three A, B, C. We arrive by such considerations at the first five main divisions.

Case 1.—Four triads like $a_1a_2a_3$, four sets like abc. Hence the schedule:

$a_1a_2a_3$, $b_1b_2b_3$, $c_1c_2c_3$, $d_1d_2d_3$;
abc, abd, acd, bcd;
Aab, Acd; Bac, Bbd; Cad, Cbc.

Cases 2 and 3.—One triad of period 1, $d_1d_2d_3$; two sets drawn from three different cycles. Consider the triad sets containing pairs aa, bb, or cc. With these may occur the letter d in 3, 2, 1, or 0 sets. If in none, then we must have (if letters are chosen suitably) aab, bbc, cca. But this leaves us to construct triads in which d_1, for example, shall be united with all nine letters a, b, c. Three of these are of course in sets Aad, Bbd, Ccd, implying sets Abc, Bac, Cab. In the remaining two sets of three, d must be united twice with each letter a, b, and c, a plain impossibility. Similar absurdity results from assuming two sets like daa, dbb. Hypotheses of one such set, or of three such, are admissible, as follows:

Case 2: Assume two sets, aad and bba. Trial of ccb leads to absurdity, whence we must have cca. In full, therefore, this schedule is:

ABC; $d_1d_2d_3$; Abc, Aad, Bac, Bbd, Cab, Ccd; aad, bba, cca; bcd, bcd.

Case 3: Assume three sets with d, aad, bbd, and ccd. The full schedule will be:

ABC; $d_1d_2d_3$; Abc, Aad, Bac, Bbd, Cad, Cbc; aad, bbd, ccd; abc, abc.

Cases 4 and 5.—With ABC and $d_1d_2d_3$ as in the preceding case, take duplicate pairs of letters with two of the isolated elements ABC; e. g.,

$$Aab,\ Acd,\ Bab,\ Bcd;\ Cac,\ Cbd.$$

There are yet to be constructed 15 ($=5\times3$) triads. Of these five sets three have to contain a doubled letter, as aa, while the other two consist of distinct letters. Listing the pairs that

must occur, we see that ad and bc must occur in triads with a double letter, so that either ac or bd, not both, will also occur in such a triad. There are accordingly two possible schedules:

Case 4: Triads ABC, ddd; Aab, Acd, Bab, Bcd; Cac, Cbd; aad, bbc, cca; abd, bcd.

It is noticeable that in this arrangement no two of the letters $abcd$ are interchangeable; the same is true in the next, the final case.

Case 5: Triads ABC, ddd; Aab, Acd, Bab, Bcd; Cac, Cbd; aad, bbd, ccb; abc, acd.

After the above distinction of five schedules there is for each case a subdivision into species by means of the pairs of subscripts attached to letters in the triads with A, B, and C. The numbers of such species, a priori, are for case 1, four; for case 2, four; for case 3, three; and for cases 4, 5, four each. Some of these are realized by two completed systems, some by one, or by none; so that, in all, 26 systems apparently distinct are found invariant under an operation or substitution $(1)^3(3)^4$.

Species in case 1.—In case 1 the sets of letters a, b, c, d are indistinguishable, and are distributed to the exchangeable letters A, B, C in all the complementary pairs. As a means of abbreviating tabulation, write these triads in the order

$$\begin{array}{ll} Aab & Acd \\ Bac & Bbd \\ Cad & Cbc, \end{array}$$

and instead of rewriting these letters with subscripts, write the subscripts only. On each letter independently the subscripts may be changed if we do not change the cyclic order 123; and this order may be reversed to 132 for all four sets of letters simultaneously. The first three pairs of subscripts may be fixed arbitrarily as 1, 1, thus: a_1b_1, c_1d_1, a_1c_1. We need only write the three remaining pairs in their relative positions, without letters:

$$(\alpha)\ \begin{array}{r|l} & 11, \\ 11 & 11 \end{array}\quad (\beta)\ \begin{array}{r|l} & 11, \\ 11 & 12 \end{array}\quad (\gamma)\ \begin{array}{r|l} & 11, \\ 12 & 12 \end{array}\quad (\delta)\ \begin{array}{r|l} & 12 \\ 11 & 13 \end{array}$$

These four cases exhaust the possibilities; for we can reduce all others to these four by permissible interchanges of letters and subscripts. There should be nine cases where indices 2, 3 do not appear in the first row or the second and we see that (α) represents one, (β) four, and (γ) four. If the pair 1 2 or 1 3 occurs in the second line, the reduction is less obvious, but not intricate, as one example will show. Let these six pairs in same order be a_1b_1, c_1d_1; a_1c_1, b_1d_2; a_1d_2, b_1c_3 (representing also, of course, their 12 conjugate pairs). Write subscripts only, and change those of d cyclically by writing 1 for 2, etc. This leaves

$$11,\ 13;\ 11,\ 11;\ 11,\ 13.$$

Next, c is written for d and vice versa, as the letters are of equal significance; then for 31 write 12, one of its conjugates. Now we have

$$11,\ 12;\ 11,\ 13;\ 11,\ 11,$$

since second and third lines have exchanged subscripts. But this is case (δ) by permutation of letters A, B, C. By such verification we confirm the completeness of this list of four species.

Species in case 2.—In case 2 there is no distinction between letters b and c, but a and d are not interchangeable with them or with each other. First we fix triads Ad_1a_1, Bd_1b_1, Cd_1c_1, so that all indices are of determinate meaning except for a choice between the orders 123 and 132. Since the combination bcd is to occur in two sets of three, these can only be $b_1c_2d_3$ and $b_1c_3d_2$; accordingly the triads Abc must include Ab_1c_1. The pairs with B and C admit some freedom of choice still, all alternatives being reducible to these four following:

$$(\alpha)\ \begin{array}{l} Ca_1b_1 \\ Ba_1c_1 \end{array}\quad (\beta)\ \begin{array}{l} Ca_1b_1 \\ Ba_1c_2 \end{array}\quad (\gamma)\ \begin{array}{l} Ca_1b_2 \\ Ba_1c_2 \end{array}\quad (\delta)\ \begin{array}{l} Ca_1b_2 \\ Ba_1c_3 \end{array}$$

Species in case 3.—Letters a, b, c are not yet distinguishable separately in case 3. Fix first, as in case 2, the triads Ad_1a_1, Bd_1b_1, Cd_1c_1. Next, two sets of triads are to contain abc,

there are three possibilities, and these serve to fix uniquely also the subscripts of the remaining sets of triads. There are then three schedules:

(α) $b_1c_1a_1$	(β) $b_1c_1a_2$	(γ) $b_1c_2a_3$
$b_1c_2a_3$	$b_2c_1a_1$	$b_1c_3a_2$
Ab_1c_3	Ab_3c_1	Ab_1c_1
Bc_1a_3	Bc_1a_3	Bc_1a_1
Ca_1b_3	Ca_1b_1	Ca_1b_1

Species in case 4.—With the letters A, B, C, there occur in case 4 twice the pairs ab and cd. With either one of these the subscripts can be taken as 11, 11; then with the other the two can be, in their order, either 12, 12, or 12, 13. In the third set, ac and bd, we are still free to fix arbitrarily one pair of subscripts, as b_1d_1, leaving two alternatives for the other pair ac. Four partial schedules result, as follows, for the pairs with A, B, and C.

	(α)	(β)	(γ)	(δ)
A	$a_1b_1 \mid c_1d_1$	11 | 11	11 | 11	11 | 11
B	$a_1b_2 \mid c_1d_2$	12 | 12	12 | 13	12 | 13
C	$a_1c_1 \mid b_1d_1$	12 | 11	11 | 11	12 | 11

Species in case 5.—The same four schedules, for the same reasons, are valid in case 5 as in case 4.

The foregoing 19 schedules can be completed to actual systems, as is readily seen by inspection, in the following 26 ways only:

System V, 1α: ABC, $a_1a_2a_3$, $b_1b_2b_3$, $c_1c_2c_3$, $d_1d_2d_3$; Aa_1b_1, Ac_1d_1, Ba_1c_1, Bb_1d_1, Ca_1d_1, Cb_1c_1; $a_1b_2c_3$, $a_1b_3d_2$, $a_1c_2d_3$, $b_1c_3d_2$.

A second system, of course, equivalent to this, differs only in the cyclic order 132 instead of 123.

System V, 1β is impossible.

System V, 1γ: ABC, $a_1a_2a_3$, $b_1b_2b_3$, $c_1c_2c_3$, $d_1d_2d_3$. Aa_1b_1, Ac_1d_1, Ba_1c_1, Bb_1d_1, Ca_1d_2, Cb_1c_2. $a_1b_3c_2$, $a_1b_2d_3$, $a_1c_3d_1$, $b_1c_1d_3$.

An equivalent system is converted into this by the substitution (ac) (bd) (23); or two others similar.

System V, 1δ: ABC, $a_1a_2a_3$, $b_1b_2b_3$, $c_1c_2c_3$, $d_1d_2d_3$. Aa_1b_1, Ac_1d_1, Ba_1c_1. Bb_1d_2, Ca_1d_1, Cb_1c_3. $a_1b_2c_2$, $a_1b_3d_3$, $a_1c_3d_2$, $b_1c_2d_3$.

System V, 2α: ABC, $d_1d_2d_3$: Aa_1d_1, Ab_1c_1, Bb_1d_1, Ba_1c_1, Cc_1d_1, Ca_1b_1; $a_1a_2d_3$, $b_1b_2a_3$, $c_1c_2a_3$; $b_1c_2d_3$, $b_1c_3d_2$.

System V, 2β: ABC, $d_1d_2d_3$; Aa_1d_1, Ab_1c_1, Bb_1d_1, Ba_1c_2, Cc_1d_1, Ca_1b_1; $a_1a_2d_3$, $b_1b_2a_3$, $c_1c_2a_2$; $b_1c_2d_3$, $b_1c_3d_2$.

System V, 2γ: ABC, $d_1d_2d_3$; Aa_1d_1, Ab_1c_1, Bb_1d_1, Ba_1c_2, Cc_1d_1, Ca_1b_2; $a_1a_2d_3$, $b_1b_2a_2$, $c_1c_2a_2$; $b_1c_2d_3$, $b_1c_3d_2$.

System V, 2δ: ABC, $d_1d_2d_3$; Aa_1d_1, Ab_1c_1, Bb_1d_1, Ba_1c_3, Cc_1d_1, Ca_1b_2; $a_1a_2d_3$, $b_1b_2a_2$, $c_1c_2a_1$; $b_1c_2d_3$, $b_1c_3d_2$.

System V, 3α: ABC, $d_1d_2d_3$; Aa_1d_1, Ab_1c_3, Bb_1d_1, Ba_1c_2, Cc_1d_1, Ca_1b_3; $a_1a_2d_3$, $b_1b_2d_3$, $c_1c_2d_3$; $a_1b_1c_1$, $a_1b_2c_3$.

System V, 3β: ABC, $d_1d_2d_3$; Aa_1d_1, Ab_1c_2, Bb_1d_1, Ba_1c_2, Cc_1d_1, Ca_1b_1; $a_1a_2d_3$, $b_1b_2d_3$, $c_1c_2d_3$; $a_1b_3c_3$, $a_1b_2c_1$.

System V, 3γ: ABC, $d_1d_2d_3$; Aa_1d_1, Ab_1c_1, Bb_1d_1, Ba_1c_1, Cc_1d_1, Ca_1b_1; $a_1a_2d_3$, $b_1b_2d_3$, $c_1c_2d_3$; $a_1b_2c_3$, $a_1b_3c_2$.

System V, 4α (1 and 2): ABC, $d_1d_2d_3$;
Aa_1b_1, Ac_1d_1, (1) $a_1a_2d_1$, $b_1b_2c_2$, $c_1c_2a_3$; $a_1b_3d_2$, $b_1c_3d_2$;
Ba_1b_2, Bc_1d_2, or
Ca_1c_1, Cb_1d_1, (2) $a_1a_2d_3$, $b_1b_2c_3$, $c_1c_2a_3$; $a_1b_3d_1$, $b_1c_1d_3$.

These two are equivalent; (1) is converted into (2) by the substitution $(b_1b_3b_2)$ $(d_1d_3d_2)$ (23) (AB).

System V, 4β (1 and 2): ABC, $d_1d_2d_3$;
Aa_1b_1, Ac_1d_1, (1) $a_1a_2d_3$, $b_1b_2c_3$, $c_1c_2a_2$; $a_1b_3d_1$, $b_1c_1d_3$.
Ba_1b_2, Bc_1d_2, or
Ca_1c_2, Cb_1d_1, (2) $a_1a_2d_1$, $b_1b_2c_2$, $c_1c_2a_2$; $a_1b_3d_2$, $b_2c_1d_3$.
System V, 4γ (1 and 2): ABC, $d_1d_2d_3$;
Aa_1b_1, Ac_1d_1, (1) $a_1a_2d_3$, $b_1b_2c_1$, $c_1c_2a_3$; $a_1b_3d_1$, $b_3c_1d_2$.
Ba_1b_2, Bc_1d_3, or
Ca_1c_1, Cb_1d_1, (2) $a_1a_2d_1$, $b_1b_2c_3$, $c_1c_2a_3$; $a_1b_3d_2$, $b_1c_1d_2$.
System V, 4δ (1 and 2): ABC, $d_1d_2d_3$;
Aa_1b_1, Ac_1d_1, (1) $a_1a_2d_3$, $b_1b_2c_1$, $c_1c_2a_2$; $a_1b_3d_1$, $b_3c_1d_2$.
Ba_1b_2, Bc_1d_3, or
Ca_1c_2, Cb_1d_1, (2) $a_1a_2d_1$, $b_1b_2c_3$, $c_1c_2a_2$; $a_1b_3d_2$, $b_1c_1d_2$.
Equivalent systems, by the substitution $(a_1a_2a_3)$ $(c_1c_3c_2)$ (23) (AB).
Systems V, 5α (1 and 2): ABC, $d_1d_2d_3$;
Aa_1b_2, Ac_1d_1, (1) $a_1a_2d_1$, $b_1b_2d_3$, $c_1c_2b_1$; $a_1b_3c_2$, $a_2c_1d_3$,
Ba_1b_2, Bc_1d_2, or
Ca_1c_1, Cb_1d_1, (2) $a_1a_2d_3$, $b_1b_2d_3$, $c_1c_2b_3$; $a_1b_3c_3$, $a_3c_1d_3$.

These two are equivalent in the same way as V, 4α, 1 and 2.
Systems V, 5β (1 and 2): ABC, $d_1d_2d_3$;
Aa_1b_1, Ac_1d_1, (1) $a_1a_2d_1$, $b_1b_2d_3$, $c_1c_2b_2$; $a_1b_3c_1$, $a_2c_1d_3$,
Ba_1b_2, Bc_1d_2, or
Ca_1c_2, Cb_1d_1, (2) $a_1a_2d_2$, $b_1b_2d_3$, $c_1c_2b_3$; $a_1b_3c_3$, $a_1c_1d_3$.

Systems V, 5γ (1 and 2): ABC, $d_1d_2d_3$;
Aa_1b_1, Ac_1d_1, (1) $a_1a_2d_3$, $b_1b_2d_3$, $c_1c_2b_1$; $a_1b_3c_2$, $a_2c_1d_2$,
Ba_1b_2, Bc_1d_3, or
Ca_1c_1, Cb_1d_1, (2) $a_1a_2d_2$, $b_1b_2d_3$, $c_1c_2b_3$; $a_1b_3c_3$, $a_3c_1d_2$.

Systems V, 5δ (1 and 2): ABC, $d_1d_2d_3$;
Aa_1b_1, Ac_1d_1, (1) $a_1a_2d_3$, $b_1b_2d_3$, $c_1c_2b_2$; $a_1b_3c_1$, $a_2c_1d_2$,
Ba_1b_2, Bc_1d_3, or
Ca_1c_2, Cb_1d_1, (2) $a_1a_2d_1$, $b_1b_2d_3$, $c_1c_2b_3$; $a_1b_3c_3$, $a_1c_1d_2$.

Two equivalent systems, as under V, 4δ.

Beside the equivalences already noted, one less obvious is that of systems V, 3α and V, 1α. which Miss Cummings will establish in Part 2. That done, we shall have invariant under this type of substitutions $(1)^3$ $(3)^4$, 21 distinct systems.

§8. TRIAD SYSTEMS WHOSE GROUP CONTAINS A SUBSTITUTION OF THE TYPE $(1)^3$ $(2)^6$.

Operations containing longer single cycles belong to fewer distinct types of triad systems, While the fifth kind of substitution, $(1)^3$ $(3)^4$, gives rise to 21, the sixth, now to be examined, will yield apparently more than 30. Actually the reduced number is the same, 21, for some systems admit two or more substitutions of the same type. This large number of systems might weary the attention, were it not that novel points of difference are developed, in themselves interesting.

Denote the 15 elements and the operation thus:

$$S \equiv (A)\ (B)\ (C)\ (a_1\ b_1)\ (a_2\ b_2)\ (a_3\ b_3)\ (a_4\ b_4)\ (a_5\ b_5)\ (a_6\ b_6)$$

Two triads conjugate under S we shall call dual to each other; there will be six triads self-dual, those containing pairs a_1b_1, a_2b_2, etc. According to the principles in section 2, ABC must be one triad and the six self-dual pairs a_ib_i must be united with A, B, or C to form triads. Denoting the three capitals generically by K, we could specify eight possible types of triads, but for present purposes conjugates combine and form four types. With (or without) the aid of diophantine equations, we find three sets of numbers for these four classes, as follows:

Type of triad.	*Number of triads in a system.*		
Ka_ia_k	5,	3,	1,
Ka_ib_k	1,	3,	5,
$a_ia_ja_k$	1,	2,	3,
$a_ia_jb_k$	7,	6,	5.

The doubles of these numbers, plus the 1 and 6 above mentioned, give the total of 35 triads for a system. The second kind will be taken as standard; the other two will be found to be reducible to this.

By definition, each of the elements A, B, C must appear with six pairs of small letters a, b. Since those not self-dual, as Aa_ib_k or Aa_ia_k, must occur in pairs, Aa_ib_k, Aa_kb_i, the self-dual triads containing A (or B, C) must also be an even number 6, 4, 2, or 0. We therefore divide systems of this section into three principal classes.

In class VI, 1: Six self-dual triads contain A;
In class VI, 2: Four self-dual triads contain A, two have B;
In class VI, 3: Two self-dual triads contain A, two have B, two have C.

We can fix, for each class, these six self-dual triads and still retain freedom to exchange symbols a, b in each pair; also to exchange certain subscripts. Beside ABC, assign to each class these fundamental triads:

Class VI, 1: Aa_1b_1, Aa_2b_2, Aa_3b_3, Aa_4b_4, Aa_5b_5, Aa_6b_6.
Class VI, 2: Aa_1b_1, ... Aa_4b_4; Ba_5b_5, Ba_6b_6.
Class VI, 3: Aa_1b_1, Aa_2b_2; Ba_3b_3, Ba_4b_4; Ca_5b_5, Ca_6b_6.

In class VI, 1 we are free to arrange that pairs with the symbol B shall be either a_ia_k or b_ib_k and that the pairs of subscripts shall be 12, 34, 56. Quite similar is the choice permitted in VI, 2.

Class VI, 1: Ba_1a_2, Bb_1b_2,; Ba_3a_4, Bb_3b_4; Ba_5a_6, Bb_5b_6.
Class VI, 2: Ba_1a_2, Bb_1b_2; Ba_3a_4, Bb_3b_4; Aa_5a_6, Ab_5b_6.

In both classes it is still optional to exchange the a_i, b_i of any conjugate pair as Ba_3a_4, Bb_3b_4. Hence it results that triads in C can all be put into a standard form Ca_ib_k. Thus in classes VI, 1 and VI, 2 the numbers of triads can be brought to agree with the second column in our tabulation; that is, there will be two triads of type $a_ia_ja_k$, and six of type $a_ia_jb_k$.

For both these classes, therefore, we write down at once the possible sets of triads containing C, leaving for separate discussion the class VI, 3.

Class VI, 1—Triads in C.

No. VI, 1_1: Ca_1b_2, Ca_2b_1, Ca_3b_4, Ca_4b_3, Ca_5b_6, Ca_6b_5.
No. VI, 1_2: Ca_1b_3, Ca_3b_1, Ca_2b_4, Ca_4b_2, Ca_5b_6, Ca_6b_5.
No. VI, 1_3: Ca_1b_4, Ca_4b_1, Ca_2b_5, Ca_5b_2, Ca_3b_6, Ca_6b_3.

Class VI, 2—Triads in C

No. VI, 2_1: Ca_1b_2, Ca_2b_1, Ca_3b_4, Ca_4b_3, Ca_5b_6, Ca_6b_5.
No. VI, 2_2: Ca_1b_3, Ca_3b_1, Ca_2b_4, Ca_4b_2; Ca_5b_6, Ca_6b_5.
No. VI, 2_3: Ca_1b_2, Ca_2b_1; Ca_3b_5, Ca_5b_3, Ba_4b_6, Ca_6b_4.
No. VI, 2_4: Ca_1b_4, Ca_4b_1, Ca_2b_5, Ca_5b_2, Ca_3b_6, Ca_6b_3.

Class VI, 3 has two self-conjngate triads in A, two in B, and two containing C. There are yet to be formed for each, four triads or two pairs of conjugates. That is, for each of these three letters we must combine four subscripts into two pairs. Notice that the six pairs are to contain each subscript twice. These may be grouped into one or more cycles; for example, if 12, 23, 31 are among them, they constitute a cycle of three. Possible are apparently

Three cycles of two pairs;
One cycle of two, one cycle of four;
Two cycles of three;
One cycle of six.

By trials it is quickly proved that the third alternative gives a schedule which can not be completed to a full system. There remain then only cycles with two, four, or six pairs. Each cycle can be divided into halves so that each half contains the same subscripts as the other (by taking alternate pairs in the cycle). This gives us twice three pairs involving all six subscripts. One set of pairs corresponding can be chosen of letters *aa* or *bb*, the other of unlike letters, *ab*. All these possibilities, together with the self-conjugate triads, are outlined in the following; exhausting the possible schedules for triads in *A*, *B*, and *C*, except for nonessential substitutions.

Class VI, 3—Triads in A, B, and C, after ABC.

Condensed tables of subscripts.

	ab	*aa, bb*	*ab, ba*
No. VI, 3_1:			
A	11, 22	34	56
B	33, 44	12, 56	....
C	55, 66	....	12, 34
No. VI, 3_2:			
A	11, 22	34	56
B	33, 44	12, 56	....
C	55, 66	....	13, 24
No. VI, 3_3:			
A	11, 22	34	56
B	33, 44	15, 26	....
C	55, 66	....	13, 24
No. VI, 3_4:			
A	11, 22	36	45
B	33, 44	15	26
C	55, 66	24	13

The cycles above described are seen in the second and third columns of these tables (under *aa, bb* and *ab, ba*). There are: One each of the first two species, two of the fourth.

Altogether in the three classes VI, 1, 2, and 3, we have therefore 11 schedules or partial systems. These 11 can all be completed to full systems, most of them in two or more ways. All the systems contain the triad *ABC;* the 18 triads containing a single *A*, *B*, or *C* are given above; and the various supplementary sets of 16 triads, of the types *aaa, bbb, aab,* and *abb* are now to be listed in full.

SUPPLEMENTARY SETS, TO COMPLETE THE FOREGOING SYSTEMS.

(Of two conjugate triads only one is given.)

VI, $1_1\alpha$. *aaa:* 135, 246.
aab: 145, 164, 326, 361, 523, 542.

VI, $1_1\alpha'$. *aaa:* 135, 246.
aab: 146, 163, 325, 362, 524, 541.
Equivalent to VI, $1_1\alpha$ by the substitution (16) (34) (25).

VI, $1_1\beta$. *aaa:* 135, 146.
aab: 235, 246, 254, 263, 361, 451.

VI, $1_1\beta'$. *aaa:* 135, 146.
aab: 236, 245, 253, 264, 361, 451.
Equivalent to VI, $1_1\beta$ by the substitution (36) (45).

VI, 1_2. *aaa:* 145, 235.
aab: 135, 162, 245, 263, 364, 461.
A second supplementary set is equivalent to this by the substitution (12) (34).

VI, $1_3\alpha$. *aaa:* 135, 246.
aab: 145, 162, 324, 361, 523, 546.
An equivalent set is derived by the substitution (12) (36) (45).

VI, $1_3\beta$. *aaa:* 136, 145.
aab: 234, 246, 253, 261, 351, 465.

VI, $2_1\alpha$. *aaa:* 135, 246.
aab: 145, 164, 326, 361, 523, 542.
An equivalent set is derived by the substitution (13) (24).

VI, $2_1\beta$. *aaa:* 135, 146.
aab: 235, 246, 254, 263, 361, 451.

VI, $2_1\gamma$. *aaa:* 135, 146.
aab: 236, 245, 253, 264, 361, 451.

VI, $2_1\delta$. *aaa:* 135, 245.
aab: 613, 624, 632, 641, 145, 235.
An equivalent set is derived by the substitution (13) (24).

VI, $2_2\alpha$. *aaa:* 145, 235.
aab: 135, 162, 245, 263, 364, 461.
A second supplementary set is transformed into this by the substitution (14) (23) (a_5b_5) (a_6b_6).

VI, $2_3\alpha$. *aaa:* 136, 145.
aab: 234, 245, 256, 263, 351, 461.
An equivalent set is reduced to this by the substitution (34) (56).

VI, $2_3\beta$. *aaa:* 135, 146.
aab: 234, 245, 256, 263, 361, 451.
Another supplementary set comes from this by the substitution (34) (56). No. VI, $2_3\beta$ is reduced to VI, $2_3\alpha$ by the substitution (a_3b_4) (a_4b_3) (a_5b_6) (a_6b_5).

VI, $2_3\gamma$. *aaa:* 135, 245.
aab: 145, 163, 234, 265, 362, 461.

VI, $2_4\alpha$. *aaa:* 135, 246.
aab: 145, 162, 324, 361, 523, 546.

VI, $2_4\beta$. *aaa:* 135, 246.
aab: 146, 165, 321, 362, 524, 543.
This is seen to arise from VI, $2_4\alpha$ by the substitution (14) (23) (56).

VI, $2_4\gamma$. *aaa:* 145, 136.
aab: 234, 246, 261, 253, 351, 465.

VI, $2_4\delta$. *aaa:* 145, 235.
aab: 612, 624, 631, 645, 135, 243.

VI, $2_4\epsilon$. *aaa:* 236, 245.
aab: 146, 165, 153, 132, 354, 462.

VI, $3_1\alpha$. *aaa:* 135, 246.
aab: 145, 164, 326, 361, 523, 542.

VI, $3_1\alpha'$. *aaa:* 135, 246.
aab: 146, 163, 325, 362, 524, 541.
Equivalent to VI, $3_1\alpha$ by the substitution (BC) $(a_1a_2b_1b_2)$ $(a_3a_6b_3b_6)$ $(a_5a_4b_5b_4)$.

VI, $3_1\beta$. *aaa:* 135, 146.
aab: 235, 246, 254, 263, 361, 451.
This comes from VI, $3_1\alpha$ by the substitution (AB) (13) (24) (a_6b_6).

VI, $3_1\beta'$. *aaa:* 135, 146.
aab: 236, 245, 253, 264, 361, 451.

VI, $3_1\gamma$. *aaa:* 135, 245.
aab: 613, 624, 632, 641, 145, 235.
This also is equivalent to VI, $3_1\alpha$, by the substitution (AC) (15) (26) (a_4b_4).

VI, $3_1\gamma'$. *aaa:* 135, 245.
aab: 614, 623, 631, 642, 145, 235.

VI, 3_2. *aaa:* 145, 235.
aab: 135, 162, 245, 263, 364, 461;
or 135, 164, 245, 261, 362, 463.
These two alternatives are equivalent by the substitution (12) (34). We shall refer to the first.

VI, $3_3\alpha$. *aaa:* 146, 235.
aab: 125, 136, 241, 362, 453, 564.

VI, $3_3\beta$. *aaa:* 146, 235.
aab: 126, 132, 245, 364, 451, 563.
Equivalent to VI, $3_3\alpha$ by the substitution (12) (34) (56).

VI, $3_3\gamma$. *aaa:* 124, 136.
aab: 251, 352, 453, 654, 236, 461.

VI, $3_3\delta$. *aaa:* 412, 465.
aab: 316, 325, 354, 362, 164, 251.

VI, $3_4\alpha$. *aaa:* 146, 235.
aab: 124, 132, 436, 453, 625, 651.

VI, $3_4\beta$. *aaa:* 146, 235.
aab: 125, 134, 432, 456, 621, 653.
Equivalent to VI, $3_4\alpha$ by the substitution (BC) (12) (36) (45).

VI, $3_4\gamma$. *aaa:* 126, 134.
aab: 234, 251, 461, 352, 456, 563.

VI, $3_4\delta$. *aaa:* 312, 345.
aab: 142, 253, 614, 625, 643, 651.
Equivalent to VI, $3_4\gamma$ by the substitution (AB) (13) (24) (56).

In the above enumeration some systems can still be omitted as redundant.

No. VI, $3_1\gamma'$ is reduced to VI, $1_1\alpha$ by the substitution (a_4b_3) (b_5b_6).

Five systems are reducible to VI, $2_1\alpha$, viz:

VI, $2_1\gamma$ by the substitution (BC) (a_1a_5) (b_1a_6) (a_2b_6) (b_2b_5) (a_4b_4);

VI, 3_1 α by the substitution (ACB) $(a_1a_5a_3)$ $(b_2b_5a_4)$ $(a_2a_6b_3b_1b_6b_4)$:

and the three whose equivalence to VI, $3_1\alpha$ has been noted already.

Further, No. VI, $3_1\beta'$ is reducible to VI, $2_1\delta$ by the substitution

$$(BC)\ (a_1a_5)\ (a_4b_3b_4)\ (a_2b_6b_1a_6b_2b_5).$$

Some of those equivalences are obvious on comparison of the structure as here described, but others would not have been found without the aid of some definite system of procedure. The method actually used was Miss Cummings's method of sequences and indices.

After these deductions for equivalence, there remain 21 systems apparently distinct, automorphic under a substitution of the type $(1)^3(2)^6$.

In writing down these supplementary sets of triads, the first step was to write the two required triads $a_ia_ja_k$ in all ways that are different as regards the schedule of triads in A, B, and C; that is, in all possible ways not transformable into one another without alteration of the preceding 19 triads of the proposed system. After each way of writing these two triads $a_ia_ja_k$, it is easy to decide from inspection whether the number of ways of filling out the six triads *aab* is 0, 1, or 2. Where possible pairs of triads $a_ia_ja_k$ have been omitted, it indicates the impossibility of filling out a system.

§9. THE SUBSTITUTION OF THE TYPE $(1)^7(2)^4$: INVARIANT TRIAD SYSTEMS.

The only remaining (reduced) type of substitutions is that which leaves unchanged 7 of the 15 elements and exchanges the others in pairs. Denote the former by numerals or digits 1, 2, 3, 4, 5, 6, 7; the latter by the pairs of letters Aa, Bb, Cc, Dd. The operation to be considered is S:

$$S\equiv(1)(2)(3)(4)(5)(6)(7)(Aa)(Bb)(Cc)(Dd).$$

The first 7 elements, as we have seen, must constitute by themselves a triad system $\triangle_7$, while each one occurs in 4 additional triads with pairs of letters from the last 8. Of these 8 letters all possible pairs occur, 4 self-conjugate and the rest in 12 conjugate pairs of pairs; 28 in all, so that every such pair is joined in a triad with one of the digits. Two pairs that are conjugate must form triads with the same digit as third element. Hence the 4 self-conjugate pairs, like Aa, are either all completed to triads by the same numeral, as 1, or else by two numerals, as 1 and 2, each joined with two pairs. Triads not self-conjugate, as $3AB$, $3ab$, occur two by two.

Assembling in columns of four the pairs associated with the several digits, we shall have a seven-by-four array. We shall find that there are five types of such arrays, aside from permutations of entire columns. To complete them to triad systems, it remains only to annex a triad system $\triangle_7$ constituted upon the seven digits. This can be done in a variety of ways, so that several systems will result from each seven-by-four array. Triads composed of one digit and two letters shall be termed *mixed*. First we tabulate the mixed triads, writing down the pairs of letters only.

PAIRS FROM MIXED TRIADS, CLASS VII 1.

1	2	3	4	5	6	7
Aa	AB	Ab	AC	Ac	AD	Ad
Bb	ab	aB	ac	aC	ad	aD
Cc	CD	Cd	BD	Bd	BC	Bc
Dd	cd	cD	bd	bD	bc	bC

PAIRS FROM MIXED TRIADS, CLASS VII 2.

	6	7
First five like the above.	AD	Ad
	ad	aD
	Bc	BC
	bC	bc

Here explanation is necessary. Any column could be selected as the second, whence the third would follow. Beside the exchange of conjugate letters in independent pairs, there are still permissible the substitutions

$$(AB)(ab),\ (CD)(cd),\ (AC)(BD)(ac)(bd),\ (AD)(BC)(ad)(bc),$$

this last a result of the others. Compared with the second or the third, any later column may be either cross-tied or not. For example, the fourth is cross-tied to the second by the triads $2AB$, $2CD$ in the one and $4AC$, $4BD$ in the other; hence also by the remaining pairs in the two columns. Notice also that when it is cross-tied to the second column it is necessarily cross-tied to its cognate column, the third. As an example of the opposite kind, the columns 6 and 7 in class VII2 are not cross-tied to columns 2, 3, 4, or 5.

If all four self-conjugate pairs stand in a single column, there are but two nonequivalent classes of seven-by-four arrays, those having the other six columns all cross-tied, and those having four all cross-tied and the two others not cross-tied with them. All others having column 1 can be reduced to either VII1 or VII2.

In the other alternative, when self-conjugate pairs of elements appear in two columns, two in each, there are three classes of schedules. Let the columns containing self-conjugate pairs be the first and second; the other two pairs in the first column are cross-tied to the self-conjugates in the second, and vice versa. Therefore also there will be another column—let it be taken for the third—cross-tied to both the first and the second. Compare the four subsequent columns with the third. Either 4, 2, or 0 are cross-tied with this third. If two are not, select them for the sixth and seventh columns. The resulting arrays are the following:

PAIRS FROM MIXED TRIADS.

For all three classes.			*For class VII3.*			
1	2	3	4	5	6	7
Aa	*AB*	*Ab*	*AC*	*Ac*	*AD*	*Ad*
Bb	*ab*	*aB*	*ac*	*aC*	*ad*	*aD*
CD	*Cc*	*Cd*	*BD*	*Bd*	*BC*	*Bc*
cd	*Dd*	*cD*	*bd*	*bD*	*bc*	*bC*

For class VII4.			
4	5	6	7
AC	*Ac*	*AD*	*Ad*
ac	*aC*	*ad*	*aD*
BD	*Bd*	*Bc*	*BC*
bd	*bD*	*bC*	*bc*

For class VII5.			
4	5	6	7
AC	*Ac*	*AD*	*Ad*
ac	*aC*	*ad*	*aD*
Bd	*BD*	*Bc*	*BC*
bD	*bd*	*bC*	*bc*

Class VII3 is reducible to class VII2. The array of VII2 has the column 1 unique, cross-tied to (interlaced with) all the others, a character not found in any other column. In the array of class VII3, the column 3 is unique in the same particular. From this clue, one finds without difficulty a transformation of the latter array into the former. This transformation does not preserve, however, the pairs of conjugate letters; in other words, it does alter the substitution S with reference to which the systems are constructed; but it changes S into another substitution S' of the same type $(1)^7(2)^4$. The transformer is this: (A) $(B\ d\ b\ a\ D\ c\ C)$ $(1\ 6\ 5\ 4\ 2\ 7\ 3)$.

Since the array of a system of class VII3 with respect to a substitution S, of type $(1)^7(2)^4$, can be transformed into an array of class VII2 with respect to a different substitution S' of the same type, all systems belonging in the one class belong also in the other; and hence class VII3 does not require a separate investigation.

Upon these arrays we are now to superpose triad systems, $\triangle_7$'s, constructed in all non-equivalent modes from the seven digits. First, for the class VII1, there is an immediate deduction available. The columns are triply cross-tied (interlaced), so that they indicate an inherent triad-system or $\triangle_7$. Compared with this inherent system, the $\triangle_7$ to be imposed must have 7, 3, 1, or 0 triads in common. As no column and no inherent triad is unique in this array, no further distinction is possible, and there are precisely four essentially different systems in this class.

SUPPLEMENTARY SETS, $\triangle_7$'S, FOR CLASS VII1.

System VII1_1: 123, 145, 167, 246, 257, 347, 356.
System VII1_2: 123, 145, 167; 247, 256, 346, 357.
System VII1_4: 123, 146, 157, 247, 256, 345, 367.
System VII1_6: 124, 136, 157, 237, 256, 345, 467.

In the array for class VII2, as has been pointed out, column 1 is unique, and the two columns 6, 7 are unlike 2, 3 and 4, 5 in relation to cross-tying or interlacing. All three of these pairs, or else only one of them, or none at all, may be united with numeral 1 in the superimposed $\triangle_7$. If only one, that one may be either 2 3 or 6 7. There are thus four cases, and each can be completed in two ways, giving apparently eight supplementary sets of triads or $\triangle_7$'s.

SUPPLEMENTARY SETS, $\triangle_7$'S, FOR CLASS VII2.

System $VII2_1$ (System $VII2_2$)	123, 145, 167;	246, 257, 347, 356. 247, 256, 346, 357.
System $VII2_3$ (System $VII2_4$)	123, 157, 146;	245, 267, 347, 365. 247, 265, 345, 367.
System $VII2_5$ (System $VII2_6$)	125, 134, 167;	236, 456, 247, 357. 237, 457, 246, 356.
System $VII2_7$ (System $VII2_8$)	124, 157, 163;	237, 453, 674; 256. 235, 456, 672; 347.

These are equivalent, two and two. For systems $VII2_1$ and $VII2_2$ the transformer is obviously $(67)(AB)(ab)(CD)(cd)$. The same transformer relates $VII2_5$ and $VII2_6$. Slightly more intricate is the transformation of $VII2_7$ and $VII2_8$, viz by the substitution $(24)(35)(67)(ABDC)(abdc)$; and that which exchanges $VII2_3$ and $VII2_4$, namely $(23)(45)(67)(ABab)(DCdc)$. We omit, therefore, the even-numbered systems in the above list, as indicated by parentheses.

In the array for class VII4, column 3 is unique, and the others are paired by being cross-tied or interlaced. The three pairs, 12, 45, 67, are distinct or unlike; for the first are triply interlaced with column 3 by sets of four letters, the second are interlaced with each other, and each by itself with column 3, while the last pair are interlaced with each other but not with column 3. It is important to observe that we can exchange simultaneously the members of all three pairs, by the substitution $(ADad)(BCbc)(12)(45)(67)$. This allows us to omit one of every two that have one of these pairs of numerals in a triad with 3, as for example 345.

SUPPLEMENTARY SETS, $\triangle_7$'S, FOR CLASS VII4.

System VII 4_1: *312*, *345*, *367;* 146, 157, 256, 247.
System VII 4_2: *312*, 346, 357; 145, 167, 247, 256.
System VII 4_3: 316, *345*, 327; 142, 157, 652, 647.
System VII 4_4: 314, 325, *367;* 126, 157, 427, 456.
System VII 4_5: 314, 326, 357; 125, 167, 427, 465.
System VII 4_6: 314, 326, 357; 127, 165, 425, 467.

This list is complete. For we need only consider the triads containing the element 3. Either all three contain pairs whose columns are interlaced in the array, ($VII4_1$), or only one, or none. That should give us $(1+6+4=)11$ systems, after making allowance for the automorphism mentioned just before the list. A further reduction is effected by observing that each of the three operations like $(AB)(CD)(ab)(cd)$ exchanges each of two pairs of columns, as (45)(67), leaving the other columns of the array unaltered. Accordingly the five systems $VII4_2$ . . . . . $VII4_6$ represent ten, and $VII4_1$ makes up eleven, the full count.

The array for class VII5 has the unique column 3, the unique pair of columns 1, 2 containing conjugate pairs, and the interchangeable pairs of columns 45, 67. We shall take account of five substitutions among letters in the array, and their effect in permuting columns and their respective digits.

T_1: $(AB)(ab)$ produces (47)(56).
T_2: $(CD)(cd)$ produces (46)(57).
T_3: $(AB)(CD)(ab)(cd)$ produces (45)(67).
T_4: $(AC)(BD)(ac)(bd)$ produces (12)(67).
T_5: $(AD)(BC)(ad)(bc)$ produces (12)(45).

Hence we distinguish only four cases, different as regards the pairs associated in triads with the unique numeral 3. Either all the pairs 12, 45, 67, or the pair 12 only, or one of the others exclusively, as 45, or none of them, must occur with 3. Each of these admits evidently two modes of completion, but two of the resulting eight systems are redundant, as will be explained.

Supplementary Sets, $\triangle_7$'s, for the Class VII5.

System VII5_1 System VII5_2	*312, 345, 367*:	146, 157, 247, 256. 147, 156, 246, 257.
System VII5_3 (System VII5_4)	*312*, 346, 357:	145, 167, 247, 265. 147, 165, 245, 267.
System VII5_5. System VII5_6.	316, *345*, 327:	142, 157, 647, 625. 147, 125, 642, 657.
System VII5_7. (System VII5_8).	314, 326, 357:	125, 167, 427, 465. 127, 165, 425, 467.

In this list two equivalences can be detected. A distinction has been pointed out, in the array of class VII5, between the set of columns, 123, and the others, columns 45 67. Substitutions can be seen which will transform any one of these latter columns into itself, permute the three others in cycle, and permute cyclically also the first three columns. Select the pair AC from column 4. Under 5, 6, 7 note the pairs containing A and C; similarly under 132.

5	6	7	1	3	2
Ac	AD	Ad	Aa	Ab	AB
Ca	Cb	CB	CD	Cd	Cc

The substitution $(A)(C)(abB)(cDd)$ is found to convert column 4 into itself, hence it is equivalent to the operation on numerals:

$$(4)\ (132)\ (567),$$

and this transforms the supplementary system VII5_4 into VII5_1. In the same way is found the operation:

$$(B)(D)(dcC)(bAa) \equiv (5)(123)(467),$$

which shows an equivalence between VII5_5 and VII5_8.

Four other relations, found by the sequence method of Miss Cummings, are readily verified.

System VII2_1 ≡ System VII1_2, by the substitution $(67)(Dd)$.
System VII4_2 ≡ System VII2_3, by the substitution $(23)(4c)(5C)(6d)(7D)$.
System VII5_1 ≡ System VII2_5, by the substitution

$$\begin{pmatrix}1234567\ Aa\ Bb\ Cc\ Dd\\ 176bCBc\ 25\ 43\ aD\ Ad\end{pmatrix}$$

System VII5_3 ≡ System VII1_4, by the substitution

$$\begin{pmatrix}1234567\ Aa\ Bb\ Cc\ Dd\\ 213ABba\ CD\ cd\ 46\ 75\end{pmatrix}$$

Accordingly there remain as distinct systems in this section the following 16:

Class VII1, systems $_{1,\ 2,\ 4,\ 6}$;
Class VII2, systems $_{3,\ 5,\ 7}$;
Class VII4, systems $_{1,\ 3,\ 4,\ 5,\ 6}$;
Class VII5, systems $_{2,\ 5,\ 6,\ 7}$.

No proof of nonequivalence has been given, save in a few special cases. It is believed that all redundances have been eliminated from each separate section. To Miss Cummings will fall the discovery of equivalences in different sections, and the proof of essential difference in the residue. As a summary result, we know that the 71 triad systems analyzed and listed in the foregoing include certainly all that admit any substitution other than the identity.

PART 2.

TRAINS FOR TRIAD SYSTEMS ON 15 ELEMENTS WHOSE GROUP IS OF ORDER HIGHER THAN UNITY.

By L. D. Cummings.

To investigate the 71 systems obtained in Part 1, and to determine the group for each system, Mr. White's method of comparison [1] for triad systems is employed. For this method the triple system is regarded as an operator and certain covariants of that operator are deduced. These covariants can be represented graphically and are called the trains of the system.

The trains show that the 71 systems are reducible to 44 noncongruent systems; of these 24 are completely known systems already fully discussed in my dissertation,[2] but the remaining 20 systems have not been described heretofore. The trains for the 44 noncongruent systems are exhibited, and for each of the 20 new systems the group is determined. The substitutions which transform the 51 systems into their equivalent systems are also given below.

A triple system on n elements consists of triads so selected that every pair of elements (or dyad) occurs once and only once in the chosen triad. If there are 15 elements, every element occurs with 7 pairs of others, and there are in the system in all 35 triads. This property qualifies the triad system to be a transformer of dyads into single elements, and since each dyad occurs once and no more this duality is unique for dyads. Thus, if the system contains the three triads 124, 135, 236, then it will transform the triad 123 which contains the pairs 12, 13, 23 into the triad 456.

From 15 elements 455 triads can be formed. Any system contains 35 of these, leaving 420 that may be called extraneous triads. Apply the system to transform them all; we shall see, as in the example worked out below, that the 35 triads in the system are transformed into themselves, but the 420 extraneous triads go either into extraneous triads or possibly into triads of the system. Some triads will transform into themselves, some will be produced more than once, and others may not be produced at all by the transformation. All that are found to be produced by the transformation are called *derivative*; all that are missing after the transformation, if any, are called *primitive*.

TRAINS OF TRIADS.

Under a given triad system as an operator, let a triad t_1 be converted into the triad t_2. Repeat the operation and continue indefinitely, so that t_2 becomes t_3; t_3 becomes t_4. Since only 455 triads exist, either a triad of the system t_r is reached or else a triad that has already appeared is repeated, namely, $t_{m+k} \equiv t_m$. In the former case the triad t_r repeats forever, while in the latter case the train beginning at t_m constitutes a recurring cycle. If the triads of the system are designated as one-term cycles, then every triad that is primitive with respect to a given triple system initiates a train terminating in a periodic cycle. Triads that do not recur in the terminal cycle are classified as forming *appendices*, and a complete train consists of one recurrent cycle and all its appendices.

Some substitution may transform the triple system into itself. Such a substitution evidently must also transform each train into itself or into a precisely similar train and therefore

[1] H. S. White: Triple systems as transformations and their paths among triads. Transactions of the American Mathematical Society, vol. 14 (1913), pp. 6–13.

[2] L. D. Cummings: On a method of comparison for triple systems. Transactions of the American Mathematical Society, vol. 15 (1914), pp. 311–327.

must leave unchanged the totality of trains connected with the system. The totality of complete trains (cycles with their appendices) forms accordingly an arrangement of triads invariant under those substitutions on the 15 elements that transform the triple system into itself and facilitates the determination of the group of the system.

Example: The triple system VI $3_4\gamma$ on 15 elements.—For convenience the system is transformed by the substitution

$$S \equiv \begin{pmatrix} A\,B\,C\; a_1 b_1\; a_2 b_2\; a_3 b_3\; a_4 b_4\; a_5 b_5\; a_6 b_6 \\ b\; a\; c\;\; 1\; 2\;\;\; 3\; 4\;\;\; d\; e\;\;\; f\; g\;\;\; 7\; 8\;\;\; 6\; 5 \end{pmatrix},$$

and is exhibited in the following 15 by 7 array:

a	b	c	d	e	f	g	1	2	3	4	5	6	7	8
bc	ac	ab	ae	ad	ag	af	$a7$	$a8$	$a6$	$a5$	$a4$	$a3$	$a1$	$a2$
de	$d5$	$d2$	$b5$	$b6$	$b8$	$b7$	$b2$	$b1$	$b4$	$b3$	bd	be	bg	bf
fg	$e6$	$e1$	$c2$	$c1$	$c3$	$c4$	ce	cd	cf	cg	$c6$	$c5$	$c8$	$c7$
17	$f8$	$f3$	$f1$	$f4$	$d1$	$d3$	df	eg	dg	$d7$	$e7$	$d8$	$d4$	$d6$
28	$g7$	$g4$	$g3$	$g2$	$e4$	$e2$	$g6$	$f5$	$e8$	ef	$f2$	$f7$	$e5$	$e3$
36	12	56	47	38	25	16	35	37	15	18	$g8$	$g1$	$f6$	$g5$
45	34	78	68	57	67	58	48	46	27	26	13	24	23	14

The transforming process is simple and may be shown in its application to a triad 458 which is extraneous to this system. Its pairs 45, 48, 58 transform, respectively, into a, 1, g, giving the transformed triad $a1g$. The triad $a1g$ transforms into the triad of the system $76f$ which repeats indefinitely. These three triads form the type of train which is exhibited graphically in figure 3. This system applied as an operator on the 455 triads yields the following set of covariants (trains):

TRAINS FOR THE SYSTEM VI$3_4\gamma$.

Six classes of trains terminating in triads of the system: (1) 11 trains, figure 1; (2) 4 trains, figure 2; (3) 12 trains, figure 6; (4) 2 trains, figure 206; (5) 2 trains, figure 210; (6) 4 trains, figure 211.

One class of trains terminating in a cycle of period 4: (7) 1 train, figure 182.

Two classes of trains terminating in cycles of period 6: (8) 5 trains, figure 183; (9) 1 train, figure 213.

Two classes of trains terminating in cycles of period 12: (10) 1 train, figure 214; (11) 4 trains, figure 215.

DETERMINATION OF THE GROUP FOR THE SYSTEM VI$3_4\gamma$.

The trains for this system separate the 35 triads into 6 distinct classes and every operation of the group that leaves the system invariant must transform any train into itself, or into another train of the same class. Since only those elements may be permuted which occur the same number of times in a class, the enumeration of the appearances of each of the 15 elements in the 6 classes of trains, as in the following table, shows the possible sets of transitive elements. An examination of the triads of the system belonging to class (1) shows that the 15 elements do not enter symmetrically as members of the triads of the class; for example, in these 11 triads the element c appears 7 times but no other element appears 7 times.

	a	b	c	d	e	f	g	1	2	3	4	5	6	7	8
(1)	1	1	7	1	1	1	1	3	3	3	3	2	2	2	2
(2)		4		1	1	1	1					1	1	1	1
(3)	4			3	3	3	3	2	2	2	2	3	3	3	3
(4)	2			1	1	1	1								
(5)		2						1	1	1	1				
(6)				1	1	1	1	1	1	1	1	1	1	1	1

The possible systems of transitivity for the group are therefore a; b; c; d, e, f, g; 1, 2, 3, 4; 5, 6, 7, 8.

The sets of possible transitive elements subdivide the classes into sets of triads which are not transformable into one another by operations of the group of the system; the subdivisions

are shown by lines separating the triads in a class. The system $VI3_4\gamma$ contains 11 nonpermutable subdivisions given in the following table:

(1)				(2)	(3)			(4)	(5)	(6)
abc	*cda* *cf3* *ce1* *cg4*	135 418 246 327	*c78* *c56*	*bd5* *bf8* *be6* *bg7*	*d68* *f76* *e57* *g85*	*dg3* *fd1* *ef4* *ge2*	*a17* *a45* *a28* *a36*	*ade* *afg*	*b12* *b43*	*d47* *f25* *e38* *g16*

In determining the group we examine first for substitutions that transform into itself one of the trains, and secondly for those that transform this train into the remaining trains of its class. Substitutions when determined must be tested on the 35 triads of the system.

The substitutions may be determined from any class of trains in the system, but most easily from the class containing the trains with the greatest number of triads since these exhibit more repetition of the elements. In the present case (4), figure 206 is selected.

(i) Examine for substitutions to transform the train *ade* into itself. The train consists of two similar parts and the substitution

$$t \equiv (a)\ (b)\ (c)\ (de)\ (fg)\ (12)\ (34)\ (56)\ (78)$$

permutes these similar parts. No other substitution exists which converts this train *ade* into itself. Therefore only a subgroup of order 2 transforms this train into itself.

(ii) Examine for substitutions to transform the train of the triad *ade* into the other train of its class. The substitution

$$s \equiv (a)\ (b)\ (c)\ (dfeg)\ (1423)\ (5867)$$

transforms the train of *ade* into the train of *afg*. Since $t \equiv s^2$ the substitution t is omitted. The substitution s applied to the 35 triads of the system transforms the system into itself. Therefore the group for this system is a cyclic group of order 4 and is generated by $s \equiv (a)\ (b)\ (c)\ (dfeg)\ (1423)\ (5867)$.

Similar detailed study determines the group for each of the following 43 systems:

Trains for the System $V4a1$.

Ten classes of trains terminating in triads of the system: (1) Seven trains, figure 1; (2) 6 trains, figure 2; (3) 3 trains, figure 3; (4) 3 trains, figure 6; (5) 1 train, figure 66; (6) 3 trains, figure 205; (7) 3 trains, figure 207; (8) 3 trains, figure 208; (9) 3 trains, figure 209; (10) 3 trains, figure 212.

One class of trains terminating in a cycle of period 12: (11) Three trains, figure 216.

Group for the System $V4a1$.

The sets of transitive elements are A; B; C; $a_1a_2a_3$; $b_1b_2b_3$; $c_1c_2c_3$; $d_1d_2d_3$. These with the trains separate the system into 13 nonpermutable subdivisions. The group is generated by $s \equiv (A)\ (B)\ (C)\ (a_1a_2a_3)\ (b_1b_2b_3)\ (c_1c_2c_3)\ (d_1d_2d_3)$, and is of order 3.

The trains which belong to the system $VI3_4\gamma$ are exhibited in Plate I, those of the system $V4a1$ in Plate II; even a casual inspection of these two plates establishes conclusively the noncongruence of the two systems.

The same type of train may occur in several systems, and in order to avoid repetition in the diagrams the 204 distinct types which occur in the remaining 42 systems are numbered and listed in a definite order. The most convenient arrangement seemed to be the following,

namely: All trains which contain a principal or maximum succession consisting of 1, 2, 3, ... , k, . . . triads are placed together and numbered consecutively; the subordinate arrangement of these is easily observed. In general, the trains are exhibited in full, but in order to save space a few of the longest trains which are divisible into parts cyclically repeated are shown in one part only. These, on account of their greater length, appear toward the end of the series of figures.

The noncongruence of the remaining 42 systems is shown in the dissimilarity in the number and in the type of trains enumerated under each system. Systems with different numbers of trains in their classes are noncongruent. The distinctiveness of trains numbered differently is evident, and the fact that no possible confusion of the trains as pictured can arise constitutes the chief merit of this method of proof of the nonconvergence of triad systems.

The trains for the 44 systems contain 216 different types, among which appear trains with cycles of period 4, 6, 9, 10, 12, 18, 20, 24, 30, and 72. Trains with cycles of period 2 or of period 3 are impossible, but while no cycle of period 5 has appeared among the trains of these systems, there is no evident reason why such a cycle may not occur in the trains of systems not yet investigated. The groups for the 44 systems have been determined and the generators for each of the 20 new groups are exhibited.

The groups for the 24 systems IA, IB, IC, IIA, IIB, IIC, IID, IIE, IIF, IIIA, IIIB, IIIC, IIID, IVA, IVB, VA, VB, VC, VD, VIA, VIB, VIC, VID, VII have already been determined,[1] and for the sake of brevity are omitted.

Mr. Cole has pointed out an error in the order of the group for the system IIIB. The order of this group is 192, and not 96, as previously stated.

Trains for the System IA.

Three classes of trains terminating in triads of the system: (1) Seven trains, figure 20 (2) 7 trains, figure 1; (3) 21 trains, figure 2.

One class of trains terminating in cycles of period 18: (4) Seven trains, figure 194.

Trains for the System IB.

Six classes of trains terminating in triads of the system: (1) Three trains, figure 24; (2) 4 trains, figure 20; (3) 3 trains, figure 4; (4) 6 trains, figure 3; (5) 9 trains, figure 2; (6) 10 trains, figure 1.

Three classes of trains terminating in cycles of period 18: (7) Three trains, figure 195; (8) 3 trains, figure 196; (9) 1 train, figure 197.

Trains for the System IC.

Six classes of trains terminating in triads of the system: (1) Six trains, figure 24; (2) 1 train, figure 20; (3) 3 trains, figure 4; (4) 12 trains, figure 3; (5) 6 trains, figure 2; (6) 7 trains, figure 1.

Three classes of trains terminating in cycles of period 18: (7) One train, figure 198; (8) 3 trains, figure 199; (9) 3 trains, figure 200.

Trains for the System IIA.

Five classes of trains terminating in triads of the system: (1) One train, figure 5; (2) 12 trains, figure 21; (3) 6 trains, figure 15; (4) 4 trains, figure 49; (5) 12 trains, figure 35.

Trains for the System IIB.

Eleven classes of trains terminating in triads of the system: (1) Two trains, figure 21; (2) 1 train, figure 14; (3) 2 trains, figure 15; (4) 4 trains, figure 17; (5) 2 trains, figure 16; (6) 4 trains, figure 85; (7) 4 trains, figure 44; (8) 4 trains, figure 77; (9) 4 trains, figure 74; (10) 4 trains, figure 39; (11) 4 trains, figure 31.

[1] Cummings, L. D. (Loc. cit.)

Trains for the System IIC.

Twenty classes of trains terminating in triads of the system: (1) One train, figure 26; (2) 1 train, figure 25; (3) 1 train, figure 21; (4) 2 trains, figure 55; (5) 2 trains, figure 56; (6) 2 trains, figure 90; (7) 1 train, figure 11; (8) 2 trains, figure 50; (9) 2 trains, figure 88; (10) 2 trains, figure 16; (11) 1 train, figure 15; (12) 2 trains, figure 86; (13) 2 trains, figure 82; (14) 2 trains, figure 44; (15) 2 trains, figure 83; (16) 2 trains, figure 36; (17) 2 trains, figure 74; (18) 2 trains, figure 40; (19) 2 trains, figure 67; (20) 2 trains, figure 65.

Trains for the System IID.

Twenty classes of trains terminating in triads of the system: (1) One train, figure 26; (2) 1 train, figure 25; (3) 1 train, figure 21; (4) 2 trains, figure 19; (5) 2 trains, figure 59; (6) 2 trains, figure 89; (7) 1 train, figure 15; (8) 1 train, figure 14; (9) 2 trains, figure 50; (10) 2 trains, figure 16; (11) 2 trains, figure 43; (12) 2 trains, figure 87; (13) 2 trains, figure 84; (14) 2 trains, figure 45; (15) 2 trains, figure 34; (16) 2 trains, figure 36; (17) 2 trains, figure 69; (18) 2 trains, figure 76; (19) 2 trains, figure 72; (20) 2 trains, figure 40.

Trains for the System IIE.

Nine classes of trains terminating in triads of the system: (1) 1 train, figure 5; (2) 4 trains, figure 26; (3) 6 trains, figure 25; (4) 4 trains, figure 21; (5) 8 trains, figure 55; (6) 2 trains, figure 14; (7) 4 trains, figure 50; (8) 2 trains, figure 15; (9) 4 trains, figure 36.

Trains for the System IIF.

Thirteen classes of trains terminating in triads of the system: (1) 2 trains, figure 21; (2) 2 trains, figure 15; (3) 2 trains, figure 51; (4) 1 train, figure 14; (5) 4 trains, figure 52; (6) 2 trains, figure 16; (7) 4 trains, figure 45; (8) 2 trains, figure 70; (9) 2 trains, figure 80; (10) 4 trains, figure 39; (11) 2 trains, figure 33; (12) 4 trains, figure 75; (13) 4 trains, figure 64.

Trains for the System IIIA.

One class of trains terminating in triads of the system: (1) 35 trains, figure 5.

Trains for the System IIIB.

Three classes of trains terminating in the triads of the system: (1) 7 trains, figure 5; (2) 24 trains, figure 25; (3) 4 trains, figure 14.

Trains for the System IIIC.

Four classes of trains terminating in triads of the system: (1) 1 train, figure 5; (2) 12 trains, figure 25; (3) 16 trains, figure 23; (4) 6 trains, figure 14.

Trains for the System IIID.

Two classes of trains terminating in triads of the system: (1) 28 trains, figure 23; (2) 7 trains, figure 14.

Trains for the System IVA.

Five classes of trains terminating in triads of the system: (1) 1 train, figure 5; (2) 8 trains, figure 25; (3) 16 trains, figure 19; (4) 6 trains, figure 15; (5) 4 trains, figure 14.

Trains for the System IVB.

Eleven classes of trains terminating in triads of the system: (1) 1 train, figure 25; (2) 4 trains, figure 23; (3) 4 trains, figure 56; (4) 4 trains, figure 59; (5) 3 trains, figure 14; (6) 3 trains, figure 15; (7) 2 trains, figure 50; (8) 4 trains, figure 52; (9) 4 trains, figure 17; (10) 2 trains, figure 36; (11) 4 trains, figure 39.

Trains for the System VA.

Three classes of trains terminating in triads of the system: (1) 1 train, figure 5; (2) 18 trains, figure 15; (3) 16 trains, figure 13.

Trains for the System VB.

Five classes of trains terminating in triads of the system: (1) 1 train, figure 14; (2) 6 trains, figure 15; (3) 12 trains, figure 47; (4) 4 trains, figure 12; (5) 12 trains, figure 39.

Trains for the System VC.

Nine classes of trains terminating in triads of the system: (1) 2 trains, figure 25; (2) 4 trains, figure 57; (3) 4 trains, figure 15; (4) 1 train, figure 14; (5) 4 trains, figure 50; (6) 8 trains, figure 54; (7) 4 trains, figure 46; (8) 4 trains, figure 36; (9) 4 trains, figure 39.

Trains for the System VD.

Seven classes of trains terminating in triads of the system: (1) 3 trains, figure 25; (2) 4 trains, figure 60; (3) 1 train, fig. 14; (4) 6 trains, figure 50; (5) 12 trains, figure 53; (6) 3 trains, figure 15; (7) 6 trains, figure 36.

Trains for the System VIA.

Six classes of trains terminating in triads of the system: (1) 4 trains, figure 20; (2) 3 trains, figure 15; (3) 6 trains, figure 132; (4) 12 trains, figure 64; (5) 6 trains, figure 109; (6) 4 trains, figure 1.

Trains for the System VIB.

Twenty-one classes of trains terminating in triads of the system: (1) 1 train, figure 26; (2) 2 trains, figure 22; (3) 2 trains, figure 20; (4) 1 train, figure 15; (5) 2 trains, figure 104; (6) 1 train, figure 16; (7) 2 trains, figure 102; (8) 1 train, figure 130; (9) 2 trains, figure 83; (10) 2 trains, figure 3; (11) 2 trains, figure 72; (12) 2 trains, figure 69; (13) 2 trains, figure 35; (14) 2 trains, figure 75; (15) 2 trains, figure 125; (16) 2 trains, figure 67; (17) 1 train, figure 113; (18) 1 train, figure 119; (19) 2 trains, figure 112; (20) 1 train, figure 109; (21) 2 trains, figure 1.

Trains for the System VIC.

Thirteen classes of trains terminating in triads of the system: (1) 3 trains, figure 22; (2) 1 train, figure 20; (3) 3 trains, figure 16; (4) 3 trains, figure 3; (5) 3 trains, figure 73; (6) 3 trains, figure 74; (7) 3 trains, figure 122; (8) 3 trains, figure 124; (9) 3 trains, figure 126; (10) 3 trains, figure 31; (11) 3 trains, figure 65; (12) 3 trains, figure 117; (13) 1 train, figure 1.

Trains for the System VID.

Thirteen classes of trains terminating in triads of the system: (1) 3 trains, figure 26; (2) 3 trains, figure 22; (3) 3 trains, figure 89; (4) 3 trains, figure 55; (5) 1 train, figure 20; (6) 3 trains, figure 103; (7) 3 trains, figure 101; (8) 3 trains, figure 3; (9) 3 trains, figure 36; (10) 3 trains, figure 76; (11) 3 trains, figure 96; (12) 3 trains, figure 98; (13) 1 train, figure 1.

Trains for the System VII.

One class of trains terminating in triads of the system: (1) 35 trains, figure 1.

Two classes of trains terminating in cycles of period 6: (2) 5 trains, figure 183; (3) 5 trains figure 186.

Trains for the System I2.

Seven classes of trains terminating in triads of the system: (1) 5 trains, figure 32; (2) 5 trains, figure 159; (3) 5 trains, figure 94; (4) 5 trains, figure 163; (5) 5 trains, figure 146: (6) 5 trains, figure 2; (7) 5 trains, figure 1.

Group for the system I2.—The sets of transitive elements are $a\ b\ c\ d\ e$; $\alpha\ \beta\ \gamma\ \delta\ \epsilon$; 1 2 3 4 5; these with the trains separate the system into 7 nonpermutable subdivisions. The group is generated by

$$s=(a\ b\ c\ d\ e)\ (1\ 2\ 3\ 4\ 5)\ (\alpha\ \beta\ \gamma\ \delta\ \epsilon)$$

and is of order 5.

Trains for the System III_2.

Three classes of trains terminating in triads of the system (1) 15 trains, figure 1; (2) 2 trins, figure 48; (3) 18 trains, figure 145.

One class of trains terminating in a cycle of period 4: (4) 9 trains, figure 182.

Group for the system III_2.—The sets of transitive elements are $a_1\ a_2\ a_3\ b_1\ b_2\ b_3$; $c_1\ c_2\ c_3\ d_1\ d_2\ d_3\ e_1\ e_2\ e_3$; these with the trains separate the system into 9 nonpermutable subdivisions. The group is generated by

$$s=(a_1)\ (a_2)\ (b_1\ b_2\ b_3)\ (c_1\ d_1\ e_1)\ (c_2\ d_2\ e_2)\ (c_3\ d_3\ c_3)\ (d_3),$$
$$t=(a_1\ a_3)\ (a_2)\ (b_1\ b_2)\ (b_3)\ (c_1\ e_3)\ (c_2\ e_2)\ (c_3\ e_1)\ (d_1\ d_3)\ (d_2),$$
$$w=(a_1\ b_1\ a_2\ b_2)\ (a_3\ b_3)\ (c_1\ e_2\ c_2\ d_1)\ (c_3)\ (d_2\ e_3\ e_1\ d_3),$$

d is of order 36.

Trains for the System III_3.

Four classes of trains terminating in triads of the system: (1) 1 train, figure 179; (2) 3 trains, figure 2; (3) 3 trains, figure 27; (4) 28 trains, figure 1.

Three classes of trains terminating respectively in cycles of periods 5, 6, and 6: (5) 1 train. figure 203; (6) 1 train, figure 188; (7) 1 train, figure 185.

Group for the system III_3.—The sets of transitive elements are $a_1\ a_2\ a_3$; $b_1\ b_2\ b_3$; $c_1\ c_2\ c_8$, $d_1\ d_2\ d_3$; $e_1\ e_2\ e_3$; these with the trains separate the system into 13 nonpermutable subdivisions. The group is generated by

$$s\equiv(a_1\ a_2\ a_3)\ (b_1\ b_2\ b_3)\ (c_1\ c_2\ c_3)\ (d_1\ d_2\ d_3)\ (e_1\ e_2\ e_3),$$

and is of order 3.

Trains for the System III_6.

Nine classes of trains terminating in triads of the system: (1) 1 train, figure 106; (2) 3 train, figure 12; (3) 6 trains, figure 148; (4) 3 trains, figure 122; (5) 3 trains, figure 6; (6) 3 trains, figure 62; (7) 3 trains, figure 2; (8) 3 trains, figure 30; (9) 9 trains, figure 1.

Two classes of trains terminating in cycles of periods 6 and 4, respectively: (10) 2 trains, figure 189; (11) 3 trains, figure 182.

Group for the system III_6.—The sets of transitive elements are $a_1\ a_2\ a_3$; $b_1\ b_2\ b_3$; $c_1\ c_2\ c_3$; $d_1\ d_2\ d_3\ e_1\ e_2\ e_3$; these with the trains separate the system into 9 nonpermutable subdivisions. The group is generated by

$$s\equiv(a_1)\ (b_1)\ (c_1)\ (a_2\ a_3)\ (b_2\ b_3)\ (c_2\ c_3)\ (d_1\ e_1)\ (d_2\ e_3)\ (d_3\ e_2),$$
$$t\equiv(a_1\ a_2\ a_3)\ (b_1\ b_2\ b_3)\ (c_1\ c_2\ c_3)\ (d_1\ d_2\ d_3)\ (e_1\ e_2\ e_3),$$

and is of order 6.

Trains for the System III_7.

Eight classes of trains terminating in triads of the system: (1) 1 train, figure 171; (2) 3 trains, figure 178; (3) 3 trains, figure 93; (4) 3 trains, figure 162; (5) 6 trains, figure 2; (7) 3 trains, figure 6; (8) 13 trains, figure 1.

Two classes of trains terminating in cycles of period 6: (9) 1 train, figure 184; (10) 1 train figure 187.

Group for the system III_7.—The sets of transitive elements are $a_1\ a_2\ a_3$; $b_1\ b_2\ b_3$; $c_1\ c_2\ c_3$; $d_1\ d_2\ d_3$; $e_1\ e_2\ e_3$; these with the trains separate the system into 13 nonpermutable subdivisions. The group is generated by

$$s\equiv(a_1\ a_2\ a_3)\ (b_1\ b_2\ b_3)\ (c_1\ c_2\ c_3)\ (d_1\ d_2\ d_3)\ (e_1\ e_2\ e_3),$$

and is of order 3.

TRAINS FOR THE SYSTEM $V1\gamma$.

Three classes of trains terminating in triads of the system: (1) 6 trains, figure 9; (2) 1 trains, figure 6; (3) 17 trains, figure 1.

One class of trains terminating in a cycle of period 72: (4) 1 train, figure 204.

Group for the system $V1\gamma$.—The sets of transitive elements are A; B C; a_1 a_2 a_3 b_1 b_2 b_3 c_1 c_2 c_3 d_1 d_2 d_3; these with the trains separate the system into five nonpermutable subdivisions. The group is generated by

$$s = (A)\ (B\ C)\ (a_1\ d_1\ b_3\ c_3\ a_2\ d_2\ b_1\ c_1\ a_3\ d_3\ b_2\ c_2),$$

and is of order 12.

TRAINS FOR THE SYSTEM $V4\beta1$.

Nine classes of trains terminating in triads of the system: (1) 3 trains, figure 160; (2) 3 trains, figure 180; (3) 9 trains, figure 2; (4) 3 trains, figure 138; (5) 1 train, figure 139; (6) 3 trains, figure 3; (7) 3 trains, figure 27; (8) 3 trains, figure 164; (9) 7 trains, figure 1.

Group for the system $V4\beta1$.—The sets of transitive elements are A; B; C; a_1 a_2 a_3; b_1 b_2 b_3; c_1 c_2 c_3; d_1 d_2 d_3; these with the trains separate the system into 13 nonpermutable subdivisions. The group is generated by

$$s = (A)\ (B)\ (C)\ (a_1\ a_2\ a_3)\ (b_1\ b_2\ b_3)\ (c_1\ c_2\ c_3)\ (d_1\ d_2\ d_3)$$

and is of order 3.

TRAINS FOR THE SYSTEM $V4\beta2$.

Seven classes of trains terminating in triads of the system: (1) 3 trains, figure 134; (2) 3 trains, figure 10; (3) 3 trains, figure 29; (4) 6 trains, figure 2; (5) 1 train, figure 95; (6) 3 trains, figure 7; (7) 16 trains, figure 1.

Two classes of trains terminating in cycles of periods 9 and 6 respectively: (8) 1 train, figure 191; (9) 3 trains, figure 190.

Group for the system $V4\beta2$.—The sets of transitive elements are A; B; C; a_1 a_2 a_3; b_1 b_2 b_3; c_1 c_2 c_3; d_1 d_2 d_3; these with the trains separate the system into 13 nonpermutable subdivisions. The group is generated by

$$s = (A)\ (B)\ (C)\ (a_1\ a_2\ a_3)\ (b_1\ b_2\ b_3)\ (c_1\ c_2\ c_3)\ (d_1\ d_2\ d_3)$$

and is of order 3.

TRAINS FOR THE SYSTEM $V4\gamma1$.

Thirteen classes of trains terminating in triads of the system: (1) 3 trains, figure 107; (2) 3 trains, figure 42; (3) 3 trains, figure 100; (4) 3 trains, figure 8; (5) 3 trains, figure 36; (6) 3 trains, figure 41; (7) 3 trains, figure 78; (8) 3 trains, figure 150; (9) 3 trains, figure 97; (10) 3 trains, figure 117; (11) 1 train, figure 95; (12) 3 trains, figure 63; (13) 1 train, figure 1.

Group for the system $V4\gamma1$.—The sets of transitive elements are A; B; C; a_1 a_2 a_3; b_1 b_2 b_3; c_1 c_2 c_3; d_1 d_2 d_3; these with the trains separate the system into 13 nonpermutable subdivisions. The group is generated by

$$s = (A)\ (B)\ (C)\ (a_1\ a_2\ a_3)\ (b_1\ b_2\ b_3)\ (c_1\ c_2\ c_3)\ (d_1\ d_2\ d_3)$$

and is of order 3.

TRAINS FOR THE SYSTEMS $V4\gamma2$.

Ten classes of trains terminating in triads of the system: (1) 3 trains, figure 161; (2) 3 trains, figure 71; (3) 3 trains, figure 8; (4) 3 trains, figure 3; (5) 3 trains, figure 68; (6) 3 trains, figure 168; (7) 3 trains, figure 158; (8) 3 trains, figure 155; (9) 1 train, figure 140; (10) 10 trains, figure 1.

Group for the system $V4\gamma2$.—The sets of transitive elements are A; B; C; a_1 a_2 a_3; b_1 b_2 b_3; c_1 c_2 c_2; d_1 d_2 d_3; these with the trains separate the system into 13 nonpermutable subdivisions. The group is generated by

$$s = (A)\ (B)\ (C)\ (a_1\ a_2\ a_3)\ (b_1\ b_2\ b_3)\ (c_1\ c_2\ c_3)\ (d_1\ d_2\ d_3)$$

and is of order 3.

Trains for the System V4δ1.

Nine classes of trains terminating in triads of the system: (1) 3 trains, figure 128; (2) 3 trains, figure 170; (3) 3 trains, figure 6; (4) 3 trains, figure 172; (5) 3 trains, figure 154; (6) 1 train, figure 66; (7) 3 trains, figure 147; (8) 3 trains, figure 137; (9) 13 trains, figure 1.

Group for the system V4δ1.—The sets of transitive elements are A; B; C; $a_1\ a_2\ a_3$; $b_1\ b_2\ b_3$; $c_1\ c_2\ c_3$; $d_1\ d_2\ d_3$; these with the trains separate the system into 13 nonpermutable subdivisions. The group is generated by

$$s = (A)\ (B)\ (C)\ (a_1\ a_2\ a_3)\ (b_1\ b_2\ b_3)\ (c_1\ c_2\ c_3)\ (d_1\ d_2\ d_3)$$

and is of order 3.

Trains for the System VI$1_3\beta$.

Nine classes of trains terminating in triads of the system: (1) 2 trains, figure 176; (2) 2 trains, figure 2; (3) 2 trains, figure 128; (4) 4 trains, figure 142; (5) 4 trains, figure 152; (6) 4 trains, figure 62; (7) 4 trains, figure 6; (8) 4 trains, figure 107; (9) 9 trains, figure 1.

One class of trains terminating in a cycle of period 4: (10) 1 train, figure 182.

Group for the system VI$1_3\beta$.—The sets of transitive elements are A; $B\ C$; $a_1\ b_1\ a_6\ b_6$; $a_2\ b_2\ a_3\ b_3$; $a_4\ b_4\ a_5\ b_5$; these with the trains separate the system into 11 nonpermutable subdivisions. The group is generated by

$$s = (A)\ (B\ C)\ (a_1\ a_6\ b_1\ b_6)\ (a_2\ b_3\ b_2\ a_3)\ (a_4\ b_5\ b_4\ a_5)$$

and is of order 4.

Trains for the System VI$2_4\alpha$.

Eleven classes of trains terminating in triads of the system: (1) One train, figure 105; (2) 1 train, figure 11; (3) 4 trains, figure 3; (4) 1 train, figure 149; (5) 2 trains, figure 8; (6) 4 trains, figure 2; (7) 2 trains, figure 181; (8) 2 trains, figure 6; (9) 1 train, figure 28; (10) 2 trains, figure 175; (11) 15 trains, figure 1.

One class of trains terminating in a cycle of period 24: (12) One train, figure 202.

Group for the System VI$2_4\alpha$.—The sets of transitive elements are A; B; C; $a_1\ b_1$; $a_2\ b_2$; $a_3\ b_3$; $a_4\ b_4$; $a_5\ b_5$; $a_6\ b_6$. These with the trains separate the system into 21 nonpermutable subdivisions. The group is generated by

$$s = (A)\ (B)\ (C)\ (a_1\ b_1)\ (a_2\ b_2)\ (a_3\ b_3)\ (a_4\ b_4)\ (a_5\ b_5)\ (a_6\ b_6),$$

and is of order 2.

Trains for the System VI$2_4\delta$.

Eighteen classes of trains terminating in triads of the system: (1) One train, figure 18; (2) 2 trains, figure 130; (3) 1 train, figure 132; (4) 2 trains, figure 133; (5) 2 trains, figure 153; (6) 1 train, figure 38; (7) 2 trains, figure 174; (8) 2 trains, figure 126; (9) 2 trains, figure 37; (10) 1 train, figure 8; (11) 2 trains, figure 62; (12) 2 trains, figure 120; (13) 3 trains, figure 2; (14) 2 trains, figure 6; (15) 2 trains, figure 156; (16) 1 train, figure 109; (17) 2 trains, figure 112; (18) 5 trains, figure 1.

One class of trains terminating in a cycle of period 4: (19) One train, figure 182.

Group for the System VI$2_4\delta$.—The sets of transitive elements are A; B; C; $a_1\ b_1$; $a_2\ b_2$; $a_3\ b_3$; $a_4\ b_4$; $a_5\ b_5$; $a_6\ b_6$. These with the trains separate the system into 21 nonpermutable subdivisions. The group is generated by

$$s = (A)\ (BC)\ (a_1\ b_1)\ (a_2\ b_2)\ (a_3\ b_3)\ (a_4\ b_4)\ (a_5\ b_5)\ (a_6\ b_6),$$

and is of order 2.

Trains for the System VI$2_4\epsilon$.

Fifteen classes of trains terminating in triads of the system: (1) One train, figure 144; (2) 1 train, figure 8; (3) 2 trains, figure 167; (4) 2 trains, figure 139; (5) 1 train, figure 165; (6) 2 trains, figure 157; (7) 4 trains, figure 2; (8) 2 trains, figure 62; (9) 1 train, figure 30; (10) 1 train, figure 6; (11) 2 trains, figure 115; (12) 1 train, figure 114; (13) 2 trains, figure 27; (14) 2 trains, figure 166; (15) 11 trains, figure 1.

Two classes of trains terminating in cycles of periods 10 and 4, respectively: (16) Two trains, figure 192; (17) 1 train, figure 182.

Group for the System $VI2_4\epsilon$.—The sets of transitive elements are A; B; C; $a_1\ b_1$; $a_2\ b_2$; $a_3\ b_3$ $a_4\ b_4$; $a_5\ b_5$; $a_6\ b_6$. These with the trains separate the system into 21 nonpermutable subdivisions. The group is generated by

$$s=(A)\ (B)\ (C)\ (a_1\ b_1)\ (a_2\ b_2)\ (a_3\ b_3)\ (a_4\ b_4)\ (a_5\ b_5)\ (a_6\ b_6),$$

and is of order 2.

TRAINS FOR THE SYSTEM $VI3_3\alpha$.

Seven classes of trains terminating in triads of the system: (1) Two trains, figure 143; (2) 2 trains, figure 110; (3) 4 trains, figure 2; (4) 4 trains, figure 61; (5) 4 trains, figure 91; (6) 4 trains, figure 136; (7) 15 trains, figure 1.

Two classes of trains terminating in cycles of periods 18 and 20, respectively: (8) Two trains, figure 193; (9) 1 train, figure 201.

Group for the System $VI3_3\alpha$.—The sets of transitive elements are A; $B\ C$; $a_1\ b_1\ a_2\ b_2$; $a_3\ b_3\ a_6\ b_6$; $a_4\ b_4\ a_5\ b_5$. These with the trains separate the system into 11 nonpermutable subdivisions. The group is generated by

$$s=(A)\ (B\ C)\ (a_1\ a_2\ b_1\ b_2)\ (a_3\ b_6\ a_6\ b_3)\ (a_4\ a_5\ b_4\ b_5),$$

and is of order 4.

TRAINS FOR THE SYSTEM $VI3_3\gamma$.

Six classes of trains terminating in triads of the system: (1) Four trains, figure 177; (2) 2 trains, figure 79; (3) 4 trains, figure 173; (4) 6 trains, figure 27; (5) 2 trains, figure 92; (6) 17 trains, figure 1.

One class of trains terminating in a cycle of period 4: (7) One train, figure 182.

Group for the System $VI3_3\gamma$.—The sets of transitive elements are A; $B\ C$; $a_1\ b_1\ a_2\ b_2$; $a_3\ b_3\ a_6\ b_6$; $a_4\ b_4\ a_5\ b_5$; these with the trains separate the system into 11 nonpermutable subdivisions. The group is generated by

$$s=(A)\ (B\ C)\ (a_1\ a_2\ b_1\ b_2)\ (a_3\ b_6\ b_3\ a_6)\ (a_4\ a_5\ b_4\ b_5),$$

and is of order 4.

TRAINS FOR THE SYSTEM $VI3_3\delta$.

Ten classes of trains terminating in triads of the system: (1) Two trains, figure 58; (2) 4 trains, figure 87; (3) 4 trains, figure 141; (4) 4 trains, figure 76; (5) 4 trains, figure 151; (6) 4 trains, figure 99; (7) 4 trains, figure 34; (8) 2 trains, figure 81; (9) 6 trains, figure 27; (10) 1 train, figure 1.

One class of trains terminating in cycle of period 4: (11) One train, figure 182.

Group for the System $VI3_3\delta$.—The sets of transitive elements are A; $B\ C$; $a_1\ b_1\ a_2\ b_2$; $a_3\ b_3\ a_6\ b_6$; $a_4\ b_4\ a_5\ b_5$; these with the trains separate the system into 11 nonpermutable subdivisions. The group is generated by

$$s=(A)\ (B\ C)\ (a_1\ a_2\ b_1\ b_2)\ (a_3\ b_6\ b_3\ a_6)\ (a_4\ a_5\ b_4\ b_5),$$

and is of order 4.

The trains show that 20 of the 71 systems obtained in Part 1 are new systems and the remaining 51 systems are each congruent to some one of the 44 systems thus far derived. The substitution which transforms each of these 51 systems into its congruent system is given below

I, 1≡VII by $s\equiv\begin{pmatrix} a & b & c & d & e & 1 & 2 & 3 & 4 & 5 & \alpha & \beta & \gamma & \delta & \epsilon \\ 4 & 7 & b & e & 1 & a & d & g & 3 & 6 & f & 2 & 5 & 8 & c \end{pmatrix}$

I, 2 a new system.

I≡IIIA by $s\equiv\begin{pmatrix} a & b & c & d & e & 1 & 2 & 3 & 4 & 5 & \alpha & \beta & \gamma & \delta & \epsilon \\ a & g & 5 & 8 & e & c & 6 & 1 & 2 & 7 & b & 4 & d & f & 3 \end{pmatrix}$

II, $1_1 \equiv$ VA by $s \equiv \begin{pmatrix} a_1 & a_2 & a_3 & b_1 & b_2 & b_3 & c_1 & c_2 & c_3 & d_1 & d_2 & d_3 & e_1 & e_2 & e_3 \\ d & 1 & 4 & a & b & c & e & 5 & 7 & f & 8 & 3 & g & 2 & 6 \end{pmatrix}$

II, 1_2 new system; III_3 a new system.

II, $5 \equiv$ IIA by $s \equiv \begin{pmatrix} a_1 & a_2 & a_3 & b_1 & b_2 & b_3 & c_1 & c_2 & c_3 & d_1 & d_2 & d_3 & e_1 & e_2 & e_3 \\ d & 4 & 8 & a & b & c & f & 2 & 5 & c & 1 & 7 & g & 3 & 6 \end{pmatrix}$

II, 1_6 new system; III_7 new system.

II, $2 \equiv$ VII by $s \equiv \begin{pmatrix} a_1 & a_2 & a_3 & b_1 & b_2 & b_3 & c_1 & c_2 & c_3 & d_1 & d_2 & d_3 & e_1 & e_2 & e_3 \\ a & 4 & f & d & 7 & 2 & c & 8 & 3 & 6 & 1 & c & b & 5 & g \end{pmatrix}$

III, $1 \equiv$ IIIA by $s \equiv \begin{pmatrix} A & a & b & c & d & e & f & g & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 5 & a & b & d & c & g & f & e & 6 & 8 & 1 & 7 & 3 & 4 & 2 \end{pmatrix}$

III, $2 \equiv$ IIID by $s \equiv \begin{pmatrix} A & a & b & c & d & e & f & g & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & a & b & e & d & g & c & f & 2 & 3 & 6 & 5 & 8 & 4 & 7 \end{pmatrix}$

III, $3 \equiv$ IA by $s \equiv \begin{pmatrix} A & a & b & c & d & e & f & g & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & a & b & c & d & e & f & g & 4 & 7 & 3 & 6 & 2 & 5 & 8 \end{pmatrix}$

V, $1\alpha \equiv$ VA by $s \equiv \begin{pmatrix} A & B & C & a_1 & a_2 & a_3 & b_1 & b_2 & b_3 & c_1 & c_2 & c_3 & d_1 & d_2 & d_3 \\ a & b & c & d & 1 & 4 & e & 2 & 3 & f & 5 & 6 & g & 8 & 7 \end{pmatrix}$

V, 1γ a new system.

V, $1\delta \equiv$ I, 2 by $s \equiv \begin{pmatrix} A & B & C & a_1 & a_2 & a_3 & b_1 & b_2 & b_3 & c_1 & c_2 & c_3 & d_1 & d_2 & d_3 \\ a_1 & a_2 & a_3 & d_1 & c_1 & e_1 & b_2 & b_1 & b_3 & d_3 & c_3 & e_3 & d_2 & c_2 & e_2 \end{pmatrix}$

V, $2\alpha \equiv$ IIIB by $s \equiv \begin{pmatrix} A & B & C & a_1 & a_2 & a_3 & b_1 & b_2 & b_3 & c_1 & c_2 & c_3 & d_1 & d_2 & d_3 \\ a & 1 & 2 & c & f & d & 3 & 8 & 6 & 5 & 4 & 7 & b & g & e \end{pmatrix}$

V, $2\beta \equiv$ VD by $s \equiv \begin{pmatrix} A & B & C & a_1 & a_2 & a_3 & b_1 & b_2 & b_3 & c_1 & c_2 & c_3 & d_1 & d_2 & d_3 \\ e & 8 & 6 & c & a & b & 3 & 7 & 4 & 2 & 1 & 5 & d & g & f \end{pmatrix}$

V, $2\gamma \equiv$ IIID by $s \equiv \begin{pmatrix} A & B & C & a_1 & a_2 & a_3 & b_1 & b_2 & b_3 & c_1 & c_2 & c_3 & d_1 & d_2 & d_3 \\ d & 1 & 5 & f & c & b & 8 & 4 & 2 & 3 & 7 & 6 & g & e & a \end{pmatrix}$

V, $2\delta \equiv$ VB by $s \equiv \begin{pmatrix} A & B & C & a_1 & a_2 & a_3 & b_1 & b_2 & b_3 & c_1 & c_2 & c_3 & d_1 & d_2 & d_3 \\ g & 5 & 3 & c & a & b & 7 & 6 & 4 & 8 & 1 & 2 & d & f & e \end{pmatrix}$

V, $3\alpha \equiv$ V, 1α by $s \equiv \begin{pmatrix} A & B & C & a_1 & a_2 & a_3 & b_1 & b_2 & b_3 & c_1 & c_2 & c_3 & d_1 & d_2 & d_3 \\ a_3 & c_2 & b_1 & b_3 & c_3 & d_3 & d_2 & a_2 & b_2 & a_1 & d_1 & c_1 & A & B & C \end{pmatrix}$

V, $3\beta \equiv$ IIIC by $s \equiv \begin{pmatrix} A & B & C & a_1 & a_2 & a_3 & b_1 & b_2 & b_3 & c_1 & c_2 & c_3 & d_1 & d_2 & d_3 \\ 6 & 1 & e & 5 & 8 & 7 & 2 & 4 & 3 & f & d & g & a & c & b \end{pmatrix}$

V, $3\gamma \equiv$ IIIA by $s \equiv \begin{pmatrix} A & B & C & a_1 & a_2 & a_3 & b_1 & b_2 & b_3 & c_1 & c_2 & c_3 & d_1 & d_2 & d_3 \\ d & 1 & 5 & c & f & g & 2 & 4 & 3 & 6 & 7 & 8 & a & c & b \end{pmatrix}$

V, $4\alpha_1$, a new system; V, $4\beta_1$, a new system.

V, $4\beta_2$, a new system; V, $4\gamma_1$, a new system.

V, $4\gamma_2$, a new system; V, $4\delta_1$, a new system.

V, $5\alpha_1 \equiv$ IC by $s \equiv \begin{pmatrix} A & B & C & a_1 & a_2 & a_3 & b_1 & b_2 & b_3 & c_1 & c_2 & c_3 & d_1 & d_2 & d_3 \\ 1 & 8 & g & 5 & 3 & 2 & f & c & e & 4 & 6 & 7 & a & d & b \end{pmatrix}$

V, $5\beta_1 \equiv$ IB by $s \equiv \begin{pmatrix} A & B & C & a_1 & a_2 & a_3 & b_1 & b_2 & b_3 & c_1 & c_2 & c_3 & d_1 & d_2 & d_3 \\ 4 & 1 & a & 3 & 2 & 7 & b & c & e & 8 & 5 & 6 & d & g & b \end{pmatrix}$

V, $5\beta_2 \equiv$ IA by $s \equiv \begin{pmatrix} A & B & C & a_1 & a_2 & a_3 & b_1 & b_2 & b_3 & c_1 & c_2 & c_3 & d_1 & d_2 & d_3 \\ 1 & 8 & g & 3 & 2 & 5 & c & e & f & 4 & 6 & 7 & a & d & b \end{pmatrix}$

V, $5\gamma_1 \equiv$ VID by $s \equiv \begin{pmatrix} A & B & C & a_1 & a_2 & a_3 & b_1 & b_2 & b_3 & c_1 & c_2 & c_3 & d_1 & d_2 & d_3 \\ 2 & 6 & g & 3 & 8 & 5 & b & d & f & 1 & 7 & 4 & a & e & c \end{pmatrix}$

V, $5\gamma_2 \equiv$ VIC by $s \equiv \begin{pmatrix} A & B & C & a_1 & a_2 & a_3 & b_1 & b_2 & b_3 & c_1 & c_2 & c_3 & d_1 & d_2 & d_3 \\ 5 & 4 & g & 8 & 6 & 3 & a & b & c & 7 & 2 & 1 & c & f & d \end{pmatrix}$

V, $5\delta_1 \equiv$ VIA by $s \equiv \begin{pmatrix} A & B & C & a_1 & a_2 & a_3 & b_1 & b_2 & b_3 & c_1 & c_2 & c_3 & d_1 & d_2 & d_3 \\ 4 & 5 & g & 7 & 2 & 1 & d & c & f & 3 & 8 & 6 & a & b & c \end{pmatrix}$

VI, $1_1\alpha \equiv$ VA by $s \equiv \begin{pmatrix} A & B & C & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \\ a & c & b & d & g & 1 & 8 & 4 & 7 & e & f & 2 & 5 & 3 & 6 \end{pmatrix}$

VI, $1_1\beta \equiv$ IVA by $s \equiv \begin{pmatrix} A & B & C & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \\ c & a & b & d & g & 1 & 2 & 5 & 6 & e & f & 3 & 4 & 8 & 7 \end{pmatrix}$

VI, $1_2 \equiv$ IIF by $s \equiv \begin{pmatrix} A & B & C & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \\ f & a & d & 4 & 3 & 2 & 1 & c & e & 5 & 6 & 8 & 7 & b & g \end{pmatrix}$

VI, $1_3\alpha \equiv$ V, 1γ by $s \equiv \begin{pmatrix} A & B & C & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \\ A & B & C & b_1 & d_1 & b_2 & d_2 & b_3 & d_3 & a_1 & c_1 & a_2 & c_2 & a_3 & c_3 \end{pmatrix}$

VI, $1_3\beta$, a new system.

VI, $2_1\alpha \equiv$ IVA by $s \equiv \begin{pmatrix} A & B & C & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \\ b & a & c & d & g & 3 & 4 & 7 & 5 & f & c & 2 & 1 & 8 & 6 \end{pmatrix}$

VI, $2_1\beta \equiv$ IIIC by $s \equiv \begin{pmatrix} A & B & C & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \\ b & a & c & 5 & 6 & d & g & 1 & 3 & 8 & 7 & f & e & 2 & 4 \end{pmatrix}$

VI, $2_1\delta \equiv$ IIIB by $s \equiv \begin{pmatrix} A & B & C & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \\ b & a & c & e & d & 1 & 2 & 6 & 7 & g & f & 3 & 4 & 5 & 8 \end{pmatrix}$

VI, $2\ \alpha \equiv$ IIA by $s \equiv \begin{pmatrix} A & B & C & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \\ f & a & d & 5 & 6 & 8 & 7 & b & g & 4 & 3 & 2 & 1 & c & e \end{pmatrix}$

VI, $2_3\alpha \equiv$ VC by $s \equiv \begin{pmatrix} A & B & C & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \\ g & f & a & d & c & 6 & 5 & 2 & 4 & e & b & 1 & 3 & 7 & 8 \end{pmatrix}$

VI, $2_4\alpha$, a new system.

VI, $2_4\gamma \equiv$ II, 1_6 by $s \equiv \begin{pmatrix} A & B & C & a_1 & b_1 & a_2 & b_2 & a_3 & b_3 & a_4 & b_4 & a_5 & b_5 & a_6 & b_6 \\ a_2 & b_2 & c_2 & a_3 & a_1 & c_2 & d_2 & c_3 & d_1 & c_3 & c_1 & b_3 & b_1 & d_3 & c_1 \end{pmatrix}$

VI, $2_4\gamma$, a new system; VI, $2_4\epsilon$, a new system.

VI, $3_2 \equiv$ VIA by $s \equiv \begin{pmatrix} A & B & C & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \\ f & c & g & 8 & 1 & 2 & 5 & a & b & 6 & 4 & 7 & 3 & d & c \end{pmatrix}$

VI, $3_3\alpha$, a new system; VI, $3_3\gamma$, a new system.

VI, $3_3\delta$, a new system; VI, $3_4\gamma$, a new system.

VI, $3_4\alpha \equiv$ VII by $s \equiv \begin{pmatrix} A & B & C & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \\ b & 5 & g & d & a & 4 & 7 & f & 2 & 3 & 6 & 1 & e & c & 8 \end{pmatrix}$

VII, $1_1 \equiv$ IIIA by $s \equiv \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & A & a & B & b & C & c & D & d \\ a & b & c & d & e & g & f & 1 & 2 & 3 & 4 & 5 & 6 & 8 & 7 \end{pmatrix}$

VII, $1_2 \equiv$ IIIB by $s \equiv \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & A & a & B & b & C & c & D & d \\ a & b & c & d & c & g & f & 1 & 2 & 3 & 4 & 5 & 6 & 8 & 7 \end{pmatrix}$

VII, $1_4 \equiv$ IIIC by $s \equiv \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & A & a & B & b & C & c & D & d \\ a & b & c & d & e & g & f & 1 & 2 & 3 & 4 & 5 & 6 & 8 & 7 \end{pmatrix}$

VII, $1_6 \equiv$ IIID by $s \equiv \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & A & a & B & b & C & c & D & d \\ a & b & c & d & e & g & f & 1 & 2 & 3 & 4 & 5 & 6 & 8 & 7 \end{pmatrix}$

VII, $2_3 \equiv$ IIE by $s \equiv \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & A & a & B & b & C & c & D & d \\ c & a & b & 4 & 2 & 1 & 3 & 8 & 5 & 7 & 6 & d & f & g & e \end{pmatrix}$

VII, $2_5 \equiv$ IVA by $s \equiv \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & A & a & B & b & C & c & D & d \\ a & d & e & f & g & c & b & 3 & 4 & 7 & 8 & 5 & 6 & 1 & 2 \end{pmatrix}$

VII, $2_7 \equiv$ IVB by $s \equiv \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & A & a & B & b & C & c & D & d \\ a & d & e & g & f & b & c & 2 & 1 & 6 & 5 & 7 & 8 & 3 & 4 \end{pmatrix}$

VII, $4_1 \equiv$ IIA by $s \equiv \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & A & a & B & b & C & c & D & d \\ b & c & a & c & g & d & f & 3 & 1 & 2 & 4 & 8 & 5 & 7 & 6 \end{pmatrix}$

VII, $4_3 \equiv$ IIF by $s \equiv \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & A & a & B & b & C & c & D & d \\ b & c & a & f & d & g & e & 2 & 4 & 3 & 1 & 8 & 5 & 6 & 7 \end{pmatrix}$

VII, $4_4 \equiv$ IIB by $s \equiv \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & A & a & B & b & C & c & D & d \\ b & c & a & f & d & g & e & 2 & 4 & 3 & 1 & 8 & 5 & 6 & 7 \end{pmatrix}$

VII, $4_5 \equiv$ IID by $s \equiv \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & A & a & B & b & C & c & D & d \\ c & b & a & d & f & g & e & 8 & 5 & 6 & 7 & 4 & 2 & 1 & 3 \end{pmatrix}$

VII, $4_6 \equiv$ IIC by $s \equiv \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & A & a & B & b & C & c & D & d \\ b & c & a & f & d & e & g & 3 & 1 & 2 & 4 & 6 & 7 & 8 & 5 \end{pmatrix}$

VII, $5_2 \equiv$ VA by $s \equiv \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & A & a & B & b & C & c & D & d \\ a & b & c & d & g & f & c & 8 & 5 & 2 & 1 & 3 & 7 & 4 & 6 \end{pmatrix}$

VII, $5_5 \equiv$ VC by $s \equiv \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & A & a & B & b & C & c & D & d \\ b & c & a & g & f & d & e & 1 & 5 & 8 & 2 & 6 & 3 & 4 & 7 \end{pmatrix}$

VII, $5_6 \equiv$ VB by $s \equiv \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & A & a & B & b & C & c & D & d \\ a & b & c & g & d & e & f & 1 & 2 & 5 & 8 & 6 & 4 & 7 & 3 \end{pmatrix}$

VII, $5_7 \equiv$ VD by $s \equiv \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & A & a & B & b & C & c & D & d \\ c & b & a & f & g & d & c & 7 & 4 & 3 & 6 & 2 & 8 & 5 & 1 \end{pmatrix}$

TRAINS FOR TRIAD SYSTEMS ON 15 ELEMENTS WHOSE GROUP IS OF ORDER UNITY.

The trains for each of the 36 noncongruent groupless systems on 15 elements have been determined. These 36 systems furnish 449 distinct types, different from the trains of the systems with a group. Among these appear trains terminating in polygoms of 4, 6, 11, 12, 13, and 14 sides, respectively.

Hence the 80 noncongruent systems applied as transformers to the 455 triads on 15 elements, produce 665 distinct covariants or trains.

PLATE I; SYSTEM VI 3 4γ.

Fig. 1.

Fig. 2

Fig. 6.

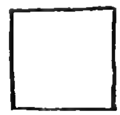


Fig. 182.

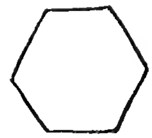


Fig. 183.

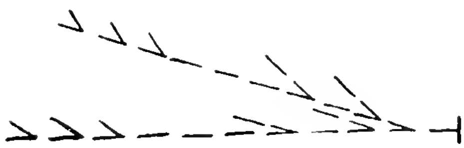


Fig. 210.

Fig. 211.

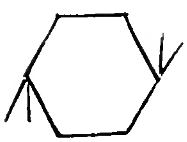


Fig. 213.

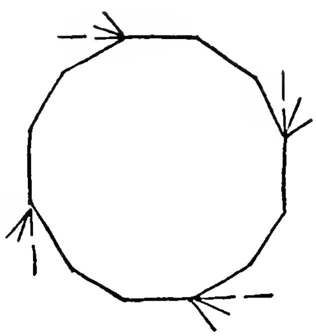


Fig. 214.

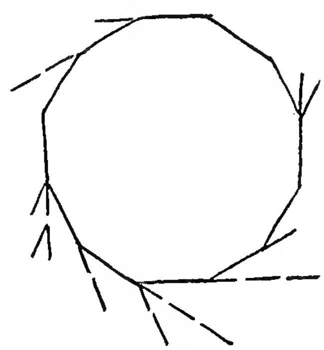


Fig. 215.

Fig. 206.

PLATE II; SYSTEM V 4 αl.

FIG. 1

FIG. 2

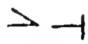


FIG. 3.

FIG. 6.

FIG. 205.

FIG. 207.

FIG. 66.

FIG. 208.

FIG 209

FIG. 212.

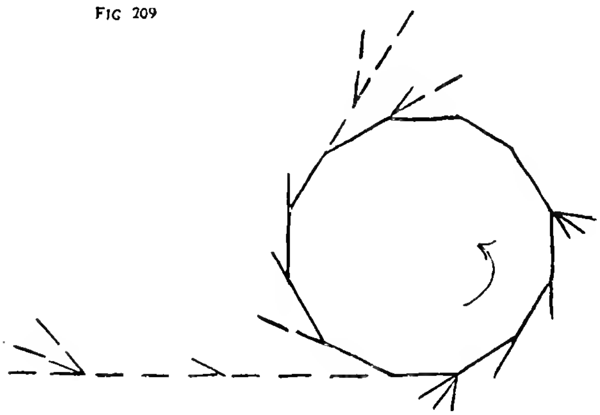


FIG 216.

Fig. 1.

Fig. 2.

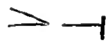


Fig. 3.

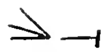


Fig. 4.

Fig. 5.

Fig 6.

Fig. 7.

Fig. 8.

Fig. 9.

Fig 10.

Fig. 11.

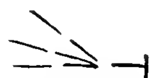


Fig. 12.

Fig. 13.

Fig. 14.

Fig. 15.

Fig. 16.

Fig. 17.

Fig. 18.

Fig. 19.

Fig. 20.

Fig. 21.

Fig. 22.

Fig. 23.

Fig. 24.

Fig. 25.

Fig. 26.

Fig. 27.

Fig. 28.

Fig. 29.

Fig. 30.

Fig. 31.

Fig. 32.

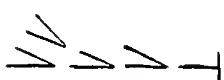


Fig. 33.

Fig. 34.

FIG. 35.

FIG. 36.

FIG. 37.

FIG. 38.

FIG. 39.

FIG. 40.

FIG. 41.

FIG. 42.

FIG. 43.

FIG. 44.

FIG. 45.

FIG. 46.

Fig. 47.

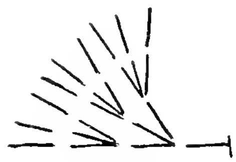


Fig. 48.

Fig. 49.

Fig. 50.

Fig. 51.

Fig. 52.

Fig. 53.

Fig. 54.

Fig. 55.

Fig. 56.

Fig. 57.

Fig. 58.

Fig. 59.

Fig. 60.

Fig. 61.

Fig. 62.

Fig. 63.

Fig. 64.

Fig. 65.

Fig. 66.

Fig. 67.

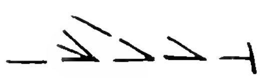


Fig. 68.

Fig. 69.

Fig. 70.


FIG. 71.


FIG. 72.


FIG. 73.


FIG. 74.


FIG. 75.


FIG. 76.

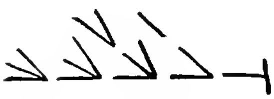


FIG. 77.

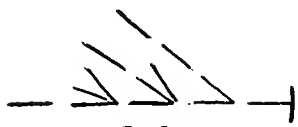


FIG. 78.

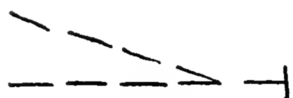


FIG. 79.


FIG. 80.


FIG. 81.


FIG. 82.

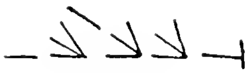


Fig. 83.

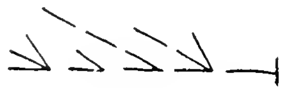


Fig. 84.

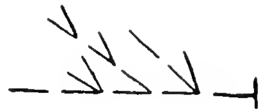


Fig. 85.


Fig. 86.

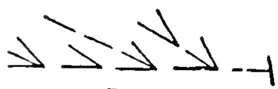


Fig. 87.


Fig. 88.

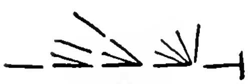


Fig. 89.


Fig. 90.

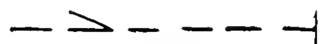


Fig. 91

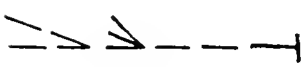


Fig. 92.

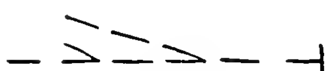


Fig. 93.


Fig. 94.

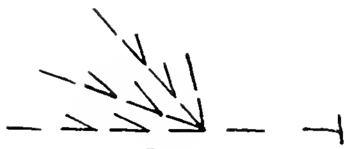


FIG. 95.

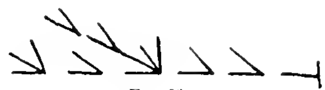


FIG. 96.

FIG. 97.

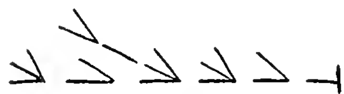


FIG. 98.

FIG. 99.

FIG. 100.

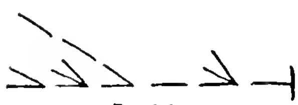


FIG. 101.

FIG. 102.

Fig. 103.

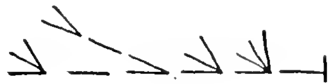


Fig. 104.

Fig. 105.

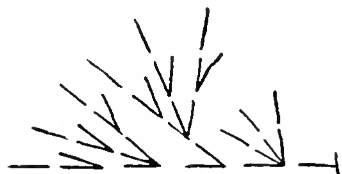


Fig. 106.

Fig. 107.

Fig. 108.

Fig. 109.

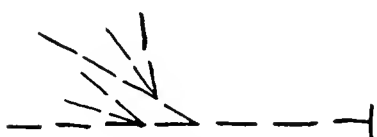


Fig. 110.

54061°—19——4

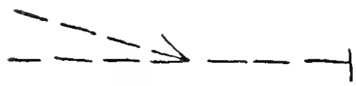


Fig. 111.

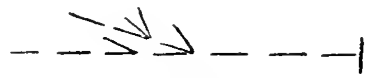


Fig. 112.

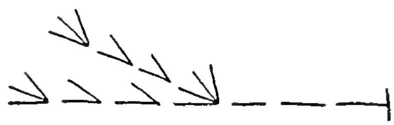


Fig. 113.

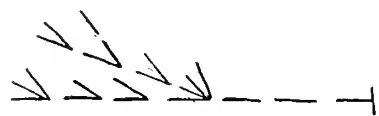


Fig. 114.

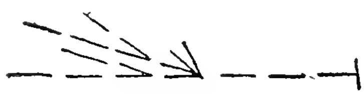


Fig. 115.

Fig. 116.

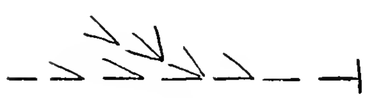


Fig. 117

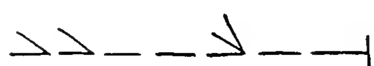


Fig. 118.

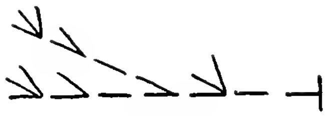


FIG. 119.

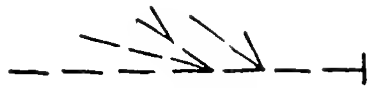
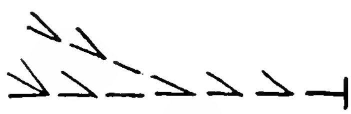


FIG. 120.

FIG. 121.

FIG. 122.

FIG. 123.



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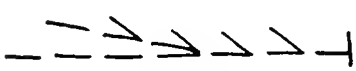

FIG. 125.

FIG. 126.


FIG. 127.

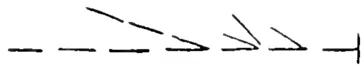
FIG. 128.

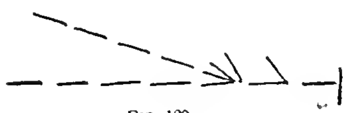
FIG. 129


FIG. 130.

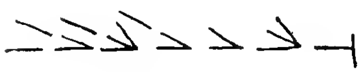
FIG. 131.

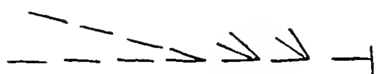
FIG. 132.

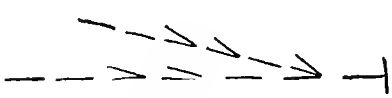
FIG. 133.

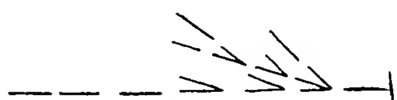
FIG. 134.

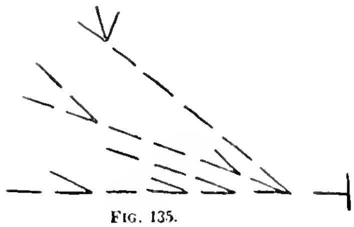

FIG. 135.

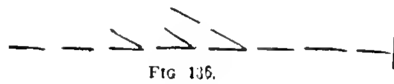

FIG. 136.

FIG. 137.

FIG. 138.

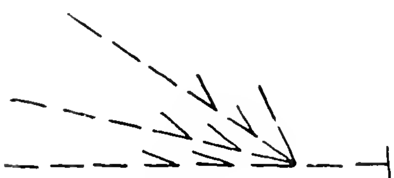

FIG. 139.

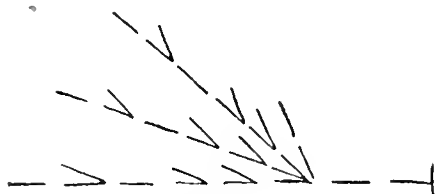

FIG. 140

FIG. 141.

FIG. 142.

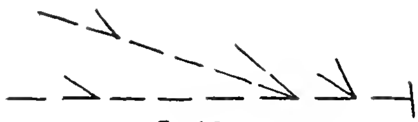

FIG. 143.

FIG. 144.

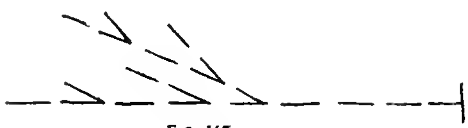

FIG. 145.

FIG. 146.

FIG. 147.

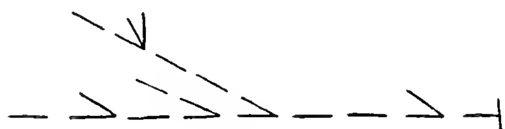

FIG. 148.

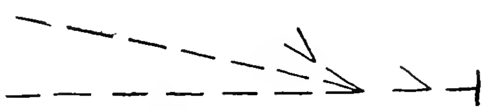

FIG. 149.

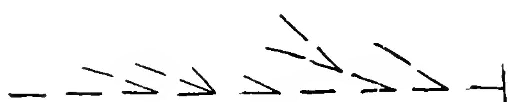

FIG. 150.

Fig. 151.

Fig. 152.

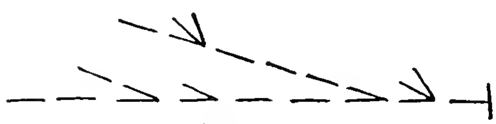

Fig. 153.

Fig. 154.

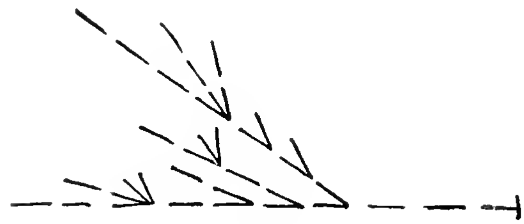

Fig. 155.

Fig. 156.

Fig. 157.

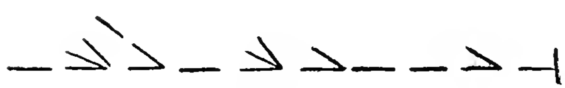

Fig. 158.

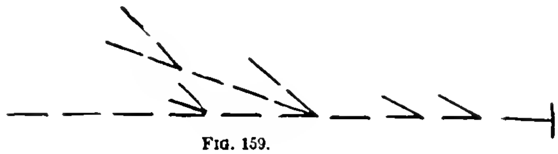

Fig. 159.

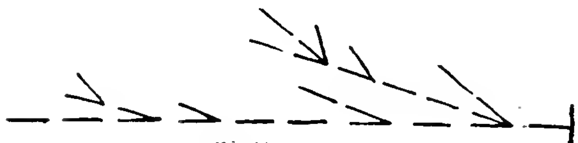

Fig. 160.

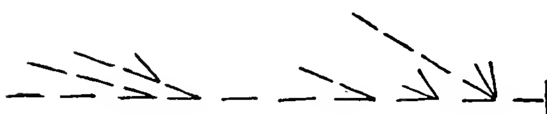

Fig. 161.

Fig. 162.

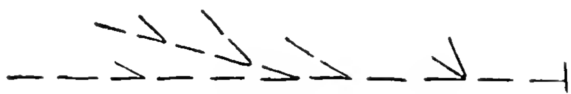

Fig. 163.

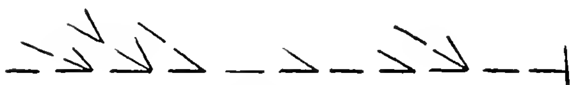

Fig. 164.

Fig. 165.

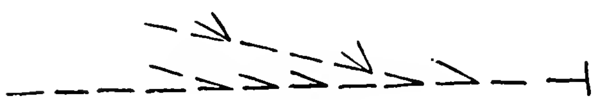

Fig. 166.

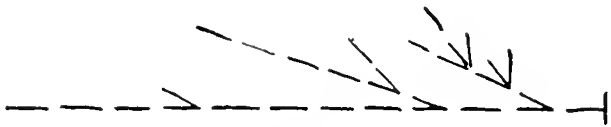

FIG. 167.

FIG. 168.

FIG. 169.

FIG. 170.

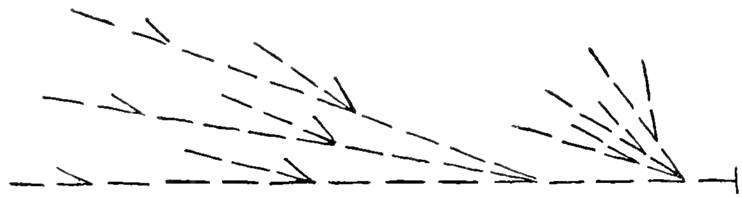

FIG. 171.

FIG. 172.

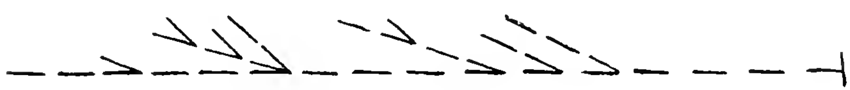

FIG. 173.

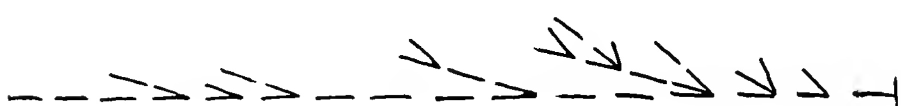

FIG. 174.

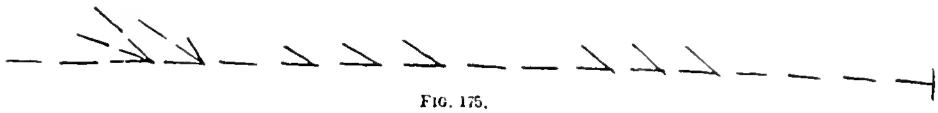

FIG. 175.

FIG. 176.

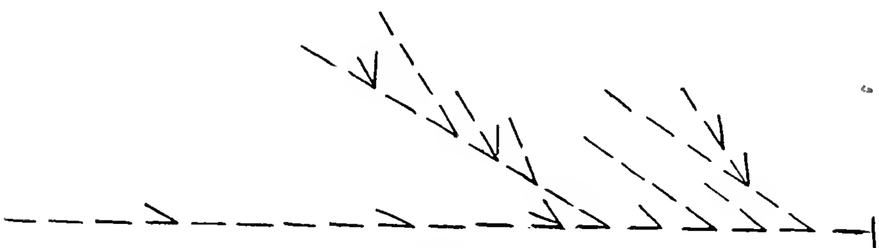

FIG. 177.

FIG. 178.

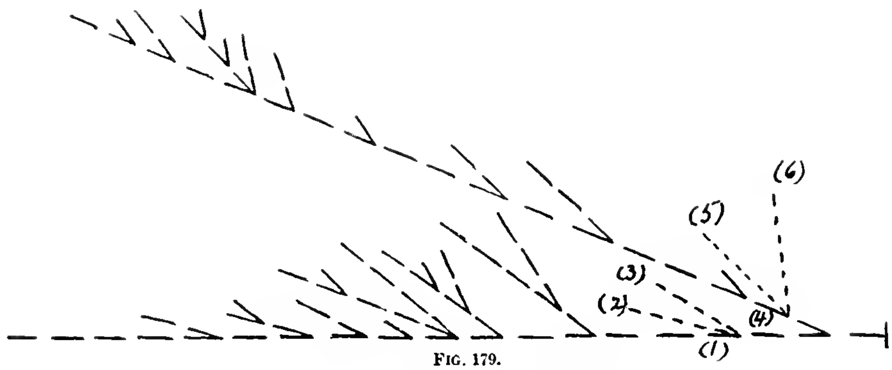

FIG. 179.
(1), (2), (3) similar; (4), (5), (6) similar.

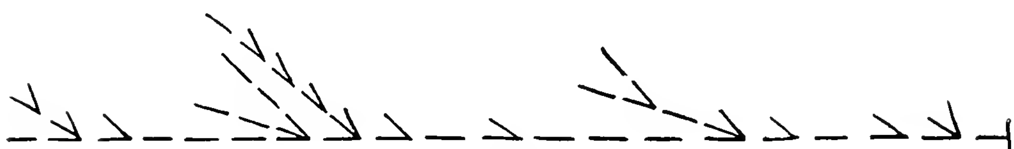

FIG. 180.

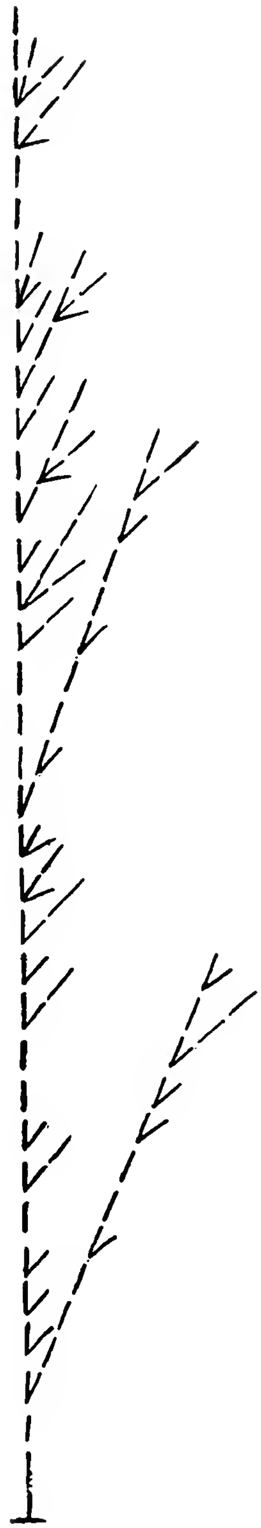

FIG. 181.

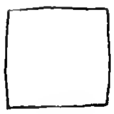

Fig. 182.

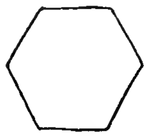

Fig. 183.

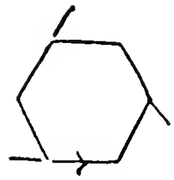

Fig. 184.

Fig. 185; ½ hexagon.

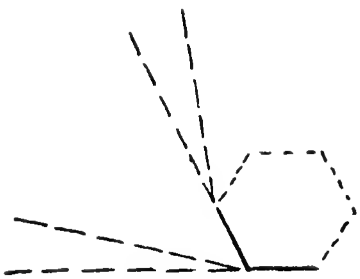

Fig. 186, ½ hexagon.

Fig. 187; ½ hexagon.

Fig. 188; ½ hexagon.

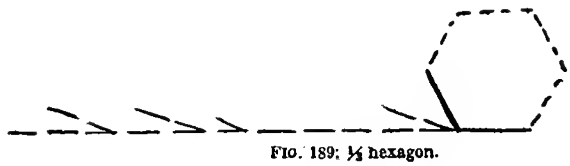

Fig. 189; ½ hexagon.

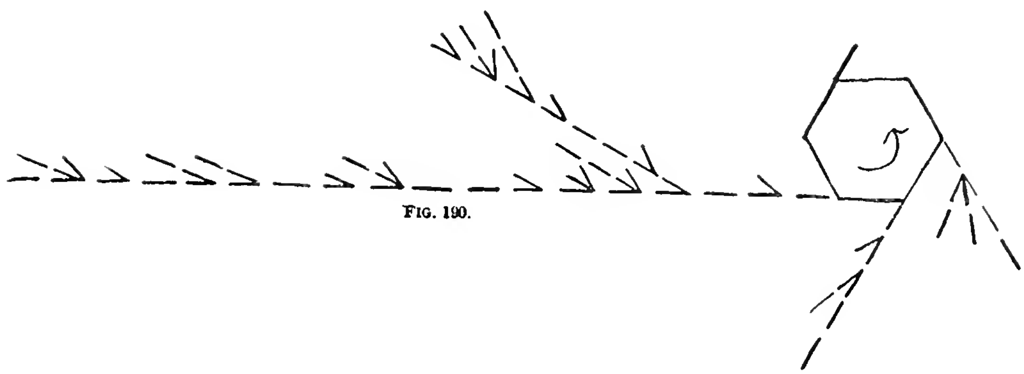

FIG. 190.

FIG. 191: ⅓ polygon of nine sides.

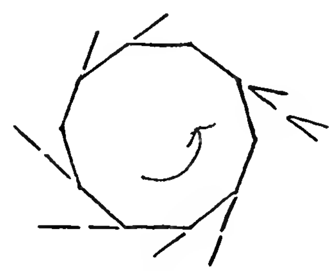

FIG. 192.

Fig. 193: ½ polygon of eighteen sides.

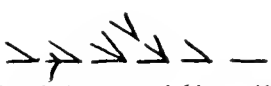

Fig. 194: ⅓ polygon of eighteen sides.

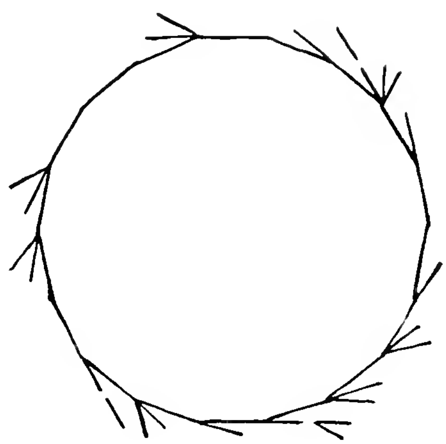

Fig. 195.

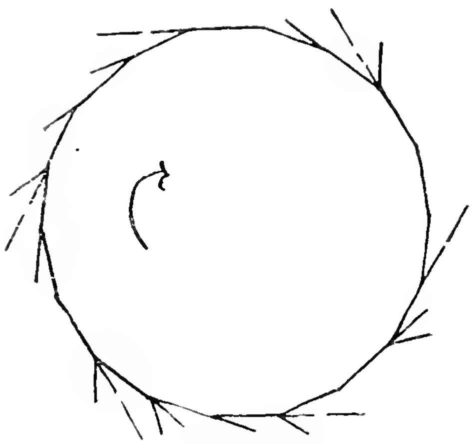

FIG. 196.

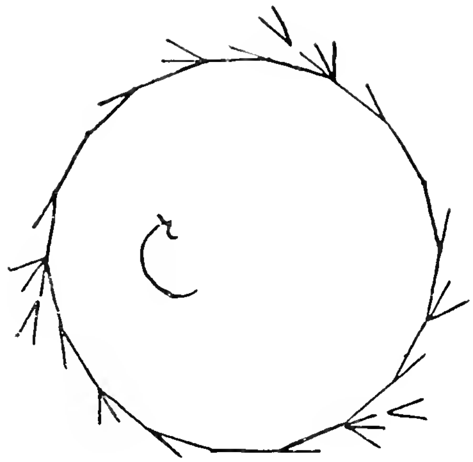

FIG. 197.

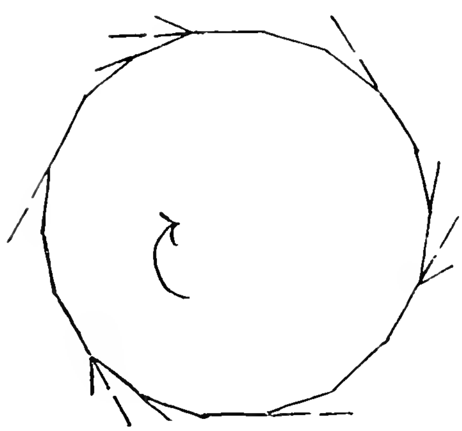

FIG. 198.

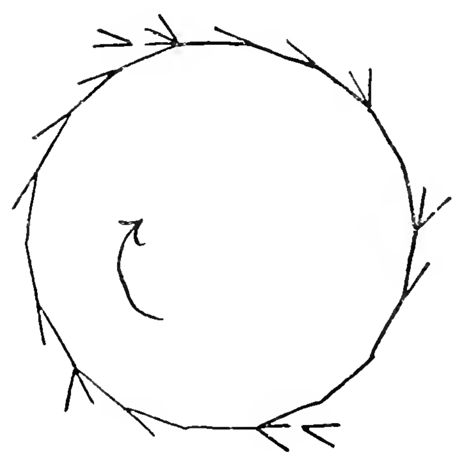

FIG. 199.

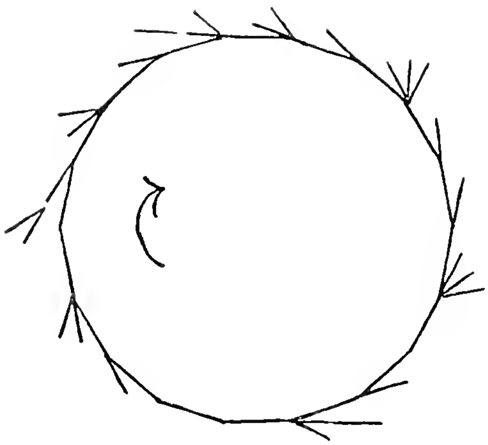

FIG. 200.

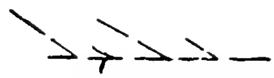

FIG 201; ¼ polygon of twenty sides

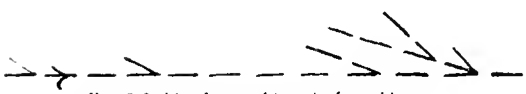

FIG 202; ⅓ polygon of twenty-four sides.

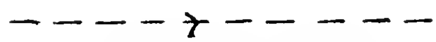

FIG. 203 ⅓ polygon of thirty sides.

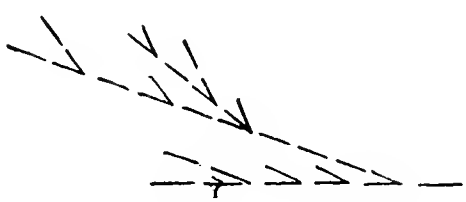

FIG. 204; $\frac{1}{12}$ polygon of seventy-two sides.

PART 3.

GROUPLESS TRIAD SYSTEMS ON 15 ELEMENTS.

By H. S. White and L. D. Cummings.

All noncongruent systems, $\triangle_{15}$, with a group having been determined in Part 1, there arises next the question concerning the possible existence of triad systems on 15 elements with the group identity. Systems whose group is identity, or groupless systems, do not exist for 7, 9, or 13 elements. In a paper [1] already published Mr. White has proved the existence of many groupless systems on 31 elements. An investigation given below in some detail has led to the discovery of a considerable number of noncongruent systems on 15 elements with the group identity. Every groupless system on 31 elements whose existence has thus far been demonstrated contains one or more systems $\triangle_{15}$ and, therefore, is a headed system. On the contrary, every groupless system on 15 elements is headless.

In any triad system the pairs of elements are more or less interconnected or interlaced. These interlacings may be determined by applying to the system under consideration a modified form of the method [2] of examination by sequences and indices. The $\triangle_{15}$ is exhibited in a 15 by 7 array. Each element heads one column; below it are placed the seven dyads, which, with the element at the head, constitute the triads of the system. Heretofore sequences and indices have been derived from the three columns of a triad in any $\triangle_{15}$; the same process is now applied to every pair of columns and yields what may be called the two-column or contracted indices for the system. Since the number of combinations of 15 columns, two at a time, is 105, this number of pairs of columns must be examined unless the group for the system is known and is different from identity. If the group contains an operator of order m, then in general m pairs of columns are examined simultaneously. The process may be illustrated in its application to a system $VI1_3\beta$, with a group of order 4 generated by $t=(a)\ (bc)\ (d8e7)\ (f3g4)\ (1536)$. Pairs of columns selected from the following table show every type of two-column or contracted index that can occur in any system.

a	b	d	1
de	df	ae	$a2$
fg	eg	bf	$b3$
12	ac	$c2$	ce
34	13	$g8$	$d5$
56	24	15	$f8$
78	57	37	$g4$
bc	68	46	67

Pairs of columns.	Index.	Contracted sequences.
ab	2^3	$de/gf/d$, 12/43/1, 56/87/5;
$b1$	3^2	$df/86/75/d$, $24/ge/ca/2$;
da	6	$c2/15/64/37/8g/fb/c$;
db	2, 4	15/73/1, $ae/g8/64/2c/a$.

The substitution t applied to the pair of columns ab gives the pair ac with the same index and similar sequences. If t is applied to the pair of columns $b1$, the three pairs $c5$, $b2$, $c6$ are obtained with index 3^2. The analysis of the 105 pairs of columns shows that the contracted indices 2^3; 2, 4; 3^2; 6 belong, respectively, to 2, 24, 4, and 75 pairs of columns.

[1] White, H. S.: Transactions of the American Mathematical Society, vol. 14 (1913), pp. 13–19.
[2] Cummings, L. D.: Transactions of the American Mathematical Society, vol. 15 (1914), pp. 311–327.

The new groupless systems are formed by interchanging duads of one column with those of another column. For example, in the pair of columns *ab*, the duads *de*, *fg*, of column *a*, may be exchanged with *df*, *eg*, of column *b*; such an interchange involving four elements, contained only in two pairs in each column, shall be designated as a *quadrangular transformation*. The columns *d*, *e*, *f*, *g* must now be rewritten in agreement with the new triads introduced into the system, and the undisturbed nine columns of $VI1_3\beta$ with the reconstructed six columns form a new system 13. The four duads 12, 34, 56, 78 of column *a* might be interchanged with the four duads on the same elements in column *b*, forming an octagonal transformation, but this is equivalent to the above quadrangular transformation followed by the interchange of the elements *a* and *b*.

In the pair of columns *b*1 the duads *df*, 68, 57 of column *b* may be interchanged with the duads *f*8, 67, *d*5 of column 1; such an interchange, involving six elements appearing exclusively in three pairs in each of two columns, shall be designated as a *hexagonal transformation*. New columns *d*, *f*, 6, 8, 5, 7 must next be constructed; these eight reconstructed columns, with the seven undisturbed columns of $VI1_3\beta$, form a system 4. The application of the second hexagonal transformation in *b*1 is equivalent to an application of the first hexagonal transformation followed by an interchange of the elements *b* and 1.

A transformation on 12 elements simply interchanges the two elements which head the columns. Therefore only the quadrangular and the hexagonal transformations which exist in a system require consideration.

By means of the operators of the group of the system, the maximum number of noncongruent transformations of each of the above types is determined—for example, in $VI1_3\beta$ the eight hexagonal transformations reduce to one, and the 30 quadrangular transformations to four noncongruent transformations.

Each of the noncongruent transformations is now applied to the system $VI1_3\beta$, and the sequences and indices are determined for the five transformed systems.

The 35 triads of the system 4, arranged in classes according to their indices, are shown in the following table:

$1^4 26$	$1^4 35$	$1^3 2^2 5$	$1^3 9$	$1^2 2^3 4$	$1^2 28$	$1^2 37$	$1^2 46$	$1^2 5^2$
157	*a*12 258	*a*78 *a*56 *cg*5	168	*a*34	*g*14 *c*47 *dg*8	*cd*2	*b*24	*b*13
$12^2 7$	1236	1245	1, 11	$2^2 4^2$	$23^2 4$	5, 7	6^2	
ade *d*46	*afg* *g*27	*e*26 *f*45 *cf*6 *c*38 *d*37	*bd*5 *bf*8 *b*67 *bcg* *ce*1	*e*48 *df*1 *g*36	*cf*7	*e*35 *f*23	*abc*	

The enumeration of the elements in the 17 classes shows that the sets of transitive elements are *a*; *b*; *c*; *d*; *e*; *f*; *g*; 1; 2; 3; 4; 5; 6; 7; 8; hence the group for the system is identity. Therefore under this hexagonal transformation the system $VI1_3\beta$, with a group of order 4, is changed into a system 4 with the group identity. The four quadrangular transformations applied to $VI1_3\beta$ yield four noncongruent systems. One of these is a new groupless system 13; the remaining three are the known systems $VI2_4\epsilon$, $VI2_4\delta$, $VI2_4\gamma$, with groups of orders 2, 2, and 6, respectively.

The headless, groupless system 4 which has been derived by a hexagonal transformation from a headless system $VI1_3\beta$ with a group of order 4, may also be derived by a quadrangular transformation from the headed system IC with a group order 3. Hence a quadrangular transformation may alter the number of systems $\triangle_7$ in a $\triangle_{15}$ and change a headed system into one without a head. Hexagonal transformations, on the contrary, leave unchanged the number of systems $\triangle_7$ in a $\triangle_{15}$ and, therefore, always transform systems with or without heads into systems with or without heads, respectively.

In further illustration of the productiveness of this method in generating new systems from those already known, we exhibit the results of its application to a groupless system 27 previously obtained by this same method. The 35 triads of the system arranged in classes according to their three-column indices are shown in the following table:

$1^3 2^2 5$	$1^3 234$	$1^3 9$	$1^2 2^3 4$	$1^2 28$	$1^2 37$	$1^2 5^2$	1236
$d47$	$f37$	cfg	def	$a35$ $bd5$ 157	acg	$af1$ $e34$ $g45$	$ae4$ $c25$ $bg3$
1245	$1, 11$	$2^3 6$	$2^2 4^2$	$2, 10$	$3, 9$	$5, 7$	6^2
$a78$ $bf8$ $e58$	$f24$ $bc7$ $b14$ $cd8$ $ce1$	$bc2$	$f56$	468 $ad2$ $c36$ $d13$ $e67$ $g27$ $g18$	$ab6$ 238	$dg6$	126

The analysis of this system by two-column indices reveals the existence of 1; 24; 15; 65 pairs of columns with the indices 2^3; $2, 4$; 3^2; 6, respectively. The group for this system is the identity and, therefore, it is possible that the application of all the quadrangular and of all the hexagonal transformations would yield 39 noncongruent systems. Since however a quadrangular transformation may lead back to a headed system, already completely determined, we apply only the 15 hexagonal transformations. These generate the following 15 systems: 30, 33, 21, 18, $VI2_4\alpha$, 28, 29, $II1_7$, 31, 9, 32, 17, 24, 17, $V4\gamma2$; 2 of these are congruent, 3 are systems with groups already determined in Part 1, while 11 are new groupless systems, which may be shown to be noncongruent by a comparison either of their indices or of their distinctive sets of trains.

The application of this method to a number of the systems given in Part 1 yielded the 33 noncongruent groupless systems tabulated below.

We make use of the notation $(a\ b)\ c\ d\ e\ f$ to denote a quadrangular transformation which occurs in the pair of columns a, b and which involves the four elements c, d, e, f. Similarly $(ab)\ c\ d\ e\ f\ g\ 1$ is a hexagonal transformation in the columns a, b which involves the six elements c, d, e, f, g, 1.

The system 1 may be derived from the system IB by the application of the quadrangular transformation $(a\ 2)\ b\ d\ 5\ 3$, and for the sake of brevity we shall write this in the form $1 = IB$, $(a\ 2)\ b\ d\ 5\ 3$.

The 33 new groupless systems are derived as follows:

$1 = IB$, $(a\ 2)\ b\ d\ 5\ 3$; $2 = IB$, $(b\ 3)\ c\ f\ 1\ 7$;
$3 = IB$, $(b\ 8)\ c\ f\ 5\ 2$; $4 = IC$, $(a\ 2)\ c\ e\ 4\ 1$;
$5 = IC$, $(a\ 8)\ c\ e\ 3\ 5$; $6 = IC$, $(a\ 5)\ f\ g\ 1\ 4$;
$7 = IC$, $(b\ 3)\ e\ 8\ g\ 6$; $8 = VIC$, $(b\ 3)\ f\ g\ 1\ 7$;
$9 = IIC$, $(c\ 3)\ d\ f\ 7\ 6$; $10 = IIC$, $(a\ 2)\ 5\ 6\ d\ g$;
$11 = IIC$, $(b\ 4)\ d\ e\ 8\ 6$; $12 = IIB$, $(c\ 5)\ e\ f\ 4\ 1$;
$13 = IIF$, $(b\ 2)\ d\ e\ 7\ 5$; $14 = IIC$, $(c\ 4)\ d\ f\ 8\ 5$;
$15 = IID$, $(c\ 3)\ e\ g\ 8\ 5$; $16 = II$, 1_3, $(a_3\ d_3)\ a_1\ b_2\ b_1\ a_2\ e_1\ e_3$.

In the system I, 2 we now replace the elements $\alpha, \beta, \gamma, \delta, \epsilon$, by $f, g, 6, 7, 8$, respectively.

$17 = I, 2$, $(f\ 1)\ 6\ 2\ 5\ 7$; $18 = 17$, $(d\ f)\ a\ 2\ 4\ 7\ 1\ 3$;
$19 = 17$, $(b\ 1)\ a\ 6\ g\ 2\ f\ 8$; $20 = 17$, $(c\ d)\ a\ 4\ 7\ b\ 5\ 2$;
$21 = 17$, $(b\ c)\ a\ 6\ e\ 3\ 1\ 4$; $22 = 17$, $(b\ 5)\ a\ 3\ e\ 8\ f\ 6$;
$23 = 17$, $(6\ 8)\ a\ b\ f\ 5\ e\ 7$; $24 = 17$, $(a\ b)\ c\ 4\ f\ 1\ 7\ 8$;
$25 = 17$, $(b\ e)\ 8\ 5\ f\ d$; $26 = 25$, $(c\ 5)\ 6\ 3\ a\ 4\ g\ f$;
$27 = 17$, $(2\ 3)\ b\ g\ 7\ f\ 4\ e$; $28 = 27$, $(c\ 5)\ a\ 4\ f\ g\ 3\ 6$;
$29 = 27$, $(e\ 3)\ b\ 2\ a\ g\ 5\ 8$; $30 = 27$, $(a\ b)\ c\ 4\ 1\ f\ 8\ 7$;
$31 = 27$, $(2\ 4)\ b\ e\ 1\ 6\ 3\ 8$; $32 = 27$, $(2\ 5)\ b\ d\ e\ 8\ a\ 3$;
$33 = 27$, $(a\ f)\ b\ 6\ 5\ 3\ 7\ 8$.

The 77 noncongruent systems thus far derived are interconnected by quadrangular and by hexagonal transformations, and in general a system is not united uniquely to another system but is derivable from several systems by different transformations; for example, the system 7 possesses the following interconnections: $7 = I\ C$, $(b\ 3)\ e\ 8\ g\ 6$; $7 = V\ 4\ \alpha\ 1$, $(d\ g)\ a\ 3\ e f\ 7\ 6$; $7 = \mathrm{II}$, 1, $(a\ 1)\ b\ c\ 4\ e\ 6\ 2$.

After this point in the investigation many other systems which were transformed by this process furnished only repetitions of the 77 noncongruent systems thus far determined, showing that the number of groupless systems was probably not much in excess of 33. An exhaustive determination of all groupless systems by this empirical method requires that every new system, as it appears, shall be subjected to each one of its possible quadrangular and hexagonal transformations. The enumeration of these transformations for the 14 new systems derived above from the system 27 shows that a complete investigation even of these systems would necessitate the application of more than 435 transformations. Hence while this empirical method for generating new systems is productive, the amount of work involved in an exhaustive investigation is prohibitive. Therefore it is evident that a new starting point and a new method are requisite to insure a complete determination of the groupless systems. This desideratum is fully met in the following Part 4.

PART 4.

STRUCTURE AS DEFINED BY INTERLACINGS, HEADS, AND SEMIHEADS; A COMPLETE CENSUS OF TRIAD SYSTEMS IN FIFTEEN ELEMENTS.

By F. N. Cole.

1. INTRODUCTION.

In forming triad systems in 15 letters, there are only four typical openings, viz:

	1 2 3	1 4 5	1 6 7	1 8 9	1 10 11	1 12 13	1 14 15
I.		2 4 6	2 5 7	2 8 10	2 9 12	2 11 14	2 13 15
II.		2 4 6	2 5 7	2 8 10	2 9 11	2 12 14	2 13 15
III.		2 4 6	2 5 8	2 7 9	2 10 12	2 11 14	2 13 15
IV.		2 4 6	2 7 8	2 9 10	2 11 12	2 13 14	2 5 15

which, from the way in which the triads containing 1 are laced with those containing 2, may be called the single tetrad, triple tetrad, hexad, and duodecad types, respectively. It turns out in the present investigation that, with a single exception, the tetrad type (single or triple) is always present, so that in the final census only openings I and II need be considered. These openings are then treated in sections 4, 5.

2. THE DUODECAD OPENING.

We show here that a triad system in the 15 letters can not be made up with duodecads alone. To this end we note that opening IV above has the following group of 24 substitutions which convert it into itself:

$$\left\{\begin{matrix}(1\ 2)\ (4\ 5\ 15\ 14\ 13\ 12\ 11\ 10\ 9\ 8\ 7\ 6),\\ (4\ 6)\ (5\ 7)\ (8\ 15)\ (9\ 14)\ (10\ 13)\ (11\ 12)\end{matrix}\right\}$$

If the triads with 1 and 2 (excluding 1 2 3) are denoted by a, b, c, d, e, f and a', b', c', d', e', f', respectively, this group is equivalent to

$$\{(af'fc'ed'dc'cb'ba'),\ (ab)(cf)(de)(b'f')(c'e')\}$$

and suffices to interchange the accented and unaccented letters with preservation of order of sequence, to move each set of letters in a cycle, and to reverse the order of each set.

If now the triads with 3 are laced through those with 1 and 2, in the duodecad manner in each case, it may happen that these new triads (1) connect two successive ones of the 1 set or 2 set, or (2) do not exhibit such a sequence. It readily follows then, with the help of the group above, that the only typical lacings are the following 14:

	abcdef	*abdcfe*	*abefcd*
	abcdfe	*abdfec*	*abfced*
A.	*abcfed*	*abdfce*	*abfdec*
	abcefd	*abecfd*	
	abcfde	*abedfc*	*acebfd*

the first 13 presenting the sequence *ab*, and the last one no such sequence.

This last one, with no sequences, may be worked out in some detail as an example of the method employed throughout this paper. We have to write down the triads with 3, lacing, say, those with 1 in the order *acebfd*, and those with 2 without sequence. We can not use 3 4 8, since this would give the sequence a' b'; we must take 3 4 9, 3 5 8, or 3 5 9. These lead to the four possibilities:

3 4 9	3 8 13	3 6 12	3 7 15	3 10 14	3 11 5
3 5 8	3 9 13	3 6 12	3 7 14	3 11 15	3 4 10
3 5 8	3 9 13	3 7 12	3 6 14	3 10 15	3 4 11
3 5 9	3 8 12	3 6 13	3 7 15	3 10 14	3 4 11

The substitution (123), with proper adjustment of the remaining numbers, converts the first of these into the other three. The first may then be taken as typical. It has the following group of six substitutions into itself: {(1 2 3) (4 8 10) (5 7 14) (6 13 11), (1 2) (11 13) (14 10) (9 15) (5 8) (4 7)}. This suggests the formation of the triad 9 15 x. Here x can not be 1, 2, 3, 4, 5, 7, 8, 9, 10, 14, or 15, since 1 9, for example, has already been used in a triad; for x we must take 6, 11, 12, or 13. But 6, 11, and 13 are equivalent under the group of order 6 above. And the remaining possibilities 9 15 6 and 9 15 12, if followed out, lead to tetrads or hexads.

Returning to the table A, we may note that in the first 13 cases the 3-triad which joins a and b must be 3 5 7, for 3 4 6 is at once excluded, and 3 4 7 and 3 5 6 involve tetrads of 1 and 4 and of 2 and 5, respectively. Starting with 3 5 7, and writing in the remaining 3-triads, we find that the first, second, sixth, and eighth cases lead directly to tetrads or hexads. The remaining cases prove to be partly equivalent to each other, and those which survive are found on continuation to the 4-triads, etc., to involve tetrads or hexads.

3. THE HEXAD OPENING.

This has the following group of 144 substitutions into itself:

{(4 7 8) (5 6 9), (4 5) (6 8) (7 9);
(10 13 14) (11 12 15), (10 11) (12 14) (13 15);
(4 10) (5 11) (6 12) (7 13) (8 14) (9 15);
(1 2) (5 6) (7 8) (11 12) (13 14)}.

The triad 3 4 x can not have $x=1$, 2, 3, 4, 5, or 6, since these have already been used; nor can $x=7$ or 8, since these give tetrads of 4 and 1 or of 4 and 2. If the triad system is to have no tetrads, we must take $x=9$, 10, 11, 12, 13, 14, or 15, and of these the last six are equivalent under the group of order 144 above. Hence the only distinct types are 3 4 9 and 3 4 10.

Starting, then, with 3 4 9, we find for 3 5 y, only $y=7$ or 10, and note that 3 5 10 is equivalent under the group above to 3 4 10, the case to be considered later. It turns out, then, that we can have only

3 4 9 3 5 7 3 6 8 3 10 15 3 11 13 3 12 14

and we note that this is invariant under the group of order 144 above. If we now write in the 4-triads, we come at once to tetrads.

The case 3 4 10 leads to 3 9 11, 3 9 13, or 3 9 15. Following each of these out in detail, we encounter everywhere tetrads, with the single exception of the Heffter system:

1 2 3	1 4 5	1 6 7	1 8 9	1 10 11	1 12 13	1 14 15
	2 4 6	2 5 8	2 7 9	2 10 12	2 11 14	2 13 15
	3 4 10	3 5 7	3 6 11	3 8 15	3 9 13	3 12 14
			4 7 12	4 8 13	4 9 14	4 11 15
			5 6 14	5 9 10	5 11 13	5 12 15
			6 8 12	6 9 15	6 10 13	8 10 14
			7 8 11	7 10 15	7 13 14	9 11 12

This, then, is the only triad system in 15 letters that can be constructed solely with hexads and duodecads.

4. THE TRIPLE-TETRAD OPENING.

This has a group of 768 substitutions:

{(4 5) (6 7), (4 6) (5 7), (4 7) (5 6);
(8 9) (10 11), (8 10) (9 11), (8 11) (9 10);
(12 13) (14 15), (12 14) (13 15), (12 15) (13 14);
(4 8 12) (5 9 13) (6 10 14) (7 11 15),
(4 8) (5 9) (6 10) (7 11);
(1 2) (5 6) (9 10) (13 14)}.

We find that the x of 4 3 x must be 7, 8, . . . , 15, and that 8, . . . , 15 are equivalent. The typical cases are then 4 3 7 and 4 3 8. Of these 4 3 7 has a group of order 128, composed of those substitutions of the group of order 768 which leave 4 and 7 unchanged or interchange them. With the aid of this group of order 128 we find that we may take as typical cases 3 5 6 or 3 5 8.

The case 3 4 7, 3 5 6 exhibits a triad system in the seven letters 1, 2, 3, 4, 5, 6, 7, included in the triad system in the 15 letters. We then speak of the latter as having a 7-head, or simply a head. The triads with 3 are now found to be the six following sets:

3 4 7	3 5 6	3 8 11	3 9 10	3 12 15	3 13 14
			3 9 12	3 10 15	3 13 14
		3 8 12	3 9 13	3 10 14	3 11 15
				3 10 15	3 11 14
			3 9 14	3 10 13	3 11 15
				3 10 15	3 11 13

Working out their continuations, we find the 22 triad systems with triple tetrads and 7-head exhibited in the previous chapters of this memoir.

The case 3 4 7, 3 5 8 presents what one may perhaps be permitted to call a semihead. The triads with 3 are here:

3 4 7	3 5 8	3 6 9	3 10 12	3 11 15	3 13 14
		3 6 11	3 9 12	3 10 15	3 13 14
		3 6 12	3 9 10	3 11 15	3 13 14
			3 9 13	3 10 14	3 11 15
				3 10 15	3 11 14
			3 9 14	3 10 13	3 11 15
				3 10 15	3 11 13

Continuing these, we find 12 triad systems with triple tetrads and semihead.

In the absence of either head or semihead, the 3-triads form one of the following sets:

3 4 8	3 6 12	3 5 9	3 7 13	3 10 14	3 11 15
				3 10 15	3 11 14
			3 7 14	3 10 13	3 11 15
				3 10 15	3 11 13
			3 7 15	3 10 13	3 11 14
				3 10 14	3 11 13
		3 5 10	3 7 15	3 9 13	3 11 14
		3 5 11	3 7 15	3 9 13	3 10 14
				3 9 14	3 10 13
		3 5 15	3 7 11	3 9 13	3 10 14

and these lead to 26 triad systems with triple tetrads but no head nor semihead.

5. ONLY SINGLE TETRADS.

The group of the single tetrad opening is of order 64:

{(4 5) (6 7), (4 6) (5 7), (4 7) (5 6);
(8 11 15 12) (9 10 14 13), (8 9) (10 12) (11 13) (14 15);
(1 2) (5 6) (9 10) (11 12) (13 14)}.

This group also leaves the triad pair 3 4 7, 3 5 6 unchanged, and the 7-head case has the following sets of 3-triads:

3 4 7	3 5 6	3 8 11	3 9 13	3 10 15	3 12 14
			3 9 15	3 10 13	3 12 14

leading to a single triad system with 7-head but no triple tetrads.

With a semihead 3 4 7, 3 5 8 there are 17 typical sets of 3-triads:

3 4 7	3 5 8	3 6 9	3 10 12	3 11 15	3 13 14
			3 10 13	3 11 15	3 12 14
			3 10 14	3 11 13	3 12 15
			3 10 15	3 11 13	3 12 14
		3 6 11	3 9 10	3 12 15	3 13 14
			3 9 13	3 10 15	3 12 14
			3 9 14	3 10 13	3 12 15
			3 9 15	3 10 13	3 12 14
		3 6 13	3 9 10	3 11 15	3 12 14
			3 9 11	3 10 15	3 12 14
			3 9 14	3 10 12	3 11 15
				3 10 15	3 11 12
		3 6 15	3 9 10	3 11 12	3 13 14
				3 11 13	3 12 14
			3 9 11	3 10 13	3 12 14
			3 9 13	3 10 14	3 11 12
			3 9 14	3 10 13	3 11 12

These lead to only three triad systems with semihead but no head or triple tetrads.

There now remains only the case of no triple tetrads, heads, or semiheads. The first 3-triad may be taken as 3 4 8, and the opening set of 14 triads is unchanged by the substitution $s=(1\ 2)\ (5\ 6)\ (9\ 10)\ (11\ 12)\ (13\ 14)$ only. The triad 4 15 x has $x=7$, 9, 10, 11, or 12, and under s 9 is equivalent to 10 and 11 to 12. We have then three typical cases: 4 7 15, 4 9 15, 4 11 15. For the first of these 8 15 x gives $x=5$ or 11, and the continuation is not unreasonably long. But the cases 4 9 15 and 4 11 15 each subdivide into four cases instead of two, making the total labor five times that of the 4 7 15 case unless some method of compression could be devised. Fortunately such a method was at hand.

The triad pairs 4 1 5, 4 2 6 and 7 1 6, 7 2 5 exhibit a tetrad. If, then, a triad system has been constructed so far as to show its triads with 1, 2, 4, and 7, it may be possible by interchanging the pairs 1, 2 and 4, 7 to throw this system into one already identified. For example, arriving at the set

4 3 8	4 9 15	4 7 11	4 10 12	4 13 14
	7 3 9	7 8 13	7 10 14	7 12 15

we find that the substitution (1 4) (2 7) (3 15 12 9 13 11) (8 14 10) throws this into a set with the original 1-triads and 2-triads, but containing 3 4 8, 4 7, 15 and therefore coming under a case already explored.

As another example, the set

4 3 8	4 11 15	4 7 10	4 9 13	4 12 14
	7 3 11	7 8 13	7 9 14	7 12 15

is thrown by (1 7 2 4) (5 6) (3 9 11 12 15 13 10) into the case 4 3 8, 4 9, 15. In fact, the nearly 150 cases under 4 11 15 reduce to only 6 by this process.

In the end there are found to be 15 triad systems with no triple tetrads, head, or semihead.

6. SUMMARY.

The following table gives the number of triad systems of each type for 15 letters:

Triple tetrad and head	22
Triple tetrad, no head, but semihead	12
Triple tetrad, no head, or semihead	26
No triple tetrad, but head	1
No triple tetrad or head, but semihead	3
No triple tetrad, head, or semihead	15
No tetrad (Heffter's system)	1
Total number of types	80

Table of Triad Systems in Fifteen Elements.

I. TRIPLE TETRADS AND HEAD.

Systems 1–20.

1 2 3	1 4 5	1 6 7	1 8 9	1 10 11	1 12 13	1 14 15
	2 4 6	2 5 7	2 8 10	2 9 11	2 12 14	2 13 15

1.	3 4 7	3 5 6	3 8 11	3 9 10	3 12 15	3 14 14
			4 8 12	4 9 13	4 10 14	4 11 15
			5 8 13	5 9 12	5 10 15	5 11 14
			6 8 14	6 9 15	6 10 12	6 11 13
			7 8 15	7 9 14	7 10 13	7 11 12

Group of order 8!/2.

2.			5 8 13	5 9 12	5 10 15	5 11 14
			6 8 15	6 9 14	6 10 13	6 11 12
			7 8 14	7 9 15	7 10 12	7 11 13

Group of order 192.

3.			5 8 14	5 9 15	5 10 12	5 11 13
			6 8 15	6 9 14	6 10 13	6 11 12
			7 8 13	7 9 12	7 10 15	7 11 14

Group of order 96.

4.			5 8 14	5 9 12	5 10 15	5 11 13
			6 8 15	6 9 14	6 10 13	6 11 12
			7 8 13	7 9 15	7 10 12	7 11 14

Group of order 8.

5.			4 8 12	4 9 13	4 10 15	4 11 14
			5 8 14	5 9 15	5 10 13	5 11 12
			6 8 13	6 9 12	6 10 14	6 11 15
			7 8 15	7 9 14	7 10 12	7 11 13

Group of order 32.

6.			5 8 14	5 9 15	5 10 13	5 11 12
			6 8 13	6 9 14	6 10 12	6 11 15
			7 8 15	7 9 12	7 10 14	7 11 13

Group of order 24.

7.			4 8 12	4 9 14	4 10 15	4 11 13
			5 8 15	5 9 13	5 10 12	5 11 14
			6 8 13	6 9 15	6 10 14	6 11 12
			7 8 14	7 9 12	7 10 13	7 11 15

Group of order 288.

8.	3 4 7	3 5 6	3 8 11	3 9 12	3 10 15	3 13 14
			4 8 13	4 9 10	4 11 14	4 12 15
			5 8 15	5 9 14	5 10 13	5 11 12
			6 8 14	6 9 15	6 10 12	6 11 13
			7 8 12	7 9 13	7 10 14	7 11 15

Group of order 4.

9.			5 8 15	5 9 14	5 10 12	5 11 13
			6 8 14	6 9 15	6 10 13	6 11 12
			7 8 12	7 9 13	7 10 14	7 11 15

Group of order 2.

10.			5 8 15	5 9 14	5 10 13	5 11 12
			6 8 14	6 9 13	6 10 12	6 11 15
			7 8 12	7 9 15	7 10 14	7 11 13

Group of order 2.

11.			4 8 13	4 9 14	4 10 12	4 11 15
			5 8 15	5 9 13	5 10 14	5 11 12
			6 8 14	6 9 10	6 11 13	6 12 15
			7 8 12	7 9 15	7 10 13	7 11 14

Group of order 2.

12.			5 8 15	5 9 13	5 10 14	5 11 12
			6 8 12	6 9 15	6 10 13	6 11 14
			7 8 14	7 9 10	7 11 13	7 12 15

Group of order 3.

Systems 1–20—Continued.

13.			4 8 13	4 9 15	4 10 12	4 11 14
			5 8 15	5 9 14	5 10 13	5 11 12
			6 8 14	6 9 10	6 11 13	6 12 15
			7 8 12	7 9 13	7 10 14	7 11 15

Group of order 8.

14.			5 8 14	5 9 10	5 11 13	5 12 15
			6 8 15	6 9 14	6 10 13	6 11 12
			7 8 12	7 9 13	7 10 14	7 11 15

Group of order 12.

15.	3 4 7	3 5 6	3 8 12	3 9 13	3 10 14	3 11 15
			4 8 15	4 9 10	4 11 12	4 13 14
			5 8 14	5 9 12	5 10 15	5 11 13
			6 8 11	6 9 14	6 10 13	6 12 15
			7 8 13	7 9 15	7 10 12	7 11 14

Group of order 4.

16.			4 8 15	4 9 14	4 10 13	4 11 12
			5 8 11	5 9 10	5 12 15	5 13 14
			6 8 14	6 9 15	6 10 12	6 11 13
			7 8 13	7 9 12	7 10 15	7 11 14

Group of order 168.

17.			5 8 11	5 9 12	5 10 15	5 13 14
			6 8 14	6 9 10	6 11 13	6 12 15
			7 8 13	7 9 15	7 10 12	7 11 14

Group of order 24.

18.	3 4 7	3 5 6	3 8 12	3 9 13	3 10 15	3 11 14
			4 8 15	4 9 10	4 11 12	4 13 14
			5 8 11	5 9 14	5 10 13	5 12 15
			6 8 14	6 9 15	6 10 12	6 11 13
			7 8 13	7 9 12	7 10 14	7 11 15

Group of order 4.

19.	3 4 7	3 5 6	3 8 12	3 9 14	3 10 13	3 11 15
			4 8 15	4 9 12	4 10 14	4 11 13
			5 8 13	5 9 10	5 11 14	5 12 15
			6 8 11	6 9 15	6 10 12	6 13 14
			7 8 14	7 9 13	7 10 15	7 11 12

Group of order 12.

20.	3 4 7	3 5 6	3 8 12	3 9 14	3 10 15	3 11 13
			4 8 15	4 9 10	4 11 12	4 13 14
			5 8 11	5 9 13	5 10 14	5 12 15
			6 8 14	6 9 12	6 10 13	6 11 15
			7 8 13	7 9 15	7 10 12	7 11 14

Group of order 3.

Systems 21–22.

1 2 3	1 4 5	1 6 7	1 8 9	1 10 11	1 12 13	1 14 15
	2 4 6	2 5 7	2 8 10	2 9 12	2 11 14	2 13 15
	3 4 7	3 5 6	3 8 11	3 9 13	3 10 15	3 12 14
	4 8 12	4 9 14	4 10 13	4 11 15		

21.			5 8 13	5 9 15	5 10 14	5 11 12
			6 8 14	6 9 10	6 11 13	6 12 15
			7 8 15	7 9 11	7 10 12	7 13 14

Group of order 3.

22.			5 8 15	5 9 11	5 10 12	5 13 14
			6 8 13	6 9 15	6 10 14	6 11 12
			7 8 14	7 9 10	7 11 13	7 12 15

Group of order 3.

II. TRIPLE TETRADS, NO HEAD, BUT SEMIHEAD.

Systems 1–12.

1 2 3	1 4 5	1 6 7	1 8 9	1 10 11	1 12 13	1 14 15
	2 4 6	2 5 7	2 8 10	2 9 11	2 12 14	2 13 15

1.	3 4 7	3 5 8	3 6 11	3 9 12	3 10 15	3 13 14
			4 8 13	4 9 14	4 10 12	4 11 15
			5 6 11	5 9 10	5 11 13	5 12 15
			6 8 12	6 9 15	6 10 13	8 11 14
			7 8 15	7 9 13	7 10 14	7 11 12

No group.

2.	5 6 12	5 9 15	5 10 13	5 11 14
	6 8 15	6 9 13	6 10 14	8 11 12
	7 8 14	7 9 10	7 11 13	7 12 15

No group.

3.	4 8 13	4 9 15	4 10 14	4 11 12
	5 8 15	5 9 13	5 10 12	5 11 14
	6 8 12	6 9 14	6 10 13	8 11 15
	7 8 14	7 9 10	7 11 13	7 12 15

No group.

4.	3 4 7	3 5 8	3 6 12	3 9 10	3 11 15	3 13 14
			4 8 13	4 9 12	4 10 15	4 11 14
			5 6 15	5 9 14	5 10 12	5 11 13
			6 8 11	6 9 13	6 10 14	8 12 15
			7 8 14	7 9 15	7 10 13	7 11 12

No group.

5.	3 4 7	3 5 8	3 6 12	3 9 13	3 10 14	3 11 15
			4 8 14	4 9 10	4 11 13	4 12 15
			5 6 11	5 9 15	5 10 12	5 13 14
			6 8 13	6 9 14	6 10 15	8 11 12
			7 8 15	7 9 12	7 10 13	7 11 14

No group.

6.	3 4 7	3 5 8	3 6 12	3 9 13	3 10 15	3 11 14
			4 8 14	4 9 10	4 11 13	4 12 15
			5 6 11	5 9 15	5 10 12	5 13 14
			6 8 15	6 9 14	6 10 13	8 11 12
			7 8 13	7 9 12	7 10 14	7 11 15

No group.

Systems 1–12—Continued.

7.	4 8 15	4 9 12	4 10 14	4 11 13
	5 6 11	5 9 10	5 12 15	5 13 14
	6 8 14	6 9 15	6 10 13	8 11 12
	7 8 13	7 9 14	7 10 12	7 11 15

Group of order 3.

8.	3 4 7	3 5 8	3 6 12	3 9 14	3 10 13	3 11 15
			4 8 14	4 9 10	4 11 13	4 12 15
			5 6 11	5 9 12	5 10 15	5 13 14
			6 8 13	6 9 15	6 10 14	8 11 12
			7 8 15	7 9 13	7 10 12	7 11 14

Group of order 2.

9.	4 8 14	4 9 12	4 10 15	4 11 13
	5 6 11	5 9 10	5 12 15	5 13 14
	6 8 13	6 9 15	6 10 14	8 11 12
	7 8 15	7 9 13	7 10 12	7 11 14

Group of order 4.

10.	4 8 13	4 9 10	4 11 14	4 12 15
	5 6 11	5 9 15	5 10 12	5 13 14
	6 8 14	6 9 13	6 10 15	8 11 12
	7 8 15	7 9 12	7 10 14	7 11 13

No group.

11.	3 4 7	3 5 8	3 6 12	3 9 14	3 10 15	3 11 13
			4 8 13	4 9 10	4 11 14	4 12 15
			5 6 11	5 9 15	5 10 12	5 13 14
			6 8 15	6 9 13	6 10 14	8 11 12
			7 8 14	7 9 12	7 10 13	7 11 15

No group.

12.	4 8 14	4 9 12	4 10 13	4 11 15
	5 6 11	5 9 15	5 10 12	5 13 14
	6 8 15	6 9 13	6 10 14	8 11 12
	7 8 13	7 9 10	7 12 15	7 11 14

No group.

III. TRIPLE TETRADS, NO HEAD OR SEMIHEAD.

Systems 1–26.

1 2 3	1 4 5	1 6 7	1 8 9	1 10 11	1 12 13	1 14 15
	2 4 6	2 5 7	2 8 10	2 9 11	2 12 14	2 13 15

1.	3 4 8	3 5 9	3 6 12	3 7 13	3 10 14	3 11 15
			4 7 14	4 9 12	4 10 15	4 11 13
			5 6 15	5 8 14	5 10 13	5 11 12
			6 8 13	6 9 10	6 11 14	8 12 15
			7 8 11	7 9 15	7 10 12	9 13 14

Group of order 3.

2.	3 4 8	3 5 9	3 6 12	3 7 13	3 10 15	3 11 14
			4 7 14	4 9 12	4 10 13	4 11 15
			5 6 15	5 8 13	5 10 14	5 11 12
			6 8 14	6 9 10	6 11 13	8 12 15
			7 8 11	7 9 15	7 10 12	9 13 14

Group of order 4.

3.	4 7 10	4 9 14	4 11 13	4 12 15
	5 6 11	5 8 15	5 10 12	5 13 14
	6 8 13	6 9 15	6 10 14	8 11 12
	7 8 14	7 9 12	7 11 15	9 10 13

Group of order 12.

4.	3 4 8	3 5 9	3 6 12	3 7 14	3 10 13	3 11 15
			4 7 10	4 9 13	4 11 14	4 12 15
			5 6 13	5 8 15	5 10 14	5 11 12
			6 8 11	6 9 14	6 10 15	8 13 14
			7 8 12	7 9 15	7 11 13	9 10 12

No group.

5.	4 7 10	4 9 14	4 11 13	4 12 15
	5 6 11	5 8 12	5 10 15	5 13 14
	6 8 15	6 9 13	6 10 14	8 11 14
	7 8 13	7 9 15	7 11 12	9 10 12

No group.

6.	4 7 13	4 9 12	4 10 15	4 11 14
	5 6 15	5 8 13	5 10 14	5 11 12
	6 8 14	6 9 10	6 11 13	8 12 15
	7 8 11	7 9 15	7 10 12	9 13 14

No group.

7.	4 7 11	4 9 12	4 10 15	4 13 14
	5 6 10	5 8 14	5 11 13	5 12 15
	6 8 15	6 9 13	6 11 14	8 11 12
	7 8 13	7 9 15	7 10 12	9 10 14

No group.

Systems 1–26—Continued.

8.	3 4 8	3 5 9	3 6 12	3 7 14	3 10 15	3 11 13
			4 7 10	4 9 13	4 11 14	4 12 15
			5 6 11	5 8 15	5 10 12	5 13 14
			6 8 14	6 9 15	6 10 13	8 11 12
			7 8 13	7 9 12	7 11 15	9 10 14

Group of order 2.

9.	4 7 10	4 9 15	4 11 12	4 13 14
	5 6 11	5 8 13	5 10 14	5 12 15
	6 8 15	6 9 14	6 10 13	8 11 14
	7 8 12	7 9 13	7 11 15	9 10 12

Group of order 6.

10.	4 7 13	4 9 12	4 10 14	4 11 15
	5 6 15	5 8 14	5 10 13	5 11 12
	6 8 13	6 9 10	6 11 14	8 12 15
	7 8 11	7 9 15	7 10 12	9 13 14

Group of order 2.

11.	3 4 8	3 5 9	3 6 12	3 7 15	3 10 13	3 11 14
			4 7 10	4 9 12	4 11 15	4 13 14
			5 6 15	5 8 14	5 10 12	5 11 13
			6 8 11	6 9 13	6 10 14	8 12 15
			7 8 13	7 9 14	7 11 12	9 10 15

No group.

12.	4 7 10	4 9 14	4 11 13	4 12 15
	5 6 11	5 8 12	5 10 15	5 13 14
	6 8 13	6 9 15	6 10 14	8 11 15
	7 8 14	7 9 13	7 11 12	9 10 12

No group.

13.	3 4 8	3 5 9	3 6 12	3 7 15	3 10 14	3 11 13
			4 7 10	4 9 13	4 11 14	4 12 15
			5 6 11	5 8 12	5 10 15	5 13 14
			6 8 14	6 9 15	6 10 13	8 11 15
			7 8 13	7 9 14	7 11 12	9 10 12

No group.

14.	3 4 8	3 5 10	3 6 12	3 7 15	3 9 13	3 11 14
			4 7 9	4 10 14	4 11 13	4 12 15
			5 6 11	5 8 12	5 9 15	5 13 14
			6 8 13	6 9 14	6 10 15	8 11 15
			7 8 14	7 10 13	7 11 12	9 10 12

No group.

Systems 1-26—Continued.

15.

4 7 14	4 9 15	4 10 12	4 11 13
5 6 13	5 8 14	5 9 12	5 11 15
6 8 11	6 9 14	6 10 15	8 12 15
7 8 13	7 9 10	7 11 12	10 13 14

No group.

16.

4 7 9	4 10 12	4 11 15	4 13 14
5 6 15	5 8 14	5 9 12	5 11 13
6 8 11	6 9 14	6 10 13	8 12 15
7 8 13	7 10 14	7 11 12	9 10 15

No group.

17.

3 4 8	3 5 11	3 6 12	3 7 15	3 9 13	3 10 14
		4 7 10	4 9 14	4 11 13	4 12 15
		5 6 9	5 8 15	5 10 12	5 13 14
		6 8 14	6 10 13	6 11 15	8 11 12
		7 8 13	7 9 12	7 11 14	9 10 15

No group.

18.

4 7 13	4 9 14	4 10 15	4 11 12
5 6 9	5 8 15	5 10 12	5 13 14
6 8 14	6 10 13	6 11 15	8 11 13
7 8 12	7 9 10	7 11 14	9 12 15

No group.

19.

4 7 12	4 9 14	4 10 15	4 11 13
5 6 9	5 10 12	5 8 15	5 13 14
6 8 14	6 10 13	6 11 15	8 11 12
7 8 13	7 9 10	7 11 14	9 12 15

No group.

20.

4 7 12	4 9 15	4 10 13	4 11 14
5 6 10	5 8 15	5 9 12	5 13 14
6 8 13	6 9 14	6 11 15	8 11 12
7 8 14	7 9 10	7 11 13	10 12 15

No group.

Systems 1-26—Continued.

21.

4 7 9	4 10 13	4 11 14	4 12 15
5 6 10	5 8 15	5 9 12	5 13 14
6 8 13	6 9 14	6 11 15	8 11 12
7 8 14	7 10 12	7 11 13	9 10 15

No group.

22.

3 4 8	3 5 11	3 6 12	3 7 15	3 9 14	3 10 13
		4 7 11	4 9 13	4 10 15	4 11 12
		5 6 9	5 8 15	5 10 12	5 13 14
		6 8 13	6 10 14	6 11 15	8 11 14
		7 8 12	7 9 10	7 11 13	9 12 15

No group.

23

4 7 10	4 9 13	4 11 14	4 12 15
5 6 9	5 8 15	5 10 12	5 13 14
6 8 13	6 10 14	6 11 15	8 11 12
7 8 14	7 9 12	7 11 13	9 10 15

No group.

24.

4 7 12	4 9 13	4 10 15	4 11 14
5 6 9	5 8 15	5 10 12	5 13 14
6 8 13	6 10 14	6 11 15	8 11 12
7 8 14	7 9 10	7 11 13	9 12 15

No group.

25.

3 4 8	3 5 15	3 6 12	3 7 11	3 9 13	3 10 14
		4 7 13	4 9 12	4 10 15	4 11 14
		5 6 14	5 8 12	5 9 10	5 11 13
		6 8 11	6 9 15	6 10 13	8 13 14
		7 8 15	7 9 14	7 10 12	11 12 15

Group of order 3.

26.

4 7 14	4 9 15	4 11 13	4 10 12
5 6 13	5 8 12	5 9 10	5 11 14
6 8 11	6 9 14	6 10 15	84 3 11
7 8 15	7 9 12	7 10 13	11 12 15

No group.

IV. NO TRIPLE TETRADS, BUT HEAD.

One system

1.

1 2 3	1 4 5	1 6 7	1 8 9	1 10 11	1 12 13	1 14 15
	2 4 6	2 5 7	2 8 10	2 9 12	2 11 14	2 13 15
	3 4 7	3 5 6	3 8 11	3 9 15	3 10 13	3 12 14
			4 8 12	4 9 11	4 10 15	4 13 14

One system—Continued.

5 8 15	5 9 13	5 10 14	5 11 12
6 8 14	6 9 10	6 11 13	6 12 15
7 8 13	7 9 14	7 10 12	7 11 15

Group of order 21.

V. NO TRIPLE TETRADS OR HEAD, BUT SEMIHEAD.

Systems 1-3.

1 2 3	1 4 5	1 6 7	1 8 9	1 10 11	1 12 13	1 14 15
	2 4 6	2 5 7	2 8 10	2 9 12	2 11 14	2 13 15

1

3 4 7	3 5 8	3 6 11	3 9 14	3 10 13	3 12 15
		4 9 10	4 8 13	4 11 15	4 12 14
		5 6 12	5 9 15	5 10 14	5 11 13
		6 8 14	6 9 13	6 10 15	8 11 12
		7 8 15	7 9 11	7 10 12	7 13 14

Group of order 3.

2.

3 4 7	3 5 8	3 6 11	3 9 15	3 10 13	3 12 14
		4 8 14	4 9 10	4 11 13	4 12 15

Systems 1-3—Continued.

5 6 12	5 9 13	5 10 14	5 11 15
6 8 13	6 9 14	6 10 15	8 11 12
7 8 15	7 9 11	7 10 12	7 13 14

Group of order 3.

3.

3 4 7	3 5 8	3 6 13	3 9 14	3 10 12	3 11 15
		4 8 15	4 9 13	4 10 14	4 11 12
		5 6 11	5 9 15	5 10 13	5 12 14
		6 8 14	6 9 10	6 12 15	8 11 13
		7 8 12	7 9 11	7 10 15	7 13 14

Group of order 3.

VI. NO TRIPLE TETRADS, HEAD, OR SEMIHEAD.

Systems 1-15.

1 2 3	1 4 5	1 6 7	1 8 9	1 10 11	1 12 13	1 14 15
	2 4 6	2 5 7	2 8 10	2 9 12	2 11 14	2 13 15

1.

3 4 8	3 5 9	3 6 14	3 7 11	3 10 13	3 12 15
		4 7 15	4 9 10	4 11 12	4 13 14
		5 6 12	5 8 15	5 10 14	5 11 13
		6 8 11	6 9 13	6 10 15	8 12 14
		7 8 13	7 9 14	7 10 12	9 11 15

No group.

2.

3 4 8	3 5 12	3 6 14	3 7 11	3 9 13	3 10 15
		4 7 15	4 9 10	4 11 12	4 13 14
		5 6 9	5 8 15	5 10 14	5 11 13
		6 8 11	6 10 13	6 12 15	8 12 14
		7 8 13	7 9 14	7 10 12	8 11 15

No group.

3.

3 4 8	3 5 10	3 6 13	3 7 12	3 9 14	3 11 15
		4 7 15	4 9 10	4 11 12	4 13 14

Systems 1-15—Continued.

5 6 11	5 8 15	5 9 13	5 12 14
6 8 12	6 9 15	6 10 14	8 11 13
7 8 14	7 9 11	7 10 13	10 12 15

No group.

4.

3 4 8	3 5 9	3 6 11	3 7 14	3 10 13	3 12 15
		4 7 15	4 9 10	4 11 12	4 13 14
		5 6 12	5 8 15	5 10 14	5 11 13
		6 8 13	6 9 14	6 10 15	8 12 14
		7 8 11	7 9 13	7 10 12	9 11 15

No group.

5.

3 4 8	3 5 10	3 6 12	3 7 13	3 11 15	3 9 14
		4 7 15	4 9 10	4 11 12	4 13 14
		5 6 11	5 8 15	5 9 13	5 12 14
		6 8 14	6 9 15	6 10 13	8 11 13
		7 8 12	7 9 11	7 10 14	10 12 15

No group.

Systems 1-15—Continued.

6. 3 4 8	3 5 14	3 6 12	3 7 11	3 9 15	3 10 13
		4 7 15	4 9 10	4 11 13	4 12 14
		5 6 13	5 8 15	5 9 11	5 10 12
		6 8 11	6 9 14	6 10 15	8 13 14
		7 8 12	7 9 13	7 10 14	11 12 15

No group.

7. 3 4 8	3 5 13	3 6 11	3 7 12	3 9 14	3 10 15
		4 7 15	4 9 10	4 11 13	4 12 14
		5 6 14	5 8 15	5 9 11	5 10 12
		6 8 12	6 9 15	6 10 13	8 13 14
		7 8 11	7 9 13	7 10 14	11 12 15

No group.

8. 3 4 8	3 5 14	3 6 13	3 7 12	3 9 11	3 10 15
		4 7 15	4 9 10	4 11 13	4 12 14
		5 6 11	5 8 15	5 9 13	5 10 12
		6 8 12	6 9 15	6 10 14	8 13 14
		7 8 11	7 9 14	7 10 13	11 12 15

No group.

9. 3 4 8	3 5 13	3 6 10	3 7 12	3 9 14	3 11 15
		4 7 15	4 9 13	4 10 14	4 11 12
		5 6 11	5 8 15	5 9 10	5 12 14
		6 8 12	6 9 15	6 13 14	8 11 13
		7 8 14	7 9 11	7 10 13	10 12 15

Group of order 4.

10. 3 4 8	3 5 11	3 6 12	3 7 14	3 10 15	3 9 13
		4 7 15	4 9 14	4 10 12	4 11 13
		5 6 13	5 8 15	5 9 10	5 12 14
		6 8 11	6 9 15	6 10 14	8 13 14
		7 8 12	7 9 11	7 10 13	11 12 15

Group of order 4.

Systems 1-15—Continued.

11. 3 4 8	3 5 13	3 6 14	3 7 12	3 9 11	3 10 15
		4 7 15	4 9 14	4 10 12	4 11 13
		5 6 11	5 8 15	5 9 10	5 12 14
		6 8 12	6 9 15	6 10 13	8 13 14
		7 8 11	7 9 13	7 10 14	11 12 15

Group of order 3.

12. 3 4 8	3 5 14	3 6 11	3 7 9	3 10 13	3 12 15
		4 7 15	4 9 10	4 11 13	4 12 14
		5 6 13	5 8 12	5 9 11	5 10 15
		6 8 14	6 9 15	6 10 12	8 11 15
		7 8 13	7 10 14	7 11 12	9 13 14

Group of order 5.

13. 3 4 8	3 5 14	3 6 9	3 7 13	3 10 15	3 11 12
		4 7 15	4 9 10	4 11 13	4 12 14
		5 6 11	5 8 13	5 9 15	5 10 12
		6 8 14	6 10 13	6 12 15	8 11 15
		7 8 12	7 9 11	7 10 14	9 13 14

Group of order 3.

14. 3 4 8	3 5 11	3 6 13	3 7 9	3 10 15	3 12 14
		4 7 15	4 9 11	4 10 12	4 13 14
		5 6 10	5 8 13	5 9 14	5 12 15
		6 8 14	6 9 15	6 11 12	8 11 15
		7 8 12	7 10 14	7 11 13	9 10 13

Group of order 4.

15. 3 4 8	3 5 10	3 6 15	3 7 13	3 9 11	3 12 14
		4 7 11	4 9 15	4 10 12	4 13 14
		5 6 9	5 8 14	5 11 13	5 12 15
		6 8 13	6 10 14	6 11 12	8 11 15
		7 8 12	7 9 14	7 10 15	9 10 13

Group of order 36.

VII. NO TETRAD.—HEFFTER'S HEADLESS SYSTEM.

1 2 3	1 4 5	1 6 7	1 8 9	1 10 11	1 12 13	1 14 15
	2 4 6	2 5 8	2 7 9	2 10 12	2 11 14	2 13 15
	3 4 10	3 5 7	3 6 11	3 8 15	3 9 13	3 12 14
			4 7 12	4 8 13	4 9 14	4 11 15

5 6 14	5 9 10	5 11 13	5 12 15
6 8 12	6 9 15	6 10 13	8 10 14
7 8 11	7 10 15	7 13 14	9 11 12

Group of order 60.

PART 5.

SEQUENCES AND INDICES FOR ALL GROUPLESS TRIAD SYSTEMS ON 15 ELEMENTS.

By L. D. Cummings.

The 80 systems described in this paper separate into the three distinct types:

(a) Twenty-three systems with a group and with a head.

(b) Twenty-one systems with a group and with no head.

(c) Thirty-six systems with no group and with no head.

The indices for type (a) and the Heffter system of type (b) have been fully discussed in an earlier paper [1] and therefore are omitted here. The indices for types (b) and (c) are given in the following tables 1 (b) and 1 (c), respectively. Now in a given system the index of a triad is invariant under all the substitutions of the symmetric group; two systems, therefore, which differ either in their indices or in the number of triads enumerated under each index are certainly incongruent. Hence tables 1 (b) and 1 (c) show conclusively the noncongruency of these 56 systems.

[1] Cummings, L. D. Transactions of the American Mathematical Society, vol. XV, No. 3, pp. 311–327.

TABLE 1 (b).

System.	$1^6 2^3$	$1^4 2^4$	$1^4 3 5$	$1^3 2^2 5$	$1^3 2 3 4$	$1^3 3^3$	$1^3 9$	$1^2 2^3 4$	$1^2 2^2 3^2$	$1^2 2 8$	$1^2 3 7$	$1^2 4 6$	$1^2 5^2$	$1 2^2 7$	$1 2 3 6$	$1 2 4 5$	$1 3^2 5$	1, 11	2^6	$2^3 6$	$2^2 4^2$	$2 3^2 4$	2, 10	3^4	3, 9	4, 8	5, 7	6^2
$V4\gamma 1$	3			3	3				3		3	3	6		3			4	1									
$VI3_3\delta$	2			4				4	8		2	4	4	6										1				
$VI2_4\delta$	1			2	1		4		3			5		1	5	4		4		2			2	1				
$VI2_4\alpha$		1		1						2		3	2		3	3		5					10	1	2	2		
$VI2_4\gamma$		1				1				6		3		3	6	6							6	3				
III		1								3				3	9			6		3	1					3	3	3
$V4\alpha 1$			4	3								6	3			3	3	7			3	3						1
$V4\gamma 2$			3						3	6			9		3			1	1				6		3			
$VI3_3\alpha$				2											4			14		4				1	6			4
$V4\beta 2$					3					3				3		6		4					6			6	3	1
$VI2_4\epsilon$					1				1	2	2	1		2	4	3	1	7					8	1			2	
III_2						2								18										9			6	
$V4\beta 1$							6								3	9		10					3					4
$VI1_3\beta$								6		4		4			4	8							8	1				
$V1\gamma$								6										12					12	1			4	
12								5		5					5			11					5					
$VI3_4\gamma$								4			2		2	4	4	4		4				4				5	2	
$V4\delta 1$									3	3				3	3		3	7	1		3		3		3	3		
$VI3_3\gamma$										4			2	8		2		2					12	1			4	
III_8										1								6			4				3	12		3

TABLE 1 (c).

System.	$1^6 3^2$	$1^5 2^2 3$	$1^4 2^4$	$1^4 2\, 6$	$1^4 3\, 5$	$1^3 2^2 5$	$1^3 2\, 3\, 4$	$1^3 9$	$1^2 2^3 4$	$1^2 2^2 3^2$	$1^2 2\, 8$	$1^2 3\, 7$	$1^2 4\, 6$	$1^2 5^2$	$1\, 2^2 7$	$1\, 2\, 3\, 6$	$1\, 2\, 4\, 5$	$1\, 3^2 5$	1, 11	2^6	$2^3 6$	$2^2 4^2$	$2\, 3^2 4$	2, 10	3, 9	4, 8	5, 7	6^2
15	1	2	....	2	3	2	....	4	2	7	2	....	1	6	....	2	....	....	....	1	....	....	....	....	....	....	....	....
11	....	3	1	....	2	3	1	1	2	2	2	....	1	5	....	6	....	....	4	....	1	1	....	....	....	....	....	....
9	....	2	1	....	2	3	1	1	....	5	2	2	4	6	....	5	....	....	....	1	....	....	....	....	....	....	....	....
10	....	2	1	....	....	5	1	....	2	5	1	1	5	2	3	3	....	....	2	....	....	1	1	....	....	....	....	....
14	....	1	....	....	1	1	1	1	....	3	3	1	1	2	....	5	3	1	5	....	2	....	1	2	....	1	....	....
6	....	....	....	2	....	2	2	1	3	....	3	....	1	1	....	2	3	....	4	....	....	....	3	4	2	....	....	2
4	....	....	....	1	2	3	....	1	1	....	3	1	1	1	2	2	5	....	5	....	....	3	1	....	....	....	2	1
7	....	....	....	1	1	1	2	1	....	....	1	1	4	....	2	3	5	....	6	....	....	3	1	....	....	....	2	1
12	....	....	....	1	1	....	1	3	2	4	4	....	....	5	2	4	1	....	3	....	1	1	....	1	1	....	....	....
13	....	....	....	1	....	1	1	2	2	6	2	....	2	2	1	5	3	....	3	....	1	1	....	1	1	....	....	....
3	....	....	....	....	2	1	....	3	1	....	1	....	2	2	1	2	4	....	5	....	2	2	....	5	....	1	....	1
5	....	....	....	....	1	3	1	....	1	....	1	....	3	1	2	2	3	....	5	....	1	1	2	1	....	2	4	1
1	....	....	....	....	1	1	1	2	1	....	3	....	....	2	....	1	3	....	8	....	2	3	....	3	1	1	1	1
8	....	....	....	....	1	1	....	1	1	1	....	....	5	3	....	7	3	....	8	....	1	1	....	1	....	....	....	1
25	....	....	....	....	1	1	....	1	....	....	1	1	3	5	1	7	2	....	8	1	1	....	1	....	....	....	....	1
35 [Cole]	....	....	....	....	....	3	....	....	1	....	3	1	3	1	1	4	3	1	2	....	....	....	1	4	3	....	1	3
2	....	....	....	....	....	2	....	2	1	....	2	....	....	....	2	2	5	1	4	....	2	3	....	5	1	1	1	1
27	....	....	....	....	....	1	1	1	1	....	3	1	....	3	....	3	3	....	5	....	1	1	....	7	2	....	1	1
20	....	....	....	....	....	1	....	....	....	....	3	1	....	2	1	3	3	1	5	....	1	1	1	4	1	1	4	2
34 [Cole]	....	....	....	....	....	1	....	....	....	....	2	....	2	....	2	2	2	2	5	....	....	....	....	6	5	1	....	5
29	....	....	....	....	....	1	....	....	....	....	1	....	....	1	1	3	....	1	8	....	1	1	....	7	2	2	5	1
24	....	....	....	....	....	....	1	....	....	....	1	....	1	3	1	2	2	2	8	....	2	1	1	1	4	1	2	2
28	....	....	....	....	....	....	....	3	....	....	5	....	1	2	2	2	1	2	4	....	1	....	3	2	3	....	....	4
26	....	....	....	....	....	....	....	2	1	....	4	1	2	3	1	2	1	1	3	....	1	2	....	6	2	1	1	1
17	....	....	....	....	....	....	....	2	1	....	3	....	2	....	1	5	3	....	9	....	1	1	1	3	2	1	....	....
22	....	....	....	....	....	....	....	2	....	....	4	1	1	2	2	1	....	....	7	....	2	3	....	5	1	1	2	1
23	....	....	....	....	....	....	....	1	2	....	3	....	2	1	....	2	1	1	5	....	....	1	....	7	5	....	3	1
21	....	....	....	....	....	....	....	1	1	....	4	....	....	3	4	1	3	....	5	....	1	1	1	3	....	3	3	1
32	....	....	....	....	....	....	....	1	....	2	4	1	2	3	....	1	1	....	7	....	1	2	....	6	2	1	1	....
31	....	....	....	....	....	....	....	1	....	....	2	....	2	1	1	5	1	1	7	1	1	....	1	4	1	2	1	3
33	....	....	....	....	....	....	....	1	....	....	2	....	....	3	2	3	1	1	4	....	1	1	....	8	1	3	1	3
19	....	....	....	....	....	....	....	1	....	....	1	....	....	1	7	3	2	1	4	....	....	2	1	4	3	....	2	3
30	....	....	....	....	....	....	....	....	....	....	2	....	1	1	2	2	3	....	5	....	....	1	1	7	2	5	2	1
18	....	....	....	....	....	....	....	....	....	....	1	....	2	2	1	3	1	1	4	....	1	1	....	7	5	4	....	2
16	....	....	....	....	....	....	....	....	....	....	1	....	1	....	2	5	3	....	6	....	1	1	....	9	1	2	1	2
36 [Cole]	....	....	....	....	....	....	....	....	....	....	....	....	....	4	1	2	....	....	9	....	1	1	....	7	3	2	3	2

The analysis by the method of sequences of the 80 noncongruent systems derived by Mr. Cole in Part 4, shows that 77 of these systems are congruent either to systems derived in Part 1 by means of operators of their groups; or to systems derived from the former, in Part 3, by means of quadrangular or hexagonal transformations.

The names or numbers of these 77 pairs of equivalent systems and the substitution which transforms each of Mr. Cole's systems into its equivalent system are given below.

I, 1≡IIIA by $s\equiv\begin{pmatrix}1&2&3&4&5&6&7&8&9&10&11&12&13&14&15\\a&b&c&d&e&g&f&1&2&3&4&5&6&8&7\end{pmatrix}$

I, 2≡IIIB by $s\equiv\begin{pmatrix}1&2&3&4&5&6&7&8&9&10&11&12&13&14&15\\a&b&c&d&e&f&g&1&2&3&4&5&6&8&7\end{pmatrix}$

I, 3≡IIIC by $s\equiv\begin{pmatrix}1&2&3&4&5&6&7&8&9&10&11&12&13&14&15\\a&b&c&d&g&f&e&1&2&3&4&5&6&8&7\end{pmatrix}$

I, 4≡IIE by $s\equiv\begin{pmatrix}1&2&3&4&5&6&7&8&9&10&11&12&13&14&15\\b&a&c&2&1&4&3&d&e&g&f&5&7&6&8\end{pmatrix}$

I, 5≡IVA by $s\equiv\begin{pmatrix}1&2&3&4&5&6&7&8&9&10&11&12&13&14&15\\a&c&b&d&g&e&f&1&2&3&4&5&6&8&7\end{pmatrix}$

I, 6≡IIA by $s\equiv\begin{pmatrix}1&2&3&4&5&6&7&8&9&10&11&12&13&14&15\\a&b&c&d&f&e&g&5&6&7&8&2&1&4&3\end{pmatrix}$

I, 7≡VA by $s\equiv\begin{pmatrix}1&2&3&4&5&6&7&8&9&10&11&12&13&14&15\\a&b&c&d&e&f&g&1&2&5&8&4&3&6&7\end{pmatrix}$

I, 8≡IVB by $s\equiv\begin{pmatrix}1&2&3&4&5&6&7&8&9&10&11&12&13&14&15\\d&f&b&c&e&a&g&5&1&3&7&4&8&6&2\end{pmatrix}$

I, 9≡IIC by $s\equiv\begin{pmatrix}1&2&3&4&5&6&7&8&9&10&11&12&13&14&15\\a&d&g&e&c&b&f&4&3&8&7&5&6&2&1\end{pmatrix}$

I, 10≡IID by $s\equiv\begin{pmatrix}1&2&3&4&5&6&7&8&9&10&11&12&13&14&15\\f&a&e&g&d&b&c&4&5&3&6&1&7&2&8\end{pmatrix}$

I, 11≡VIB by $s\equiv\begin{pmatrix}1&2&3&4&5&6&7&8&9&10&11&12&13&14&15\\b&c&a&f&g&d&e&7&1&5&6&2&3&4&8\end{pmatrix}$

I, 12≡VID by $s\equiv\begin{pmatrix}1&2&3&4&5&6&7&8&9&10&11&12&13&14&15\\c&b&d&f&g&e&a&2&4&3&8&7&5&1&6\end{pmatrix}$

I, 13≡VC by $s\equiv\begin{pmatrix}1&2&3&4&5&6&7&8&9&10&11&12&13&14&15\\f&d&c&b&e&a&g&2&7&6&5&4&8&1&3\end{pmatrix}$

I, 14≡VD by $s\equiv\begin{pmatrix}1&2&3&4&5&6&7&8&9&10&11&12&13&14&15\\d&e&c&a&b&g&f&4&1&5&7&8&3&6&2\end{pmatrix}$

I, 15$\equiv$IIF by $s\equiv\begin{pmatrix}1&2&3&4&5&6&7&8&9&10&11&12&13&14&15\\ d&f&a&c&g&b&e&1&6&7&3&2&5&8&4\end{pmatrix}$

I, 16$\equiv$IIID by $s\equiv\begin{pmatrix}1&2&3&4&5&6&7&8&9&10&11&12&13&14&15\\ a&d&b&f&e&g&c&1&2&5&6&3&4&8&7\end{pmatrix}$

I, 17$\equiv$VB by $s\equiv\begin{pmatrix}1&2&3&4&5&6&7&8&9&10&11&12&13&14&15\\ f&c&d&g&a&b&e&1&3&7&2&4&8&5&6\end{pmatrix}$

I, 18$\equiv$IIB by $s\equiv\begin{pmatrix}1&2&3&4&5&6&7&8&9&10&11&12&13&14&15\\ a&f&b&g&e&d&c&4&3&5&6&2&1&8&7\end{pmatrix}$

I, 19$\equiv$V1A by $s\equiv\begin{pmatrix}1&2&3&4&5&6&7&8&9&10&11&12&13&14&15\\ a&d&g&b&e&f&c&5&8&1&2&4&3&7&6\end{pmatrix}$

I, 20$\equiv$VIC by $s\equiv\begin{pmatrix}1&2&3&4&5&6&7&8&9&10&11&12&13&14&15\\ b&c&e&d&a&f&g&4&8&2&3&6&5&1&7\end{pmatrix}$

I, 21$\equiv$IC by $s\equiv\begin{pmatrix}1&2&3&4&5&6&7&8&9&10&11&12&13&14&15\\ g&d&e&b&c&a&f&8&1&4&5&6&3&7&2\end{pmatrix}$

I, 22$\equiv$IB by $s\equiv\begin{pmatrix}1&2&3&4&5&6&7&8&9&10&11&12&13&14&15\\ d&c&e&g&f&a&b&6&1&7&5&3&2&8&4\end{pmatrix}$

II, 1$\equiv$10 by $s\equiv\begin{pmatrix}1&2&3&4&5&6&7&8&9&10&11&12&13&14&15\\ 6&d&1&f&3&c&7&b&8&e&4&g&a&2&5\end{pmatrix}$

II, 2$\equiv$11 by $s\equiv\begin{pmatrix}1&2&3&4&5&6&7&8&9&10&11&12&13&14&15\\ a&f&5&1&2&7&8&d&e&g&b&c&6&4&3\end{pmatrix}$

II, 3$\equiv$ 9 by $s\equiv\begin{pmatrix}1&2&3&4&5&6&7&8&9&10&11&12&13&14&15\\ d&6&1&a&g&5&2&8&4&b&e&c&7&f&3\end{pmatrix}$

II, 4$\equiv$15 by $s\equiv\begin{pmatrix}1&2&3&4&5&6&7&8&9&10&11&12&13&14&15\\ a&f&e&d&c&g&b&8&7&2&1&3&4&6&5\end{pmatrix}$

II, 5$\equiv$13 by $s\equiv\begin{pmatrix}1&2&3&4&5&6&7&8&9&10&11&12&13&14&15\\ f&a&d&g&e&b&c&2&8&1&7&5&4&6&3\end{pmatrix}$

II, 6$\equiv$12 by $s\equiv\begin{pmatrix}1&2&3&4&5&6&7&8&9&10&11&12&13&14&15\\ a&d&e&g&c&f&b&1&2&6&5&4&3&8&7\end{pmatrix}$

II, 7$\equiv$V4γ1 by $s\equiv\begin{pmatrix}1&2&3&4&5&6&7&8&9&10&11&12&13&14&15\\ c_2&b_2&b_3&d_1&B&c_3&a_1&a_2&C&A&d_2&b_1&d_3&c_1&a_3\end{pmatrix}$

II, 8$\equiv$VI$2_4\gamma$ by $s\equiv\begin{pmatrix}1&2&3&4&5&6&7&8&9&10&11&12&13&14&15\\ b_6&b_3&a_5&b_2&a_1&a_6&C&a_3&a_2&A&a_4&B&b_5&b_4&b_1\end{pmatrix}$

II, 9$\equiv$VI$3_3\delta$ by $s\equiv\begin{pmatrix}1&2&3&4&5&6&7&8&9&10&11&12&13&14&15\\ a_4&a_1&a_2&a_6&a_5&b_4&B&b_1&b_6&A&a_3&C&b_2&b_3&b_5\end{pmatrix}$

II, 10$\equiv$14 by $s\equiv\begin{pmatrix}1&2&3&4&5&6&7&8&9&10&11&12&13&14&15\\ 1&3&b&4&c&a&2&g&8&5&e&f&7&6&d\end{pmatrix}$

II, 11$\equiv$ 8 by $s\equiv\begin{pmatrix}1&2&3&4&5&6&7&8&9&10&11&12&13&14&15\\ a&d&b&e&f&g&c&7&6&4&3&1&2&5&8\end{pmatrix}$

II, 12$\equiv$25 by $s\equiv\begin{pmatrix}1&2&3&4&5&6&7&8&9&10&11&12&13&14&15\\ f&1&a&g&c&8&e&4&3&b&d&7&2&5&6\end{pmatrix}$

III, 1$\equiv$V4α2 by $s\equiv\begin{pmatrix}1&2&3&4&5&6&7&8&9&10&11&12&13&14&15\\ C&A&B&a_1&c_1&b_1&d_1&b_2&d_2&a_2&c_2&a_3&c_3&b_3&d_3\end{pmatrix}$

III, 2$\equiv$VI$1_3\beta$ by $s\equiv\begin{pmatrix}1&2&3&4&5&6&7&8&9&10&11&12&13&14&15\\ A&B&C&a_5&b_5&a_6&b_6&b_2&a_2&b_1&a_1&b_3&a_3&b_4&a_4\end{pmatrix}$

III, 3$\equiv$V1γ by $s\equiv\begin{pmatrix}1&2&3&4&5&6&7&8&9&10&11&12&13&14&15\\ A&B&C&d_1&c_1&b_1&a_1&a_3&b_3&c_3&d_3&c_2&d_2&a_2&b_2\end{pmatrix}$

III, 4$\equiv$28 by $s\equiv\begin{pmatrix}1&2&3&4&5&6&7&8&9&10&11&12&13&14&15\\ d&3&1&5&b&6&g&7&4&f&e&2&a&8&c\end{pmatrix}$

III, 5$\equiv$ 6 by $s\equiv\begin{pmatrix}1&2&3&4&5&6&7&8&9&10&11&12&13&14&15\\ g&1&8&3&6&c&d&e&b&2&7&5&f&4&a\end{pmatrix}$

III, 6$\equiv$ 4 by $s\equiv\begin{pmatrix}1&2&3&4&5&6&7&8&9&10&11&12&13&14&15\\ g&8&1&d&c&4&5&6&3&b&e&2&7&f&a\end{pmatrix}$

III, 7$\equiv$ 7 by $s\equiv\begin{pmatrix}1&2&3&4&5&6&7&8&9&10&11&12&13&14&15\\ g&8&1&5&4&c&d&f&a&2&7&3&e&6&b\end{pmatrix}$

III, 8$\equiv$VI$2_4\alpha$ by $s\equiv\begin{pmatrix}1&2&3&4&5&6&7&8&9&10&11&12&13&14&15\\ A&B&C&a_4&b_4&a_3&b_3&b_1&a_1&b_2&a_2&b_6&b_5&a_6&a_5\end{pmatrix}$

III, 9$\equiv$II1_6 by $s\equiv\begin{pmatrix}1&2&3&4&5&6&7&8&9&10&11&12&13&14&15\\ a_1&b_1&c_1&c_3&c_2&d_3&e_2&a_2&a_3&e_1&d_1&d_2&b_2&e_3&b_3\end{pmatrix}$

III, 10$\equiv$VI$2_4\epsilon$ by $s\equiv\begin{pmatrix}1&2&3&4&5&6&7&8&9&10&11&12&13&14&15\\ A&B&C&a_4&b_4&a_3&b_3&b_1&a_1&b_2&a_2&b_6&b_5&a_6&a_5\end{pmatrix}$

III, 11=a new groupless system. (Reference number 34.)

III, 12$\equiv$25 by $s\equiv\begin{pmatrix}1&2&3&4&5&6&7&8&9&10&11&12&13&14&15\\ b&d&5&c&4&8&7&2&g&a&6&e&3&f&1\end{pmatrix}$

III, 13$\equiv$ 5 by $s\equiv\begin{pmatrix}1&2&3&4&5&6&7&8&9&10&11&12&13&14&15\\ g&1&8&7&2&b&e&a&f&4&5&6&3&d&c\end{pmatrix}$

III, 14≡21 by $s \equiv \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\ e & b & 1 & a & g & 4 & 2 & f & d & 8 & 5 & c & 3 & 7 & 6 \end{pmatrix}$

III, 15≡20 by $s \equiv \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\ a & c & 2 & 3 & 5 & 6 & b & 8 & 7 & d & 4 & 1 & f & e & g \end{pmatrix}$

III, 16≡19 by $s \equiv \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\ 1 & d & 3 & b & 4 & 5 & 7 & e & c & f & 8 & a & 6 & 2 & g \end{pmatrix}$

III, 17≡26 by $s \equiv \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\ 4 & d & 7 & g & c & 6 & 8 & 3 & f & 1 & b & e & 2 & 5 & a \end{pmatrix}$

III, 18≡32 by $s \equiv \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\ 7 & e & 6 & f & 3 & d & 4 & 5 & 1 & b & c & g & 2 & a & 8 \end{pmatrix}$

III, 19≡ 1 by $s \equiv \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\ 3 & b & 4 & e & 1 & f & 7 & 2 & 5 & d & a & 6 & g & 8 & c \end{pmatrix}$

III, 20≡ 3 by $s \equiv \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\ 2 & c & 4 & e & 1 & d & 3 & 7 & g & 6 & a & 8 & 5 & f & b \end{pmatrix}$

III, 21≡27 by $s \equiv \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\ 5 & f & 6 & c & 2 & g & 4 & 3 & a & 7 & 1 & d & b & e & 8 \end{pmatrix}$

III, 22≡22 by $s \equiv \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\ f & b & 6 & 8 & 5 & e & d & 4 & 3 & 1 & a & 7 & 2 & c & g \end{pmatrix}$

III, 23=a new groupless system. (Reference number 35.)

III, 24≡ 2 by $s \equiv \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\ b & 3 & 4 & c & 1 & f & 7 & 2 & 5 & d & a & 6 & 8 & g & e \end{pmatrix}$

III, 25≡V4γ2 by $s \equiv \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\ B & C & A & b_1 & a_1 & d_3 & c_2 & d_1 & c_3 & b_2 & a_2 & b_3 & a_3 & d_2 & c_1 \end{pmatrix}$

III, 26≡31 by $s \equiv \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\ 5 & f & 6 & b & d & 8 & e & a & 3 & 1 & 7 & 2 & c & 4 & g \end{pmatrix}$

IV, 1≡1A by $s \equiv \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\ g & f & b & a & e & c & d & 4 & 5 & 6 & 3 & 1 & 8 & 7 & 2 \end{pmatrix}$

V, 1≡V4β2 by $s \equiv \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\ e_3 & e_1 & a_1 & C & a_2 & a_3 & e_2 & d_1 & B & A & d_3 & d_2 & b_1 & b_2 & b_3 \end{pmatrix}$

V, 2≡V4δ1 by $s \equiv \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\ e_1 & e_2 & a_2 & C & a_3 & a_1 & e_3 & d_1 & A & B & d_3 & d_2 & b_3 & b_1 & b_2 \end{pmatrix}$

V, 3≡I14β1 by $s \equiv \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\ e_1 & C & a_3 & e_2 & a_2 & a_1 & e_3 & d_1 & A & b_1 & d_3 & B & d_2 & b_3 & b_2 \end{pmatrix}$

VI, 1≡23 by $s \equiv \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\ f & 6 & b & 7 & 2 & a & 1 & c & g & 3 & 4 & d & e & 8 & 5 \end{pmatrix}$

VI, 2=a new groupless system. (Reference number 36.)

VI, 3≡18 by $s \equiv \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\ d & 8 & c & e & f & 5 & b & 1 & a & g & 6 & 7 & 2 & 4 & 3 \end{pmatrix}$

VI, 4≡33 by $s \equiv \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\ f & 5 & 3 & g & c & 4 & 2 & b & 6 & d & e & a & 1 & 8 & 7 \end{pmatrix}$

VI, 5≡16 by $s \equiv \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\ d_3 & e_1 & c_3 & b_2 & a_1 & d_2 & d_1 & a_2 & b_1 & a_3 & e_3 & c_1 & c_2 & e_2 & b_3 \end{pmatrix}$

VI, 6≡17 by $s \equiv \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\ e & 2 & 5 & g & f & b & 7 & 4 & a & c & 1 & d & 8 & 6 & 3 \end{pmatrix}$

VI, 7≡30 by $s \equiv \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\ b & 2 & e & 8 & 7 & 3 & g & 5 & d & c & 4 & a & 6 & f & 1 \end{pmatrix}$

VI, 8≡29 by $s \equiv \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\ d & 1 & 3 & b & 5 & 4 & 7 & 2 & a & 6 & g & f & e & 8 & c \end{pmatrix}$

VI, 9≡VI$3_3\alpha$ by $s \equiv \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\ a_3 & b_4 & b_5 & a_4 & A & B & b_3 & b_6 & a_1 & b_1 & C & b_2 & a_6 & a_2 & a_5 \end{pmatrix}$

VI, 10≡VI$3_4\gamma$ by $s \equiv \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\ B & a_1 & a_5 & b_2 & a_6 & a_4 & b_4 & b_6 & a_2 & b_5 & b_1 & a_3 & b_3 & A & C \end{pmatrix}$

VI, 11≡III$_7$ by $s \equiv \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\ e_3 & e_1 & a_3 & d_2 & b_2 & c_3 & b_3 & c_2 & a_2 & d_3 & b_1 & d_1 & a_1 & c_1 & e_2 \end{pmatrix}$

VI, 12≡12 by $s \equiv \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\ f & 1 & a & 6 & 2 & 5 & 7 & b & 8 & 4 & 3 & g & c & d & e \end{pmatrix}$

VI, 13≡III$_3$ by $s \equiv \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\ d_1 & a_1 & c_2 & d_2 & a_3 & d_3 & a_2 & c_3 & b_3 & e_2 & e_1 & c_1 & b_2 & b_1 & e_3 \end{pmatrix}$

VI, 14≡VI$3_3\gamma$ by $s \equiv \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\ a_3 & b_4 & b_5 & b_3 & B & A & a_4 & a_2 & b_6 & C & b_1 & a_1 & a_6 & b_2 & a_5 \end{pmatrix}$

VI, 15≡III$_2$ by $s \equiv \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\ d_3 & a_3 & e_3 & a_2 & a_1 & d_1 & d_2 & c_3 & e_1 & b_3 & c_2 & c_1 & e_2 & b_2 & b_1 \end{pmatrix}$

VII, ≡VI1 by $s \equiv \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\ a & f & 4 & 1 & 3 & 7 & g & 2 & c & 5 & 8 & d & e & 6 & b \end{pmatrix}$

In Part 4 of this paper Mr. Cole distinguishes four varieties of interlacing of duads in a system, namely, the single tetrad or oktad, the triple tetrad, the hexad, and the dodekad. These types correspond to what we have designated in Part 3 as two-column or contracted indices. The triple tetrad, the oktad, the hexad, and the dodekad corresponding, respectively, to the indices 2^3; 2, 4; 3^2; 6.

Since these four types of interlacing form the basis for Mr. Cole's derivation of the 80 systems, it seemed probable that the two-column indices might furnish a sufficient and unique characterization for a triad system.

The two-column indices for each of the 80 systems were, therefore, determined and are exhibited in the following Table 2:

TABLE 2.

System.	2^3	2, 4	3^2	6	System.	2^3	2, 4	3^2	6	System.	2^3	2, 4	3^2	6
IIIA	105	0	0	0	4	4	27	7	67	31	1	18	18	68
IIIB	57	48	0	0	7	4	24	8	69	24	1	18	14	72
IIIC	49	24	0	32	IB	3	42	7	53	20	1	18	12	74
IIID	49	0	0	56	VIA	3	42	4	56	19	1	15	12	77
IIE	29	60	0	16	12	3	36	7	59	36 [Cole]	1	12	11	81
IVA	29	60	0	16	VI, $2_4\delta$	3	33	6	63	IA	0	42	7	56
IVB	25	36	0	44	13	3	33	5	64	V, $1\beta 1$	0	30	10	65
VD	25	36	0	44	VI, $2_4\gamma$	3	21	6	75	I, 2	0	30	5	70
VA	21	36	0	48	6	2	30	10	63	17	0	27	12	66
VC	21	36	0	48	3	2	27	13	63	22	0	24	12	69
VB	21	12	0	72	1	2	24	9	70	VI, $3_4\gamma$	0	24	7	74
IIA	15	66	0	24	VI, $1_3\beta$	2	24	4	75	II, 1_7	0	21	15	69
VID	13	57	1	31	26	2	21	15	67	34 [Cole]	0	21	14	70
IID	13	54	0	38	28	2	21	13	69	V, $4\beta 2$	0	21	7	77
IIC	13	54	0	38	V, 1γ	2	12	12	79	V, $4\delta 1$	0	21	7	77
IIB	11	42	0	52	25	2	10	30	63	VI, $3_3\alpha$	0	18	21	66
IIF	11	42	0	52	V, $4\gamma 2$	1	36	16	52	23	0	18	14	73
15	11	36	4	54	14	1	36	9	59	VI, $3_3\gamma$	0	18	13	74
9	9	33	4	59	8	1	33	8	63	33	0	18	11	76
V, $4\gamma 1$	7	36	4	58	5	1	27	9	68	II, 1_2	0	18	9	78
VIB	5	54	4	42	27	1	24	15	65	18	0	15	16	74
IC	5	45	7	48	35 [Cole]	1	24	12	68	29	0	15	16	74
V, $4\alpha 1$	5	24	10	66	32	1	24	12	68	16	0	15	14	76
VIC	4	48	4	49	VI, $2_4\alpha$	1	21	20	63	30	0	15	11	79
11	4	45	5	51	21	1	21	10	73	II, 1_3	0	6	15	84
10	4	42	3	56	VI, $2_4\epsilon$	1	21	8	75	VII	0	0	15	90
VI, $3_3\delta$	4	42	1	58	2	1	21	7	76					

This table shows that the two-column indices suffice to establish the noncongruency of 72 of the 80 systems, but fail to distinguish uniquely the remaining 8 systems. The systems not uniquely determined by their two-column indices consist of the five pairs of headed systems IIC, IID; IIB, IIF; IIE, IVA; IVB, VD; VA, VC; one pair with a group but no head V4δ1, V4β2; and two pairs of groupless systems 18, 29; 32, 35 [Cole].

Perfect discrimination is possible by the use of a double entry table, 15 by 15, showing not merely the number, but the exact distribution of triple tetrads, oktads, hexads, and dodekads in each of the eight pairs of apparently duplicate systems.

The investigation for one of these pairs of apparently duplicate systems, IIF and IIB, is given below in some detail.

The system IIF is arranged in a 15-by-7 array. Each element heads one column; below it are placed the seven duads of elements that occur with it in triads of the system. The two-column indices for the 105 pairs of columns are determined, and we find that the indices 2^3; 2, 4; 3^2; 6 belong, respectively, to 11, 42, 0, and 52 pairs of columns.

To exhibit more evidently the types of interlacings existing amongst the duads in the 105 pairs of columns, we now arrange the two-column indices for the system IIF in the following 15-by-15 array of Table 3.

TABLE 3.

	a	*b*	*c*	*d*	*e*	*f*	*g*	1	2	3	4	5	6	7	8
a	......	2	2	2	4	2	4	6	6	6	6	6	6	6	6
b	2	......	2	4	4	4	4	6	4	6	4	6	6	6	6
c	2	2	......	4	4	4	4	6	6	6	6	4	6	6	4
d	2	4	4	......	4	2	4	4	6	4	6	6	4	4	6
e	4	4	4	4	......	4	2	6	6	6	6	4	6	6	4
f	2	4	4	2	4	......	4	6	6	6	6	6	6	6	6
g	4	4	4	4	2	4	......	6	4	6	4	6	6	6	6
1	6	6	6	4	6	6	6	......	4	2	4	4	6	6	4
2	6	4	6	6	6	6	4	4	......	4	2	6	4	4	6
3	6	6	6	4	6	6	6	2	4	......	4	4	6	6	4
4	6	4	6	6	6	6	4	4	2	4	......	6	4	4	6
5	6	6	4	6	4	6	6	4	6	4	6	......	4	4	2
6	6	6	6	4	6	6	6	6	4	6	4	4	......	2	4
7	6	6	6	4	6	6	6	6	4	6	4	4	2	......	4
8	6	6	4	6	4	6	6	4	6	4	6	2	4	4	......

In this table the two-column indices 2^3; 2, 4; 3^2; 6 are replaced, for the sake of brevity, by the single figures 2, 4, 3, 6, respectively. The figure placed at the intersection of any row with any column shows the index for the pair of columns formed from the two elements which lead the row and head the column, respectively. For example, in Table 3, the indices for the pairs of columns *bc*, *bd*, *b1* are 2^3; 2, 4; 6, respectively.

In the 15 by 7 rectangular array for the system IIF, the column headed by the element *a* may be united with each of the columns headed by one of the remaining 14 elements to form 14 pairs of columns. The figures tabulated under the element *a* in table 3 shows that of these 14 pairs of columns there are 4 pairs with the index 2^3; 2 pairs with the index 2, 4; and 8 pairs with the index 6. Interpreted in terms of interlacings these two-column indices place in evidence the fact that the duads of column *a* are united with the duads in the remaining 14 columns by 4 triple tetrads, 2 oktads, and 8 dodekads. Similar results for each of the 15 elements are briefly summed up in the following Table 4, which we shall designate as the table of interlacings for the system IIF.

TABLE 4.—*Interlacings for system IIF.*

	a	*b*	*c*	*d*	*e*	*f*	*g*	1	2	3	4	5	6	7	8
Index 2^3........	4	2	2	2	1	2	1	1	1	1	1	1	1	1	1
Index 2, 4.......	2	6	6	8	7	4	7	5	6	5	6	6	5	5	6
Index 6.........	8	6	6	4	6	8	6	8	7	8	7	7	8	8	7

Since only those elements whose duads are similarly interlaced through a system may belong to the same set of transitive elements, Table 4 shows clearly that the possible sets of transitive elements for the system IIF are *a*, *bc*, *d*, *eg*, *f*, 1367, 2458. A fact in exact accordance with the results obtained previously in the examination of this system by the method of trains and also by the method of the three-column indices.

The interlacings for the system IIB are exhibited in Table 5.

TABLE 5.—*Interlacings for the system IIB.*

	a	*b*	*c*	*d*	*e*	*f*	*g*	1	2	3	4	5	6	7	8
Index 2^3........	4	2	2	2	1	2	1	1	1	1	1	1	1	1	1
Index 2, 4.......	2	6	6	6	9	6	5	5	6	5	6	6	5	5	6
Index 6.........	8	6	6	6	4	6	8	8	7	8	7	7	8	8	7

A comparison of Table 4 with Table 5 demonstrates conclusively the noncongruency of the systems IIF and IIB. Table 4 shows that there is a column *d* in the system IIF the interlacings of whose duads with the duads of other columns in IIF are represented by the numbers 2, 8, and 4, corresponding, respectively, to two triple tetrads, eight oktads, and four dodekads. Table 5 shows no column in IIB with similar interlacings, therefore these two systems IIB and IIF are certainly incongruent. The reader will observe other distinctive columns in these two tables.

The tables of interlacings for the remaining seven apparently duplicate pairs of systems are adjoined below and establish the noncongruency of the two systems in each pair.

Table of interlacings for the system IIC.

	a	*b*	*c*	*d*	*e*	*f*	*g*	1	2	3	4	5	6	7	8
Index 2^3......	4	2	2	4	1	2	1	1	1	1	1	1	2	2	1
Index 2, 4.....	6	6	8	8	7	8	9	6	8	6	8	8	6	6	8
Index 6.......	4	6	4	2	6	4	4	7	5	7	5	5	6	6	5

Table of interlacings for the system IID.

	a	*b*	*c*	*d*	*e*	*f*	*g*	1	2	3	4	5	6	7	8
Index 2^3......	4	2	2	2	3	2	1	1	1	1	1	2	1	1	2
Index 2, 4.....	6	6	8	6	9	8	9	6	8	6	8	5	9	9	5
Index 6.......	4	6	4	6	2	4	4	7	5	7	5	7	4	4	7

The system IID contains no column similar to the column *d* of IIC, therefore IIC and IID are noncongruent.

Table of interlacings for the system IIE.

	a	*b*	*c*	*d*	*e*	*f*	*g*	1	2	3	4	5	6	7	8
Index 2^3......	8	4	6	4	3	4	3	2	4	2	4	3	4	4	3
Index 2, 4.....	6	10	8	8	9	8	9	8	6	8	6	9	8	8	9
Index 6.......	0	0	0	2	2	2	2	4	4	4	4	2	2	2	2

Table of interlacings for the system IVA.

	a	*b*	*c*	*d*	*e*	*f*	*g*	1	2	3	4	5	6	7	8
Index 2^3......	6	6	2	4	4	4	4	4	4	4	4	3	3	3	3
Index 2, 4.....	8	8	12	6	6	6	6	6	6	6	6	11	11	11	11
Index 6.......	0	0	0	4	4	4	4	4	4	4	4	0	0	0	0

The system IVA contains no column similar to the column *a* of IIE, hence IIE and IVA are noncongruent.

Table of interlacings for the system IVB.

	a	*b*	*c*	*d*	*e*	*f*	*g*	1	2	3	4	5	6	7	8
Index 2^3......	10	2	2	4	4	4	4	4	4	4	4	3	3	3	3
Index 2, 4.....	0	8	4	6	2	6	2	5	5	5	5	6	6	6	6
Index 6.......	4	4	8	4	8	4	8	5	5	5	5	5	5	5	5

Table of interlacings for the system VD.

	a	*b*	*c*	*d*	*e*	*f*	*g*	1	2	3	4	5	6	7	8
Index 2^3......	2	2	2	3	7	3	3	3	3	4	4	3	4	4	3
Index 2, 4.....	8	8	8	7	3	7	7	6	6	6	6	6	6	6	6
Index 6.......	4	4	4	4	4	4	4	5	5	4	4	5	4	4	5

No column in VD is similar to the column *a* of IVB, hence IVB and VD are noncongruent.

Table of interlacings for the system VA.

	a	*b*	*c*	*d*	*e*	*f*	*g*	1	2	3	4	5	6	7	8
Index 2^3......	2	2	2	3	3	3	3	3	3	3	3	3	3	3	3
Index 2, 4.....	12	12	12	3	3	3	3	3	3	3	3	3	3	3	3
Index 6.......	0	0	0	8	8	8	8	8	8	8	8	8	8	8	8

Table of interlacings for the system VC.

	a	b	c	d	e	f	g	1	2	3	4	5	6	7	8
Index 2^3.......	2	2	2	3	3	3	3	3	3	3	3	3	3	3	3
Index 2, 4......	1	8	8	11	11	11	11	5	5	5	5	5	5	5	5
Index 6........	8	4	4	0	0	0	0	6	6	6	6	6	6	6	6

No column of VA agrees with any column in VC, therefore VA and VC are noncongruent.

Table of interlacings for the system V4δ1.

	a	b	c	d	e	f	g	1	2	3	4	5	6	7	8
Index 2, 4......	3	0	3	2	2	2	2	2	2	4	4	4	4	4	4
Index 3^2.......	0	1	1	1	1	1	2	2	2	1	1	1	0	0	0
Index 6........	11	13	10	11	11	11	10	10	10	9	9	9	10	10	10

Table of interlacings for the system V4β2.

	a	b	c	d	e	f	g	1	2	3	4	5	6	7	8
Index 2, 4......	0	3	0	2	2	2	2	2	2	5	5	5	4	4	4
Index 3^2.......	7	0	4	0	0	0	1	1	1	1	1	1	1	1	1
Index 6........	7	11	10	12	12	12	11	11	11	8	8	8	9	9	9

The two systems V4δ1 and V4β2 show dissimilar columns *a*, and therefore are noncongruent.

Table of interlacings for the system 18.

	a	b	c	d	e	f	g	1	2	3	4	5	6	7	8
Index 2, 4......	0	3	3	4	3	2	0	0	2	2	2	2	2	3	2
Index 3^2.......	1	3	3	4	1	1	3	2	2	3	1	3	2	1	2
Index 6........	13	8	8	6	10	11	11	12	10	9	11	9	10	10	10

Table of interlacings for the system 29.

	a	b	c	d	e	f	g	1	2	3	4	5	6	7	8
Index 2, 4......	2	1	2	3	2	2	2	1	1	2	5	3	0	3	1
Index 3^2.......	1	1	1	2	4	2	0	3	3	4	3	3	3	1	1
Index 6........	11	12	11	9	8	10	12	10	10	8	6	8	11	10	12

The tables of interlacings for the systems 18 and 29 show many dissimilar columns, therefore these systems are noncongruent.

Table of interlacings for the system 32.

	a	b	c	d	e	f	g	1	2	3	4	5	6	7	8
Index 2^3.........	0	0	0	0	1	0	0	0	0	0	0	0	0	1	0
Index 2, 4........	3	4	5	3	3	5	5	2	3	3	4	3	0	2	3
Index 3^2.........	2	1	1	2	3	2	0	1	2	4	2	1	3	0	0
Index 6..........	9	9	8	9	7	7	9	11	9	7	8	10	11	11	11

Table of interlacings for the system 35 [Cole].

	a	b	c	d	e	f	g	1	2	3	4	5	6	7	8
Index 2^3.........	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0
Index 2, 4........	4	3	3	4	4	3	1	3	2	2	3	4	2	4	5
Index 3^2.........	1	4	2	1	1	1	1	0	3	2	5	0	2	1	1
Index 6..........	9	7	9	9	9	10	12	10	8	10	6	10	10	9	8

The column e of 32 has no duplicate in 35; hence the two systems are noncongruent.

We have derived, then, in Part 5 a new method of comparison for triad systems by means of the two-column indices and the table of interlacings for the system.

This method of comparison, since it naturally yields at least a partial, in some cases a complete, separation into sets of transitive elements for the system, will also facilitate the determination of the group belonging to the system.

CONCLUSION.

Looking toward the census of triad systems in more than 15 elements, we have in the foregoing memoir four modes of classification which would be applicable to the construction and comparison of systems. Of these we venture to express the belief that the method of indices will be found most convenient for comparisons, while for construction there is no doubt that a group, where one can be prescribed, is the most direct auxiliary. Any exhaustive census, certainly for 31 or more elements, is out of the question in finite time; but systems admitting, for example, certain cyclic groups are not numerous nor difficult of construction, the method of indices showing very quickly their noncongruency. In the present state of the theory the most desirable forward step would be a demonstration that some one of these methods is (or is not) a sufficient means of proving congruency for triad systems of any number of elements above 15.

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TABLES OF MINOR PLANETS DISCOVERED BY JAMES C. WATSON.

PART II.

ON v. ZEIPEL'S THEORY OF THE PERTURBATIONS OF THE MINOR PLANETS OF THE HECUBA GROUP.

BY

ARMIN O. LEUSCHNER, ANNA ESTELLE GLANCY, AND
SOPHIA H. LEVY.

CONTENTS.

PREFACE.

Part I of "Tables of Minor Planets Discovered by James C. Watson," containing tables for 12 of the 22 Watson planets, was published in 1910 in the Memoirs of the National Academy of Sciences, Volume X, Seventh Memoir, with a preface by Simon Newcomb, in which he gives an account of the early history of the investigations of the perturbations of the Watson planets under the auspices of the Board of Trustees of the Watson Fund.

In the introduction to Part I [1] reference is made to the Watson planets of the *Hecuba* group, for which it was found necessary to construct special tables on the plan of Bohlin's tables for the group 1/3. A comparison of these tables with similar tables by v. Zeipel remained to be made before applying either of them to the development of perturbations of planets of the *Hecuba* group. This comparison was completed in 1913 with the assistance of Miss A. Estelle Glancy and Miss Sophia H. Levy, with the results set forth in the following pages.

Publication of these results was delayed, partly because it seemed desirable to verify the tables by application to a number of planets and partly on account of interruptions caused in recent years by war conditions. Miss Glancy, in particular, had undertaken to test the accuracy of our tables, which we had applied to v. Zeipel's example, (10) *Hygiea*, by further investigations on this example after joining the Observatorio Nacional at Córdoba in 1913. This test has now been completed with highly satisfactory results. The tables have also been successfully applied to the Watson planets of the *Hecuba* group, including (175) *Andromache*, which, on account of unusually large perturbations and other unfavorable conditions, forms so far the most striking example of the applicability of the Bohlin-v. Zeipel method and of our revised tables for the *Hecuba* group.

The plan of work included conferences, in which Miss Glancy and Miss Levy took a leading part, for the discussion of the Bohlin-v. Zeipel method, involving verification of all mathematical developments and formulation of plans for the construction of tables, and, after the appearance of v. Zeipel's tables, for the comparison of v. Zeipel's original, and our revised tables. The numerical work was carried out by Miss Glancy and Miss Levy, who have also contributed very largely to the theoretical part of the work, and have prepared the principal details of the manuscript.

To avoid confusion v. Zeipel's notation and method of procedure have been followed throughout in completing our tables for the *Hecuba* group, which were well under way when v. Zeipel's memoir appeared.

To aid computers in the use of the formulæ and of the revised tables, Miss Glancy has prepared detailed directions illustrated by an application to (10) *Hygiea*, the example first chosen by v. Zeipel. These are contained in the first section of the present memoir.

Miss Glancy's contributions to this investigation and her work on *(10) Hygiea* were accepted by the University of California in partial fulfillment of the requirements for the degree of doctor of philosophy.

Miss Levy's contributions and her work on *(175) Andromache* were similarly accepted for the same degree.

It seems highly desirable to make the tables for the development of the perturbations of minor planets of the *Hecuba* group at once available to astronomers. They are therefore published herewith, in advance of the perturbations and tables of the remaining Watson planets, as Part II of "Tables of Minor Planets Discovered by James C. Watson." One or two parts, which are to follow, will contain all the numerical results for the perturbations and tables of Watson planets not published in Part I (1910).

This memoir is presented in two sections. The first section, entitled "Formulæ and Tables for the *Hecuba* Group, according to the Theory of Bohlin-v. Zeipel, and an Example of their

[1] Pp. 200–201.

Use," contains a collection of the formulæ to be used for any planet of the *Hecuba* group, the general tables of the perturbations which must be employed, and a more complete application of the formulæ and the revised tables to the planet (10) *Hygiea*, than v. Zeipel gives. The second and more extensive section, entitled "Tables for the Determination of the Perturbations of the *Hecuba* Group of Minor Planets," concerns the construction of the tables and their discussion with reference to the corresponding tables by v. Zeipel. It forms the preliminary part of the investigation but is presented last as supplementary to the final results given in the first section.

In the second section the tabular values which differ from the corresponding numbers in v. Zeipel's tables are placed in brackets. The general Tables XXXV, XXXVIII, XLIII, LIV, LVi, LVii, LVI, LVII, of the second section, which, in order, are required to compute the perturbations of any planet of the *Hecuba* group, are repeated without brackets at the end of the first section as Tables A, B, C, D, E_1, E_2, F, G, so that the first section is complete in itself for use in developing the perturbations of any planet of this group without the necessity of reference to the second section.

A general account of the investigations of the perturbations of the Watson planets was presented to the Academy on April 16, 1916, and is published in the "Proceedings of the National Academy of Sciences," Volume 4, No. 12, March, 1919.

Armin O Leuschner.

Washington, D. C., *1918, December.*

TABLES OF MINOR PLANETS DISCOVERED BY JAMES C. WATSON.

By Armin O. Leuschner, Anna Estelle Glancy, and Sophia H. Levy.

INTRODUCTION.

Those planets whose mean daily motions are approximately 600″ are classed with the planet *Hecuba*, or, in the group for which

$$\mu = \frac{n'}{n} = \tfrac{1}{2}(1-w)$$

where n' and n are the mean daily motions of Jupiter and the planet, respectively, and w is a small quantity.

Among the minor planets discovered by James C. Watson there are several of this type. In the course of the general program of determining the perturbations of the Watson asteroids, there arose the necessity of computing special tables for the *Hecuba* group in preparation for the application of Bohlin's method to individual planets.

General tables for the group ½ were in the process of construction, under the direction of Professor Leuschner,[1] according to the method of Bohlin,[2] when tables for this group were published by H. v. Zeipel.[3] The computers, Dr. Sidney D. Townley and Miss Adelaide M. Hobe, made a comparison of their tables with those of v. Zeipel and found certain discrepancies Because of this fact the completion of the tables for the *Hecuba* group was deferred. These discrepancies have been explained, as a result of a careful investigation, and the tables have been completed by Miss A. Estelle Glancy and Miss Sophia H. Levy, under the direction of Professor Leuschner.

In the completion of the tables, v. Zeipel's method and order of procedure have generally been followed. There are numerous discrepancies between our tables and v. Zeipel's. As far as possible, with the aid of the original manuscript, kindly forwarded by the author, we have traced the source of these disagreements. In some of the more complicated functions it was not possible to do so, and these discrepancies remain unexplained. Our own results, however, are substantiated by the employment of independent developments. Further, where we found terms omitted which were of the same order as those which were included, we frequently extended the tables. In this connection, it is pertinent to remark that it is very difficult to set up a consistent criterion for the omission of terms. With the exception of a few scattered negligible terms, our tables are published in full. They contain terms which may ordinarily be omitted, yet their numerical magnitudes depend upon the elements of the particular planet under consideration, and their use is left to the computer's judgment. Many of them are incomplete, i. e., the tabulated coefficients do not necessarily include all the terms of a given degree in the eccentricities or mutual inclination or of the small quantity w, which depends upon the difference between the planet's and twice *Jupiter's* mean motion. In other words, the coefficients may not contain all the terms of a given degree having the factors

$$w^s,\ \eta^p,\ \eta'^q,\ j^{2t}$$

which are defined on page 12. But, assuming certain numerical limits for the fundamental auxiliary functions, the coefficients are of this magnitude. The value of the additional terms will be shown best in an application of our tables to the same planet for which v. Zeipel computed the perturbations.

Unless stated otherwise, the references to Bohlin refer to the French edition and are designated by *B*. References to v. Zeipel are designated by *Z*.

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[1] Memoirs of the National Academy of Sciences, Vol. X, Seventh Memoir, p. 200.

[2] Formeln und Tafeln zur gruppenweisen Berechnung der allgemeinen Störungen benachbarter Planeten (Upsala, 1896). Sur le Développement des Perturbations Planétaires (Stockholm, 1902).

[3] Angenäherte Jupiterstörungen für die Hecuba-Gruppe (St. Pétersbourg, 1902).

I. FORMULAE AND TABLES FOR THE HECUBA GROUP, ACCORDING TO THE THEORY OF BOHLIN-v. ZEIPEL, AND AN EXAMPLE OF THEIR USE.

DETERMINATION OF CONSTANT ELEMENTS AND OF PERTURBATIONS OF THE MEAN ANOMALY.

The planet (10) *Hygiea* was selected by v. Zeipel as an example of the use of his tables for the group $\frac{1}{2}$. We have used it as a preliminary example for the application of our own tables, so as to provide further comparison of our tables with those of v. Zeipel.

This example is presented with the direct purpose of meeting the needs of the computer. For this reason, no attempt is made to explain the significance of the functions involved, yet their use will be less mechanical, if, in a general way, some of the essential principles underlying their development are understood. The theory of v. Zeipel is taken up in the second section of this memoir.

The method proposed by v. Zeipel is a practical adaptation of Bohlin's method of computing the perturbations by *Jupiter* upon planets whose mean motions bear nearly commensurable ratios to that of *Jupiter.* In particular, the formulae are derived for the planets of the *Hecuba* group. Tracing the history of this method one step further back, Bohlin's method is a modification of the theory of Hansen for the indeterminate case of nearly commensurable mean motions. Or, concisely, in v. Zeipel's own words, "Die benutzte Methode kann einfach dadurch charakterisirt werden, dass die Differentialgleichungen von Hansen mittels des Integrationsverfahrens des Herrn K. Bohlin gelöst worden sind."[1]

Certain principles of Hansen are fundamental to an understanding of some of the important equations. Briefly, the perturbations are reckoned in the plane of the orbit and perpendicular to it. In the plane of the orbit $n\delta z$ signifies the displacement in the planet's mean anomaly (δz is the perturbation in the time); ν gives the disturbed radius vector through the relation

$$r=\bar{r}(1+\nu)$$

and the displacement in the third coordinate is denoted by $\dfrac{u}{\cos i}$. With Hansen's choice of ideal coordinates, the fundamental analytical relations are:

$$\begin{aligned} &\varepsilon-e\sin\varepsilon=nt+c+n\delta z\\ &\bar{r}\cos f=a\ (\cos\varepsilon-e)\\ &\bar{r}\sin f=a\ \sqrt{1-e^2}\sin\varepsilon\\ &r=\bar{r}(1+\nu)\end{aligned}\tag{1}$$

$$\begin{aligned} d\beta&=\frac{u}{\cos i}\,a\sin 1''\\ \Delta x&=d\beta\cos a\\ \Delta y&=d\beta\cos b\\ \Delta z&=d\beta\cos c\end{aligned}\tag{2}$$

$$\begin{aligned} x&=r\sin a\sin(A'+f)+\Delta x\\ y&=r\sin b\sin(B'+f)+\Delta y\\ z&=r\sin c\sin(C'+f)+\Delta z\end{aligned}\tag{3}$$

where $\varepsilon, f, \bar{r}$ are fictitiously disturbed coordinates, which, in connection with constant elements and the perturbations $n\delta z$, ν, and $\dfrac{u}{\cos i}$ give the true position of the body. A', B', C', $\sin a$, $\sin b$, $\sin c$ are the constants for the equator. The notation for the eccentric anomaly and the true anomaly is v. Zeipel's; in Hansen's notation they would be written $\bar{\epsilon}, \bar{f}$.

[1] Angenäherte Jupiterstörungen für die Hecuba-Gruppe, p. I.

When *Jupiter's* mean motion and that of the planet are nearly commensurable, the integration of Hansen's differential equations becomes impracticable through the presence of large integrating factors. The integrals are of the form:

$$\frac{1}{n^2\left(i-\frac{i'n'}{n}\right)^2}\,\begin{matrix}\sin\\ \cos\end{matrix}\left\{(in-i'n')t\right\}\quad\begin{matrix}-\infty<i<+\infty\\ 0<i'<+\infty\end{matrix}$$

For the *Hecuba* group the mean motion is approximately twice the mean motion of *Jupiter*. Hence, for exact commensurability,

$$\mu=\frac{n'}{n}=\frac{1}{2};\quad \frac{1}{n^2\left(i-\frac{i'n'}{n}\right)^2}=\infty;\ i'=2,\ i=1.$$

By introducing the exponential in place of the sine and cosine, the indeterminateness can be removed, for if $in-i'n'=0$, then $e^{\sqrt{-1}\,(in-i'n')t}=1$. This is one of Bohlin's modifications.

For any given planet the ratio is not exactly commensurable, and the developments are originally made for the case of exact commensurability. They are then expressed, for a given case, by Taylor's series in ascending powers of a small quantity w, which depends upon the difference between the real ratio and exact commensurability. In addition to positive powers of w there will occur negative powers. They are due to the following causes. An argument θ is introduced (see p. 13), from which the mean anomaly of *Jupiter* is eliminated through the introduction of w. It is a necessary consequence of the form of the partial differential equations in which $\frac{d\theta}{d\varepsilon}$ appears, that the integration of first-order terms shall contain w^{-1} and that higher order terms shall contain other negative powers. Hence the integrals are series in both positive and negative powers of w.

In distinction to the method of Hansen the elements appear explicitly in the arguments or as factors in the terms of the series.

An important feature of v. Zeipel's theory is his treatment of the constants of integration. Since the method is essentially Hansen's, the constants of integration must be determined consistently with that method. Given osculating elements, the constants of integration are determined by the condition that, at the date of osculation, $(t=0)$, the perturbations and their velocities shall be equal to zero.

v. Zeipel adopts osculating elements as his initial elements. With these elements and the perturbations and their velocities at the date of osculation, he computes elements, designated by the subscript unity, in which the constants of integration are absorbed. They are analogous to Hansen's constant elements, i. e., the fundamental equations of Hansen are valid.

Our transformations of the elements differ from v. Zeipel's in two respects. First, the constants in $\frac{u}{\cos i}$, and in its velocity have not been introduced into the elements i, Ω, but are treated in the usual Hansen manner. Second, v. Zeipel introduces certain terms in the perturbations which have the same period as the planet (argument ε), into the elements to form mean elements. This has not been done.

The general tables, XXXV, XXXVIII, XLIII, LIV, LV_I, LV_II, LVI, LVII, which are required in computing the perturbations, are given at the conclusion of the formulae. The formulae for any planet of the group $\frac{1}{2}$ are given completely, and they are supplemented by numerical values for the planet *(10) Hygiea*.

The references to v. Zeipel's paper are indicated briefly by Z, followed by the number of the page.

The osculating elements of the planet are taken from Z 139; the elements for *Jupiter* are taken from Astronomical Papers of the United States Nautical Almanac Office, Vol. VII, p. 23.

(10) *Hygiea.*

Epoch, 1851, Sept. 17.0, Ber. M. T.

OSCULATING ELEMENTS.

$$n_0 = 634''.850 = 0°.176347$$
$$\varphi_0 = 5° 46'.28 = 5°.7713$$
$$\pi_0 = 227\ 46.61 = 227.7768$$
$$\Omega_0 = 287\ 37.19 = 287.6198$$
$$\omega_0 = 300\ \ 9.42 = 300.1570$$
$$i_0 = 3\ 47.14 = 3.7857$$
$$c_0 = 126\ 59.81 = 126.9968$$

Jupiter.

Epoch, 1851, Sept. 17.0, Ber. M. T.

MEAN ELEMENTS.

$$n' = 299''.1284 = 0°.0830912$$
$$\varphi' = 2° 45'.95 = 2°.7658$$
$$\pi' = 11\ 54.45 = 11.9075$$
$$\Omega' = 98\ 55.97 = 98.9328$$
$$\omega' = 272\ 58.48 = 272.9747$$
$$i' = 1\ 18.70 = 1.3117$$
$$c' = 199\ 57.70 = 199.9617$$

Mean equinox and ecliptic, 1850.0.

Epoch, 1851, Sept. 16.96279 Gr. M. T.

The following notes in regard to these elements are of importance:

Jupiter's elements were first taken from *Z* 139. They were used only in the equations numbered (1). In these equations either set of elements may be used with sufficient accuracy. In fact, it is not necessary to know *Jupiter's* elements as accurately as those of the planet, for they appear only in the arguments of the perturbations. We have adopted Hill's values of the elements and Newcomb's value of the mass of *Jupiter.* The tables of the perturbations are based, however, on Bessel's value for m'. To correct the perturbations for the improved value, it is only necessary to multiply them by 1.0005, and this is done in the formulæ which follow.

The original epoch of Jupiter's elements was 1850.0 Gr. M. T. It was changed by the formula

$$c' = 148°\ 1'.97 + n't \tag{4}$$

The elements of *Hygiea* are very good osculating elements, computed by Zech. They include perturbations by *Jupiter, Saturn,* and *Mars* and are based on five oppositions. The reference for these elements is doubtful, for in Astronomische Nachrichten **39, 347**, the elements given by Zech are not identically the same, although the differences are very small. The values given by v. Zeipel were probably taken from Zech's manuscript, to which he had access. They may, therefore, contain some later corrections.

The auxiliary quantities Ψ, Φ, J are first computed by the formulae:

$$\begin{aligned}
\sin\tfrac{1}{2}J \sin\tfrac{1}{2}(\Psi+\Phi) &= \sin\tfrac{1}{2}(\Omega_0-\Omega')\sin\tfrac{1}{2}(i_0+i') \\
\sin\tfrac{1}{2}J \cos\tfrac{1}{2}(\Psi+\Phi) &= \cos\tfrac{1}{2}(\Omega_0-\Omega')\sin\tfrac{1}{2}(i_0-i') \\
\cos\tfrac{1}{2}J \sin\tfrac{1}{2}(\Psi-\Phi) &= \sin\tfrac{1}{2}(\Omega_0-\Omega')\cos\tfrac{1}{2}(i_0+i') \\
\cos\tfrac{1}{2}J \cos\tfrac{1}{2}(\Psi-\Phi) &= \cos\tfrac{1}{2}(\Omega_0-\Omega')\cos\tfrac{1}{2}(i_0-i')
\end{aligned} \tag{5}$$

$$\text{Check: } \frac{\sin\Psi}{\sin i_0} = \frac{\sin\Phi}{\sin i'} = \frac{\sin(\Omega_0-\Omega')}{\sin J}$$

Then follow

$$\Pi_0 = \pi_0 - \Omega_0 - \Phi;\ \eta_0 = \frac{e_0}{2}; \qquad \Pi' = \pi' - \Omega' - \Psi;\ \eta' = \frac{e'}{2}.$$

$$j^2 = \sin^2\frac{J}{2}\cos^2\frac{1}{2}\varphi_0\cos^2\frac{1}{2}\varphi';\ \iota = \sin J\cos^2\frac{1}{2}\varphi \tag{6}$$

$$\mathcal{A}_0 = \Pi_0 - \Pi';\ \Sigma_0 = \Pi_0 + \Pi'$$

$$w_0 = \frac{n_0 - 2n'}{n_0}$$

and the arguments for the date of osculation:

$$\theta_0 = \frac{1}{2} c_0 - g' \text{ where } g' = c' + [n'\delta z']; \ [n'\delta z'] = (9.5215) \sin 115^\circ\!.326, \tag{7}$$

where the coefficient in parentheses is logarithmic in degrees.

$$\varepsilon_0 - e_0 \sin \varepsilon_0 = c_0; \ \Gamma = \frac{1}{2} \varepsilon_0 + \theta_0 + \Delta_0 \tag{8}$$

---

(10) Hygiea.

$\Psi = 186^\circ\!.4792$	$\Pi_0 = 302^\circ\!.3984$	$\Delta_0 = 215^\circ\!.8679$
$\Phi = 357.7586$	$\Pi' = 86.5305$	$\Sigma_0 = 28.9289$
$J = 5.0856$		
$\log \eta_0 = 8.70139$	$\log j^2 = 7.29275$	$\theta_0 = 223.2334$ (a)
$\log \eta' = 8.38238$	$\log \iota = 8.94739$	$\frac{1}{2} c_0 - (c' + [n'\delta z']) = 223.2445$ (b)
$\log w_0 = 8.76072$		$\frac{1}{2} c_0 - c' = 223.5448$ (c)
		See footnote.[1]

$$\varepsilon_0 = 131^\circ\!.3236; \ \Gamma = 145^\circ\!.0746$$

With these initial quantities all the arguments and factors in Table LVI or F are computed. The required function, $w - w_0$, is computed by successive approximations, the first approximation being

$$w = w_0$$

In the first trial the smallest terms and the last digit may be omitted; the second trial should be accurate; a third trial, if necessary, will require only corrections to the largest terms.

The mean motion n is then given by

$$n = \frac{2n'}{1 - w} \tag{9}$$

---

(10) Hygiea.

The three successive trials for w give

$w - w_0$	
$+0.00388$	$w = +0.061208$
$+0.003541$	$\log w = 8.78681$
$+0.003568$	$n = 637''\!.2633$

Designating by C and S series to be computed next from Table LVII or G, it is evident by inspection of Table LVII that

$$C \cos \psi + S \sin \psi = \Sigma c \cos (\psi + X) = \Sigma c \cos X \cos \psi - \Sigma c \sin X \sin \psi$$

from which

$$C = \Sigma c \cos X; \ S = -\Sigma c \sin X \tag{10}$$

[1] Three numerical values for the argument θ_0 are given. According to the theory (see footnote, Part 2, p. 147), (a) is rigid; (b) is rigid within the accuracy of the developments by v. Zeipel; (c) is an approximation which v. Zeipel used and which is used here. The value (b) is preferable.

In equation (b), $[n'\delta z'] = +0^\circ\!.3114$ and is the complete perturbation of Jupiter by Saturn taken from Hill; in all other parts of the computation $[n'\delta z']$ is only the long period term used by v. Zeipel.

To make the order of computation evident, the successive steps for a group of terms for *Hygiea* are given.

X	c	X	$-S$	$+C$
	$''$	$°$	$''$	$''$
$-5\Gamma+6\theta_0+6J_0$	+ 0.4	111.10	+ 0.4	− 0.1
$-4\Gamma+6\theta_0+6J_0$	+ 1.9	256.18	− 1.8	− 0.5
$-3\Gamma+6\theta_0+6J_0$	+ 4.6	41.25	+ 3.0	+ 3.5
$-2\Gamma+6\theta_0+6J_0$	+ 6.8	186.33	− 0.7	− 6.8
$-\Gamma+6\theta_0+6J_0$	+21.5	331.40	−10.3	+18.8
$6\theta_0+6J_0$	−63.0	116.48	−56.4	+28.1
$\Gamma+6\theta_0+6J_0$	− 4.0	261.55	+ 4.0	+ 0.6
$2\Gamma+6\theta_0+6J_0$	− 3.1	46.63	− 2.2	− 2.1
$3\Gamma+6\theta_0+6J_0$	− 1.9	191.70	+ 0.4	+ 1.9

The second column contains the sum of the numerical coefficients multiplied by their respective factors $w^s\eta^p\eta'^q j^{2t}$. The columns $-S$ and $+C$ contain the required terms from this group in the table. They can be computed at the same time if a traverse table is used.[1]

From S and C the elements π and φ can be computed by the formulæ:

$$\begin{aligned} e \sin (\pi-\pi_0) &= S \cos \varphi_0 \\ e \cos (\pi-\pi_0) &= e_0 + C \cos {}^2\varphi_0 \\ e &= \sin \varphi \end{aligned} \tag{11}$$

In place of η_0, J_0, Σ_0 the following are used hereafter:

$$\eta = \frac{e}{2} \qquad J = J_0 + (\pi-\pi_0) \qquad \Sigma = \Sigma_0 + (\pi-\pi_0) \tag{12}$$

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(10) Hygiea.

$S = +1215''.0$ $\qquad \pi-\pi_0 = +3°.0203$ $\qquad \log \eta = 8.74517$

$C = +2191.1$ $\qquad \tau = 230.7971$ $\qquad \log \sin\varphi = 9.04620$

$\Delta = 218°.8882$ $\qquad \Sigma = 31°.9492$ $\qquad \varphi = 6°.3858$

There remains one more element to determine, namely, c, but the computation must be deferred until we know the perturbation $n\delta z$ at $t=0$. (See equation (1), page 10 or page 16.)

The fictitiously disturbed eccentric anomaly at the time $t=0$ denoted by ε_1, is determined through the relations:

$$\varepsilon_0 - e_0 \sin \varepsilon_0 = c_0 \tag{13}$$

where ε_0 is calculated with the aid of Astrand's table [2];

$$tg\tfrac{1}{2}(v_0-\pi_0) = \sqrt{\frac{1+e_0}{1-e_0}}\, tg\tfrac{1}{2}\varepsilon_0;\; tg\tfrac{1}{2}\varepsilon_1 = \sqrt{\frac{1-e}{1+e}}\, tg\tfrac{1}{2}(v_0-\pi) \tag{14}$$

---

(10) Hygiea.

$\varepsilon_0 = 131°.3236$ $\qquad v_0 = 3°.2968$ $\qquad \varepsilon_1 = 127°.6064$ $\qquad \varepsilon_1 - e \sin \varepsilon_1 = 122°.5578$

The perturbation $n\delta z$ is computed as follows:

The function $1+\Phi(\vartheta)$ is computed from Table XXXVIII or B. The coefficients are multiplied by their respective factors, the trigonometric functions of the arguments are expanded, and the coefficients of $\begin{smallmatrix}\cos\\ \sin\end{smallmatrix} j\vartheta$ are collected, (j is the numerical coefficient of ϑ).

[1] Memoirs of the National Academy of Sciences, Vol. X, Seventh Memoir, p. 218.

[2] Hülfstafeln zur leichten und genauen Auflösung des Kepler'schen Problems (Leipzig, 1890).

(10) Hygiea.

$$1+\Phi(\vartheta)=(1-0.008064)\{1-0.055937 \sin 2\vartheta+0.017170 \cos 2\vartheta$$
$$+0.016057 \sin 4\vartheta+0.012244 \cos 4\vartheta$$
$$+0.000905 \sin 6\vartheta-0.005081 \cos 6\vartheta+\;.\;.\;.\;.\;.$$
$$+(\vartheta-\vartheta_0)(+0.000007-0.000490 \sin 2\vartheta-0.001266 \cos 2\vartheta$$
$$-0.000361 \sin 4\vartheta+0.000409 \cos 4\vartheta+\;.\;.\;.\;.\;.)\}$$

where the coefficients are in radians, and ϑ_0 is the value of ϑ at $t=0$.

Let $1+\sigma$ be the nontrigonometrical term in $1+\Phi(\vartheta)$, take it out as a common factor, and denote the numerical coefficients by A_2, B_2, A_4, B_4, A_6, B_6, b_0, a_2, b_2, a_4, b_4, respectively.

With these coefficients the following are computed:

$$K=\frac{1}{1-w}\,\frac{1}{\sin 1''}$$

$$S_2=K\left\{A_2-\frac{5}{4}(A_2A_4+B_2B_4)+\frac{1}{4}A_2(A_2^2+B_2^2)+\frac{1}{2}a_2\right\} \qquad S_2'=K\frac{w}{2}\,b_2$$
$$C_2=K\left\{-B_2+\frac{5}{4}(A_2B_4-B_2A_4)-\frac{1}{4}B_2(A_2^2+B_2^2)+\frac{1}{2}b_2\right\} \qquad C_2'=-K\frac{w}{2}\,a_2$$
$$S_4=K\left\{\frac{1}{2}A_4+\frac{1}{4}(A_2^2-B_2^2)\right\} \qquad S_4'=K\frac{w}{4}\,(b_4+A_2b_2-B_2a_2) \qquad (15)$$
$$C_4=K\left\{-\frac{1}{2}B_4-\frac{1}{2}A_2B_2\right\} \qquad C_4'=-K\frac{w}{4}\,(a_4+A_2a_2+B_2b_2)$$
$$S_6=K\left\{\frac{1}{3}A_6+\frac{5}{12}(A_2A_4-B_2B_4)-\frac{1}{12}A_2(3B_2^2-A_2^2)\right\} \qquad C_0''=K\frac{w^2}{4}\,(b_0-A_2b_2-B_2a_2)$$
$$C_6=K\left\{-\frac{1}{3}B_6-\frac{5}{12}(A_2B_4+B_2A_4)-\frac{1}{12}B_2(3A_2^2-B_2^2)\right\}$$

There are check formulae for these quantities in Z 134, equation (153), (161′). In equation (153) there is a misprint; in equation (161′) there are two misprints. The errors and their corrections are noted in the list of errata which accompanies the second section of this paper.

A part of the long period terms in $n\delta z$, denoted by $[n\delta z]_1$, is expressed by

$$[n\delta z]_1=S_2 \sin 2\zeta+C_2 \cos 2\zeta+S_4 \sin 4\zeta+C_4 \cos 4\zeta+S_6 \sin 6\zeta+C_6 \cos 6\zeta+\;.\;.\;.\;.$$
$$+\frac{2}{w}(\zeta-\zeta_0)(S_2' \sin 2\zeta+C_2' \cos 2\zeta+S_4' \sin 4\zeta+C_4' \cos 4\zeta+\;.\;.\;.\;)+\left(\frac{2}{w}\right)^2(\zeta-\zeta_0)^2C_0''+\;.\;.\;. \qquad (16)$$

---

(10) Hygiea.

$$1+\Phi(\vartheta)=(1-0.008064)\;\{1-0.056384 \sin 2\vartheta+0.017308 \cos 2\vartheta+0.016186 \sin 4\vartheta$$
$$+0.012342 \cos 4\vartheta+0.000912 \sin 6\vartheta-0.005122 \cos 6\vartheta+\;.\;.\;.$$
$$+(\vartheta-\vartheta_0)(+0.000007-0.000494 \sin 2\vartheta-0.001276 \cos 2\vartheta-0.000364 \sin 4\vartheta$$
$$+0.000412 \cos 4\vartheta+\;.\;.\;.\;.\;)+\;.\;.\;.\;.\;\}$$

	$b_0=+0.000007$
$A_2=+0.017308$	$a_2=-0.000494$
$B_2=-0.056384$	$b_2=-0.001276$
$A_4=+0.012342$	$a_4=-0.000364$
$B_4=+0.016186$	$b_4=+0.000412$
$A_6=-0.005122$	
$B_6=+0.000912$	

Unit of A_2, etc., is one radian

$$[n\delta z]_1 = (3.59592)\sin 2\zeta + (4.09785)\cos 2\zeta + (3.0783)\sin 4\zeta + (3.2230_n)\cos 4\zeta + (2.4390_n)\sin 6\zeta + (1.494_n)\cos 6\zeta + \dots\dots$$
$$+\frac{2}{w}(\zeta-\zeta_0)\,[(0.933_n)\sin 2\zeta + (0.521)\cos 2\zeta + (0.085)\sin 4\zeta + (0.005)\cos 4\zeta + \dots\dots] + \dots\dots$$

in which the coefficients are logarithmic in seconds of arc. For this planet it is not necessary to include C_0''.

In equation (16) let

$$S_n = k\cos K \qquad C_n = k\sin K$$
$$S'_n = -k'\sin K' \qquad C'_n = k'\cos K' \tag{17}$$

Then

$$[n\delta z]_1 = \Sigma k\sin(n\zeta+K) + \frac{2}{w}(\zeta-\zeta_0)\Sigma k'\cos(n\zeta+K') + \dots\dots \tag{18}$$

The argument ζ is given by the relation:

$$\zeta = \frac{1+\sigma}{1+\frac{1}{2}(A_2^2+B_2^2)}\left(\frac{w}{2}\varepsilon - [n'\delta z']\right) + \frac{1-w}{2}\,c - c' \tag{19}$$

and ζ_0 is the value of ζ at $t=0$, in which, $[n'\delta z']$, the long period term between *Jupiter* and *Saturn* is:

$$[n'\delta z'] = (9.5215)\sin\{(9.58539)\,T + 115°326\} \tag{20}$$

where the numerical coefficients are logarithmic in degrees, and T is measured from the date of osculation in Julian years.

The complete expression for the long period term in $n\delta z$ is:

$$[n\delta z] = [n\delta z]_1 + \frac{2}{1-w}\,\frac{\sigma-\frac{1}{2}(A_2^2+B_2^2)}{1+\frac{1}{2}(A_2^2+B_2^2)}\left(\frac{w}{2}\varepsilon - [n'\delta z']\right) \tag{21}$$

It is important to remark that, in equations (19), (21), the eccentric anomaly is computed by the usual formula,

$$\varepsilon - e\sin\varepsilon = c + nt + n\delta z \tag{1}$$

in which the multiples of 2π must be retained, for ε is used here as if it were the time. Since $n\delta z$ is unknown, the computation is by successive approximations.

(10) Hygiea.

$$[n\delta z]_1 = (4.11837)\sin(2\zeta + 72°5246) + (3.3130)\sin(4\zeta + 305.627) + (2.442)\sin(6\zeta + 186.48) + \dots\dots$$
$$+\frac{2}{w}(\zeta-\zeta_0)\{(0.963)\cos(2\zeta + 68.83) + (0.199)\cos(4\zeta + 309.75) + \dots\} + \dots$$

in which the coefficients are logarithmic in seconds of arc.

$$\log\frac{1+\sigma}{1+\frac{1}{2}(A_2^2+B_2^2)} = 9.99572;\quad \log\left\{\frac{2}{1-w}\,\frac{\sigma-\frac{1}{2}(A_2^2+B_2^2)}{1+\frac{1}{2}(A_2^2+B_2^2)}\right\} = 8.31918_n$$

The argument ϑ in $(n\delta z - [n\delta z])$, the short period part of $n\delta z$, is given by

$$\vartheta = \frac{1-w}{2}\,[n\delta z]_1 + \zeta \tag{22}$$

and the function itself is computed from Table XXXV or A.

The numerical coefficients in Table XXXV or A are multiplied by their respective factors

$$w^s \eta^p \eta'^q j^{2t}$$

and the terms are then collected in the form

$$n\delta z-[n\delta z]=\Sigma C \begin{smallmatrix}\sin\\ \cos\end{smallmatrix} \left(i\frac{1}{2}\varepsilon+j\vartheta+k\varDelta-l\Sigma\right) \tag{23}$$

By expanding the trigonometric functions, the known part of the argument, namely,

$$k\varDelta-l\Sigma$$

is incorporated in the coefficients, and the terms are collected in the form:

$$\begin{aligned} n\delta z-[n\delta z] &= \Sigma a\sin\chi+\Sigma b\cos\chi+(\vartheta-\vartheta_0)\,(\Sigma a'\sin\chi+\Sigma b'\cos\chi) \\ &\quad+(\vartheta-\vartheta_0)^2(\Sigma a''\sin\chi+\Sigma b''\cos\chi) \\ &\quad+\ \ldots\ldots \end{aligned} \tag{24}$$

where

$$\chi=i\frac{1}{2}\varepsilon+j\vartheta \tag{25}$$

Let

$$\begin{array}{ll} a\ =\ k\ \cos K & b\ =k\ \sin K \\ a'\ =-k'\ \sin K' & b'\ =k'\ \cos K' \\ a''=\ k''\cos K'' & b''=k''\sin K'' \end{array} \tag{26}$$

Then

$$\begin{aligned} n\delta z-[n\delta z] &= \Sigma k\sin(\chi+K)+(\vartheta-\vartheta_0)\,\Sigma k'\cos(\chi+K') \\ &\quad+(\vartheta-\vartheta_0)^2\Sigma k''\sin(\chi+K'')+\ \ldots\ldots \end{aligned} \tag{27}$$

The tabulation of $n\delta z-[n\delta z]$ for (*10*) *Hygiea* is given on page 27.

Finally, the complete perturbation in the mean anomaly is:

$$n\delta z=[n\delta z]+(n\delta z-[n\delta z]) \tag{28}$$

It is now possible to determine c by successive approximations from equations (20), (19), (18), (21), (22), (27), (28).

From equation (1), which holds for any time t,

$$\begin{aligned} c&=\varepsilon_1-e\sin\varepsilon_1-n\delta z \\ t&=o \\ \varepsilon&=\varepsilon_1 \end{aligned} \tag{29}$$

As a first approximation

$$n\delta z=0 \qquad\qquad c=\varepsilon_1-e\sin\varepsilon_1$$

Introducing this value of c in equation (19), a first approximation for $n\delta z$ is made. For $t=0$,

$$\begin{aligned} (\zeta-\zeta_0)&=0 \\ (\vartheta-\vartheta_0)&=0 \end{aligned} \tag{30}$$

Substituting the value of $n\delta z$ in equation (29), and computing a new value of c, the process of solution by trials is repeated until a satisfactory agreement is reached.

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(10) *Hygiea.*

Below is the last approximation for the constant c.

(See tabulation of $n\delta z-[n\delta z]$ on pg. 27.)

		$\chi=i\frac{\epsilon}{2}+j\vartheta$		$\chi+K$	$\log\sin(\chi+K)$	$k\sin(\chi+K)$
Approx. $n\delta z$	$+0°6124$					
$\epsilon_1-e\sin\epsilon_1$	122. 5578	$\frac{1}{2}\epsilon+\vartheta$	285°683	323°619	9.7732_n	− 282″
Approx. c, equ. (1), p. 10	121. 9454	$\frac{1}{2}\epsilon+3\vartheta$	9. 443	291. 021	9.9701_n	− 680
		$\frac{1}{2}\epsilon+5\vartheta$	93. 203	258. 21	9.991_n	− 260
$\frac{1-w}{2}c$, p. 13	57. 240					
		$-\frac{1}{2}\epsilon+\vartheta$	158. 077	183. 00	8.719_n	− 6
$\frac{1-w}{2}c-c'$, p. 12	217. 278	$-\frac{1}{2}\epsilon+3\vartheta$	241. 837	335. 37	9.620_n	− 17
		ϵ	127. 606	135. 14	9. 848	+ 25″
$\frac{w}{2}\epsilon_1$, p. 13	+3. 9053	$\epsilon+2\vartheta$	211. 366	288. 414	9.9772_n	−3403
		$\epsilon+4\vartheta$	295. 126	256. 179	9.9872_n	− 723
$[n'\delta z']$, equ. (20), p. 16	+0. 3003	$\epsilon+6\vartheta$	18. 886	223. 38	9.837_n	− 168
		$\epsilon+8\vartheta$	102. 646	186. 8	9.073_n	− 5
$(9.99572)\left(\frac{w}{2}\epsilon_1-[n'\delta z']\right)$	+3. 5697	$-\epsilon+2\vartheta$	316. 154	14. 11	9. 387	+ 23
		$-\epsilon+4\vartheta$	39. 914	129. 91	9. 885	+ 3
		$\frac{3}{2}\epsilon+3\vartheta$	137. 049	74. 51	9. 984	+ 121
$(8.3192_n)\left(\frac{w}{2}\epsilon_1-[n'\delta z']\right)$	−0. 0752	$\frac{3}{2}\epsilon+5\vartheta$	220. 809	39. 53	9. 804	+ 44
		$\frac{3}{2}\epsilon+7\vartheta$	304. 569	3. 25	8. 754	+ 1
ζ, equ. (19), p. 16	220. 848	$2\epsilon+2\vartheta$	338. 972	236. 180	9.9195_n	− 86
2ζ	81. 696	$2\epsilon+4\vartheta$	62. 732	209. 15	9.688_n	− 19
4ζ	163. 392	$2\epsilon+6\vartheta$	146. 492	183. 0	8.72_n	− 1
6ζ	245. 088	$\frac{5}{2}\epsilon+5\vartheta$	348. 415	2. 4	8. 62	0
		$\frac{5}{2}\epsilon+7\vartheta$	72. 175	327. 9	9.72_n	− 4
$2\zeta+72°525$, p. 16	154. 221					+ 217″−5654″
$4\zeta+305.627$	109. 019					
$6\zeta+186.48$	71. 57					
					$n\delta z-[n\delta z]$	{ − 5437″ − 1°5103
log sin	9. 6384				$(8.3192_n)\left(\frac{w}{2}\epsilon_1-[n'\delta z']\right)$	− 0. 0752
log sin	9. 9756					
log sin	9. 9771				$[n\delta z]_1$	+ 2. 1994
					$n\delta z$, equ. (21)	+ 0. 6139
	+ 5712″				$c=c_1$	121. 9439
	+ 1944					
	+ 262				$(6.8050_n)c_1$	− 0. 0778
					c_2, p. 19	121. 8661
$[n\delta z]_1$	{ +7918″ +2°1994					
$(9.67154)[n\delta z]_1$	+1. 032				$\frac{1-w}{2}c_1$	+57. 240
ϑ, equ. (22), p. 16	221°880				$\frac{1-w}{2}c_1-c'$	217. 278
2ϑ	83. 760					
3ϑ	305. 640				$(9.6715)[n\delta z]_1$	+ 1. 032
					ϑ_0, equ. (22)	221. 880
4ϑ	167. 520					
5ϑ	29. 400					
6ϑ	251. 280					
7ϑ	113. 160					
8ϑ	335. 040					
$\frac{1}{2}\epsilon$, p. 14	63. 803					
ϵ	127. 606					
$\frac{3}{2}\epsilon$	191. 409					
2ϵ	255. 212					
$\frac{5}{2}\epsilon$	319. 015					

Collecting the elements, and adopting a change of notation, introduced at this point by v. Zeipel, namely, the addition of the subscript unity to the elements just now determined,

$$n_1=637\overset{\prime\prime}{.}2633=0°17701758$$
$$\varphi_1=\quad 6°3858$$
$$\pi_1=230.7971$$
$$c_1=121.9439$$

These elements are constants; they differ from constant osculating elements only by the constants of integration in $n\delta z$ and ν. They are to be used in the same manner as Hansen uses constant osculating elements.

It is possible, in a similar manner, to absorb the constants of integration in the third coordinate in the elements i_0 and Ω_0, but this transformation will be omitted.

It is a convenience to the computer to have n_1 and c_1 transformed to mean elements. The last term in equation (21) increases in magnitude, progressively with the time. The computation of this term of large magnitude may be avoided by modifications of the elements n_1 and c_1.

The method of transformation can be clearly shown from the example *(10) Hygiea*,

$$\frac{2}{1-w}\frac{\sigma-\frac{1}{2}(A_2{}^2+B_2{}^2)}{1+\frac{1}{2}(A_2{}^2+B_2{}^2)}\left(\frac{w}{2}\varepsilon-[n'\delta z']\right)=(6.80497_n)\varepsilon+(8.3192)\ [n'\delta z'] \tag{31}$$

By equation (1)

$$\begin{aligned}(6.80497_n)\varepsilon+(8.3192)\ [n'\delta z']=(6.80497_n)\ c_1-0''.4067\ t-14''.6\sin\varepsilon\\ +(6.80497_n)n\delta z+(8.3192)[n'\delta z']\end{aligned} \tag{32}$$

It is evident from equations (1), (21), and (23) that the first term on the right-hand side of equation (32) may be combined with the mean anomaly at the epoch to form a mean mean anomaly, given by

$$c_2=c_1+(6.80497_n)c_1$$

Furthermore, the second term on the right-hand side of equation (32) may be combined with nt in equation (1). A mean mean motion is thereby introduced, which is given by

$$n_2=n_1-0''.4067=636''.8566$$

Again, the third term on the right-hand side of equation (32) may be combined with a term in $(n\delta z-[n\delta z])$ which has the argument ε. In the construction of $(n\delta z-[n\delta z])$ there occurred the terms

$$+34''.8\sin\varepsilon+4''.6\cos\varepsilon=(1.545)\sin(\varepsilon+7^\circ.53)$$

The addition of $-14''.6\sin\varepsilon$ from equation (32) gives

$$+20''.2\sin\varepsilon+4''.6\cos\varepsilon=(1.320)\sin(\varepsilon+12^\circ.74)$$

These two values for the argument $\chi=\varepsilon$ are tabulated in the body of the table given on p. 27.

Further, since it is intended to improve the perturbations by the use of Newcomb's value for the mass of *Jupiter*, $n\delta z$ must be multiplied by the factor 1.00050. The combination of the correction for the mass of *Jupiter* with the term of the same form in equation (32) gives

$$(+0.00050-0.00064)n\delta z=-0.00014\ n\delta z$$

This correction is the last step in the determination of $n\delta z$, since it depends upon the perturbation itself.

Without change of notation for $n\delta z$, the collected results are:

$$\varepsilon-e\sin\varepsilon=c_2+n\delta z+n_2t \tag{33}$$

where

$$n\delta z=[n\delta z]_1+(n\delta z-[n\delta z])-0.00014\ n\delta z+(8.319)\ [n'\delta z'] \tag{34}$$

It must be remembered that $[n\delta z]_1$ and $(n\delta z-[n\delta z])$ are numerically different from their original values, but there should be no confusion if this transformation is not made before the constant c has been determined.

The constant elements are now:

Epoch and Osculation, 1851, Sept. 17.0, Ber. M. T.

$$\begin{aligned}n_2&=636''.8566=0^\circ.17690461\\ c_2&=121^\circ.8661\\ \varphi_1&=6.3858\\ \pi_1&=230.7971\\ \Omega_0&=287.6198\\ i_0&=3.7857\end{aligned}$$

Equinox and ecliptic, 1850.0

$$a_2^3=\frac{k^2}{n_2^2} \qquad e_1=\sin\varphi_1 \qquad p_2=a_2(1-e_1^2)$$

$$\log k^2=9.98741 \qquad \log e_1=9.04620 \qquad \log p_2=0.49191$$

$$\log n_2^2=8.49548 \qquad \log\sqrt{\frac{1-e_1}{1+e_1}}=9.95150$$

$$\log a_2^3=1.49193 \qquad \log a_2=0.49731$$

Certain other transformations of the elements which v. Zeipel makes are omitted. Those terms of the perturbations which have the argument ε have the same period as the planet and can, therefore, be absorbed in the elements. It would be necessary to set up formulae for this transformation to mean elements, and it is not profitable to do so.

PERTURBATIONS OF THE RADIUS VECTOR.

The perturbations in the radius vector are computed in a manner similar to that for $(n\delta z-[n\delta z])$. In Table XLIII the numerical coefficients are multiplied by their respective factors w^s, η^p, η'^q, j^{2t}, the terms are collected, the known parts of the arguments are incorporated in the coefficients, and the terms are grouped in the form:

$$\begin{aligned}\nu= &\ \Sigma a\ \sin\chi+\Sigma b\ \cos\chi+\cdot\cdot\cdot\cdot\\ &+(\vartheta-\vartheta_0)\{\Sigma a'\ \sin\chi+\Sigma b'\ \cos\chi+\cdot\cdot\cdot\cdot\}\\ &+(\vartheta-\vartheta_0)^2\{\Sigma a''\ \sin\chi+\Sigma b''\ \cos\chi+\cdot\cdot\cdot\cdot\}+\cdot\cdot\cdot\cdot\end{aligned} \tag{35}$$

Let

$$\begin{aligned} a &=-k\ \sin K & b &=k\ \cos K\\ a' &= k'\ \cos K' & b' &=k'\ \sin K'\\ a'' &=-k''\sin K'' & b'' &=k''\cos K''\end{aligned} \tag{36}$$

Then

$$\nu=\Sigma k\cos(\chi+K)+(\vartheta-\vartheta_0)\Sigma k'\sin(\chi+K')+(\vartheta-\vartheta_0)^2\Sigma k''\cos(\chi+K'')+\cdot\cdot\cdot\cdot \tag{37}$$

and to correct the perturbation for the use of the improved value of the mass, ν should be multiplied by 1.00050.

If the mean motion n_2 is adopted, the constant in ν must be corrected by

$$+\frac{2}{3}\ \frac{n_2-n_1}{n_1}\ \frac{1}{\sin 1''} \tag{38}$$

This correction of the constant in ν permits the use of the relation

$$n_2^2a_2^3=k^2$$

in the computation of a geocentric place; without this correction it would be necessary to use the relation

$$n_1^2a_1^3=k^2$$

in the determination of the parameter p. In the computation of the eccentric anomaly it is permissible to use either n_1 or n_2, for the difference is taken up in the modification of $n\delta z$, but the theory of Hansen demands the use of constant elements. Hence, strictly speaking, n_1 must be used in computing a geocentric place. The modification of the constant in ν renders the employment of n_2 equivalent to the use of n_1.

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(10) *Hygiea.*

$$\frac{2}{3}\ \frac{n_2-n_1}{n_1}\ \frac{1}{\sin 1''}=-\frac{2}{3}\ \frac{0''.4067}{637''.3}\ \frac{1}{\sin 1''}=-87''.8$$

The constant in Table XLIII or C is $+47''.6$. Therefore, the new constant is:

$$+47''.6-87''.8=-40''.2=(1.604)\cos 180^\circ.00$$

where the coefficient is logarithmic in seconds of arc.

The perturbation is tabulated on page 27.

PERTURBATIONS OF THE THIRD COORDINATE.

The perturbations of the third coordinate are derived from Tables LIV, LV_I, LV_{II} or D, E_1, E_2. The first of these is of the same form as the tables for $(n\delta z-[n\delta z])$ and ν. After making analogous transformations and multiplying by the factor $\iota \cos i$, (ι is defined by equation (6)),

$$\iota \cos i \Sigma U_{p\cdot q}\, \eta^p \eta'^q \sin A = \Sigma k \sin (\chi + K) \tag{39}$$

Both Table LV_I or E_1 and Table LV_{II} or E_2 lead to a single numerical quantity, since all the factors and arguments are known constants.

The perturbation u is given by

$$u = \iota \cos i \left[\Sigma U_{p\cdot q}\, \eta^p \eta'^q \sin A + n_2 t\{K_1 (\cos \varepsilon - e_1) + K_2 \sin \varepsilon\} + c_1 (\cos \varepsilon - e_1) + c_2 \sin \varepsilon\right] \tag{40}$$

in which c_1, c_2, the constants of integration, have not been determined.

The constants c_1 and c_2 are determined by Hansen's conditions:

$$\left.\begin{matrix} u = 0 \\ \dfrac{du}{d\varepsilon} = 0 \end{matrix}\right\} t = 0 \tag{41}$$

Substituting these relations and equation (39) in equation (40), the determination of c_1 and c_2 is given by the solution of

$$C_1 (\cos \varepsilon - e_1) + C_2 \sin \varepsilon = -\Sigma k \sin (\chi + K); \quad C_1 \sin \varepsilon - C_2 \cos \varepsilon = \Sigma k \frac{d\chi}{d\varepsilon} \cos (\chi + K) \tag{42}$$

where

$$\begin{aligned} C_1 &= \iota \cos i . c_1 \\ C_2 &= \iota \cos i . c_2 \end{aligned} \tag{43}$$

and

$$\frac{d\chi}{d\varepsilon} = \frac{i}{2} + j \frac{d\vartheta}{d\varepsilon}$$

where

$$\frac{d\vartheta}{d\varepsilon} = \frac{1+\sigma}{1+\frac{1}{2}(A_2{}^2 + B_2{}^2)} \frac{w}{2}\left(1 + \frac{1-w}{2} \frac{d[n\delta z]_1}{d\zeta}\right) \tag{44}$$

A double notation is used here, for $\cos i$ is the cosine of the inclination of the orbit, and $\frac{i}{2}$ is the numerical coefficient of ε in the argument χ, but this should cause no confusion.

Dividing and multiplying the factor

$$\iota \cos i \cdot n_2 t$$

by 365.25

$$\iota \cos i \cdot n_2 t = \frac{\iota \cos i \cdot n_2}{365.25} T \tag{45}$$

where T is the interval in Julian years, measured from the date of osculation.

It is evident that

$$C_1 (\cos \varepsilon - e_1) + C_2 \sin \varepsilon$$

can be incorporated in

$$\Sigma k \sin (\chi + K)$$

in the same manner as similar terms were treated in $(n\delta z-[n\delta z])$.

For symmetry of form, let

$$\iota \cos i \cdot n_2 t\{K_1 (\cos \varepsilon - e_1) + K_2 \sin \varepsilon\} = \Sigma k' \cos (\chi + K') \tag{46}$$

Finally, then, without change of notation,

$$u = \Sigma k \sin (\chi + K) + T \Sigma k' \cos (\chi + K') \tag{47}$$

in which the constants of integration are absorbed in the first term. The perturbation u is tabulated on page 27.

The perturbations in the heliocentric coordinates are computed from equations (3) The signs of $\cos a$, $\cos b$, $\cos c$ are determined as follows:

$$\begin{aligned} &\cos a > 0 \text{ if } 0 < \Omega < 180^\circ \\ &\cos b < 0 \text{ if } -90^\circ < \Omega < +90^\circ \\ &\cos b < 0 \text{ in any case, if } \varepsilon > i \\ &\cos c > 0 \text{ if } \sin i \cos \Omega < \cos i \\ &\cos c > 0 \text{ in any case if } i < 45^\circ \end{aligned}$$

(10) *Hygiea.*

$$t=0$$

$$\begin{aligned}\frac{d}{d\zeta}[n\delta z]_1 = [&(4.41940)\cos(2\zeta_0+\ \ 72°5246)\\ &+(3.9150)\ \ \cos(4\zeta_0+305°627)\\ &+(3.220)\ \ \ \cos(6\zeta_0+186°48)]\sin 1''\end{aligned}$$

$$\log\frac{w}{2}\frac{1+\sigma}{1+\frac{1}{2}(A_2^2+B_2^2)}=8.48150 \qquad \frac{d\vartheta}{d\varepsilon}=0.0285$$

$$\log\frac{1-w}{2}=9.67154 \qquad \frac{d\chi}{d\varepsilon}=\frac{i}{2}+0.0285j$$

$$\Sigma k\sin(\chi+K)=-\ \ 70''5$$

$$\Sigma k\frac{d\chi}{d\epsilon}\cos(\chi+K)=+101''6$$

$$C_1=+\ \ 35''9$$
$$C_2=+120''8$$

From Table LIV, multiplied by $\iota\cos i$ we have three terms in

$$\Sigma k\sin(\chi+K)=-4''2-1''9\sin\varepsilon+2''7\cos\varepsilon$$

which, added to

$$C_1(\cos\varepsilon-e_1)+C_2\sin\varepsilon=+120''8\sin\varepsilon+35''9(\cos\varepsilon-e_1)$$

gives for two terms in $\Sigma k\sin(\chi+K)$

$$-7''8+118''9\sin\varepsilon+38''6\cos\varepsilon=(0.89)\sin 270°0+(2.0970)\sin(\varepsilon+17°99)$$

CHECK COMPUTATION.

After the elements have been determined and the final tabulation of the perturbations is ready, the following checks should be performed, even if the computation has been duplicated.

$$t=0$$

$$\theta_0=\tfrac{1}{2}(\varepsilon_1-e_1\sin\varepsilon_1)-g'$$

$$g'=c'+[n'\delta z'] \qquad \theta_0=\vartheta_0+\frac{1-w}{2}(n\delta z-[n\delta z])-\eta w\sin\varepsilon$$

where the necessary quantities are to be taken from the last approximation for c.

Secondly, the heliocentric coordinates

$$x-\Delta x,\ y-\Delta y,\ z-\Delta z$$

for $t=0$ must check when computed by the usual formulae for two body motion and osculating elements, and when computed with the final set of elements and the corresponding perturbations, $n\delta z$ and ν, taken from the final tabulation.

The final tabulation of the perturbation in the third coordinate is checked by the test

$$t=0 \quad ; \quad u=0$$

COMPUTATION OF THE PERTURBATIONS FOR THE TIME t.

It is well to emphasize here the distinction between the elements n_1 and c_1 and the elements n_2 and c_2 in their relation to the perturbations. Let $n\delta z_1$ denote the perturbation in the mean anomaly computed according to equations (20), (19), (18), (21), (22), (27), (28), and let $n\delta z_2$ signify the perturbation computed according to equations (20), (19), (18), (22), the final tabulation of $(n\delta z-[n\delta z])$, and an equation analogous to (34). (It must be remembered that equation (34) is for *(10) Hygiea* only. The numerical coefficients are determined for each planet individually.)

Before the determination of c there can be no confusion, for there is but one way to compute the perturbation $n\delta z$. Later, when both c_1 and c_2 are given, the computation may be performed in either manner. The latter method is, of course, adopted. The question then arises, what values of ε and c are to be used in equation (19)?

Clearly, there is only one value of ε, for

$$\varepsilon - e_1 \sin \varepsilon = c_1 + n_1 t + n\delta z_1 = c_2 + n_2 t + n\delta z_2$$

and both $n\delta z$ and ε must be found by trials. Further, since the introduction of n_2 and c_2 arises merely from a transfer of certain terms in the perturbation, the argument of the perturbation is independent of this transformation. Therefore c_1 is the constant in equation (19).

For any time t the order of computation is: equation (33), neglecting $n\delta z$, (20), (19), (18), (22), final tabulation of $n\delta z - [n\delta z]$, and the equation analogous to (34). Since the perturbations are large, the argument ε is not sufficiently accurate when $n\delta z$ is neglected. It is, therefore, always necessary to make a second approximation for $n\delta z$. In the first trial the small terms may be omitted.

(10) *Hygiea.*

PERTURBATIONS $n\delta z$, ν, u, FOR 1873, SEPT. 20.4491, BER. M. T.

$\log e_1$ (degrees)	0.80432
$\log \frac{w}{2}$	8.48578
$\log \frac{1}{57.30} \cdot \frac{2}{w}$	9.7561
$\frac{1-w}{2} \cdot c - c'$	217°278
ζ_0	220°848
ϑ_0	221°880
c_2	121°8661
	1873
Ber. M. T.	Sept.[1] 20.4491
t	+ 8039ᵈ4491
$n_2 t$	+ 1422°2156
$c_2 + n_2 t$	1544.0817 104.0817+1440°
$n\delta z$	− [2] 3.666
$M = c_2 + n_2 t + n\delta z$	100.416
ε	[3] 106.526+1440° 1546.526
$\log \varepsilon$	3.18935
$\log \frac{w}{2}\varepsilon$	1.67513
$\frac{w}{2}\varepsilon$	+ 47°329
$\frac{w}{2}\varepsilon - [n'\delta z']$	+ 47°053
$\log \left(\frac{w}{2}\varepsilon - [n'\delta z']\right)$	1.67259
$\log (9.99572)\left(\frac{w}{2}\varepsilon - [n'\delta z']\right)$	1.66831
$(9.99572)\left(\frac{w}{2}\varepsilon - [n'\delta z']\right)$	+ 46°592
ζ	263°870
2ζ	167°740
4ζ	335.480
6ζ	143.220
$\zeta - \zeta_0$	43.022
$2\zeta + 72°5246$	240°265
$4\zeta + 305.627$	281.107
$6\zeta + 186.48$	329.70
$2\zeta + 68°83$	236.57
$4\zeta + 309.75$	285.23

log sin	9.9387_n
log sin	9.9918_n
log sin	9.703_n
log cos	9.741_n
log cos	9.419
$\log (\zeta - \zeta_0)$	1.6337
$\log \frac{2}{w}(\zeta - \zeta_0)$	1.3898
$\log \frac{2}{w}(\zeta - \zeta_0) \cos$	1.131_n
$\log \frac{2}{w}(\zeta - \zeta_0) \cos$	0.809
	− 11405″
	− 2017
	− 140
	− 124
	+10
$[n\delta z]_1$	− 13676″
$\log [n\delta z]_1$ (secs)	4.13596_n
$\log [n\delta z]_1$ (degrees)	0.57966_n
$[n\delta z]_1$	−3°.7989
$\log (9.6715)\, [n\delta z]_1$	0.2512_n
$(9.6715)\, [n\delta z]_1$	−1°783
ϑ	262°087
$\vartheta - \vartheta_0$	40°207
ϑ	262°087
2ϑ	164.174
3ϑ	66.261
4ϑ	328.348
5ϑ	230.435
6ϑ	132.522
7ϑ	34.609
8ϑ	296.696
(4)	
$\frac{1}{2}\varepsilon$	53°.263
ε	106.526
$\frac{3}{2}\varepsilon$	159.789
2ε	213.052
$\frac{5}{2}\varepsilon$	266.315

$\chi = i \cdot \frac{\varepsilon}{2} + j\vartheta$	
$\frac{1}{2}\varepsilon + \vartheta$	315°350
$\frac{1}{2}\varepsilon + 3\vartheta$	119.524
$\frac{1}{2}\varepsilon + 5\vartheta$	283.698
$-\frac{1}{2}\varepsilon + \vartheta$	208.824
$-\frac{1}{2}\varepsilon + 3\vartheta$	12.998
ε	106.526
$\varepsilon + 2\vartheta$	270.700
$\varepsilon + 4\vartheta$	74.874
$\varepsilon + 6\vartheta$	239.048
$\varepsilon + 8\vartheta$	43.222
$-\varepsilon + 2\vartheta$	57.648
$-\varepsilon + 4\vartheta$	221.822
$\frac{3}{2}\varepsilon + \vartheta$	61.876
$\frac{3}{2}\varepsilon + 3\vartheta$	226.050
$\frac{3}{2}\varepsilon + 5\vartheta$	30.224
$\frac{3}{2}\varepsilon + 7\vartheta$	194.398
$-\frac{3}{2}\varepsilon + \vartheta$	102.298
2ε	213.052
$2\varepsilon + 2\vartheta$	17.226
$2\varepsilon + 4\vartheta$	181.400
$2\varepsilon + 6\vartheta$	345.574
$\frac{5}{2}\varepsilon + 5\vartheta$	136.750
$\frac{5}{2}\varepsilon + 7\vartheta$	300.924

[1] Corr. for aberr. [2] From previous approx. [3] From Astrand's table. [4] See eq. (1), page 16.

(10) *Hygiea.*

Perturbations $n\delta z$, ν, u, for 1873, Sept. 20.4491, Ber. M. T.—Continued.

$\chi = i\frac{\epsilon}{2} + j\vartheta$	$n\delta z - [n\delta z]$			ν			u		
i j	$\chi+K$	log sin $(\chi+K)$	k sin $(\chi+K)$	$\chi+K$	log cos $(\chi+K)$	k cos $(\chi+K)$	$\chi+K$	log sin $(\chi+K)$	k sin $(\chi+K)$
				°			°		
0 0				180. 00	0. 000$_n$	− 40″	270. 00	0. 000$_n$	− 8″
0 2				58. 608	9 7167	+375″	296. 33	9. 952$_n$	−12
0 4				98. 841	9. 1867$_n$	− 33	238	9. 928$_n$	0
0 6	°			146. 28	9. 920$_n$	− 52			
1 1	353. 286	9. 0679$_n$	− 56″	173. 425	9. 9971$_n$	− 137	80. 40	9. 994	+ 11″
1 3	41. 102	9. 8178	+ 479″	221. 824	9. 8723$_n$	− 193	110. 78	9. 971	+ 14
1 5	88. 71	0. 000	+ 265	268. 86	8. 2988$_n$	− 2	156. 04	9. 609	+ 3
−1 1	233 74	9. 907$_n$	− 85	192. 38	9. 9898$_n$	− 3	122. 28	9. 927	+ 11
−1 3	106. 53	9. 982	+ 41	111. 41	9. 562$_n$	− 13	172. 10	9 138	+ 1
2 0	119. 27	9. 941	+ 18	300. 02	9. 699	+ 3	124. 52	9. 916	+103
2 2	347. 748	9. 3268$_n$	− 761	167. 726	9. 9900$_n$	−1852	79. 94	9. 993	+ 59
2 4	35. 927	9. 7686	+ 437	215. 194	9. 9123$_n$	− 329	104. 99	9. 985	+ 25
2 6	83. 53	9. 997	+ 243	263. 148	9. 0767$_n$	− 15	150. 50	9. 692	+ 5
2 8	127. 4	9. 90	+ 35	309. 92	9. 807	+ 27			
−2 2	115. 61	9. 955	+ 84	88. 6	8. 39	0	0. 51	7. 948	+ 1
−2 4	311. 82	9. 872$_n$	− 3	349. 03	9. 992	+ 6			
3 1				276. 65	9. 064	+ 1	225. 83	9. 856$_n$	− 5
3 3	163. 51	9. 453	+ 36	341. 94	9. 978	+ 85	257. 49	9. 990$_n$	− 3
3 5	208. 94	9. 685$_n$	− 34	28. 42	9. 944	+ 34	278. 56	9. 995$_n$	− 1
3 7	253. 08	9. 981$_n$	− 13	54. 02	9. 769	+ 1			
−3 1				136. 98	9. 864$_n$	− 2	4. 98	8. 939	0
4 0							348. 8	9. 288$_n$	0
4 2	274. 434	9. 9987$_n$	− 104				40. 38	9. 811	+ 1
4 4	327. 82	9. 726$_n$	− 21	156. 79	9. 963$_n$	− 9	85. 1	9. 998	+ 1
4 6	22. 1	9. 58	+ 5	198. 96	9. 976$_n$	− 4	92. 97	9. 999	+ 1
5 5	150. 8	9. 69	+ 5	330. 79	9. 941	+ 10			
5 7	196. 6	9. 46$_n$	− 2	16. 85	9. 981	+ 8			
			+1648″ −1079″			+550″ −2684″			+236″ −29″

$(\vartheta-\vartheta_0)$[1] or T	$\chi+K'$	log cos $(\chi+K')$	k' cos $(\chi+K')$	$\chi+K'$	log sin $(\chi+K')$	k' sin $(\chi+K')$	$(\chi+K')$	log cos $(\chi+K')$	k' cos $(\chi+K')$
				°					
0 0				270. 00	0. 000$_n$	− 6″	180. 00	0. 000$_n$	0
0 2				232. 94	9. 902$_n$	− 8			
0 4	°			281. 51	9. 991$_n$	− 7			
2 0	292. 53	9. 583	+ 371″	292. 573	9. 965$_n$	− 447	47. 67	9. 828	+ 6″
2 2	4. 7	0. 00	+ 2	351. 93	9. 147$_n$	0			
2 4	41. 3	9. 88	+ 6	41. 29	9. 819	+ 3″			
−2 2	123. 85	9. 746$_n$	− 2″	305. 02	9. 913$_n$	− 1			
4 0	219. 90	9. 885$_n$	− 20						
4 2	104. 1	9. 39$_n$	− 1	104. 23	9. 986	+ 4			
4 4	154. 82	9. 957$_n$	− 1	147	9. 736	+ 1			
			+ 379″ − 24″			+ 8″ − 469″			+ 6″

$(\vartheta-\vartheta_0)^2$	$\chi+K''$	log sin $(\chi+K'')$	k'' sin $(\chi+K'')$	$\chi+K''$	log cos $(\chi+K'')$	k'' cos $(\chi+K'')$
	°			°		
2 0	296. 23	9. 953$_n$	− 3″	112. 79	9. 588$_n$	− 1″
4 0	227. 15	9. 865$_n$	− 1	47	9. 834	0
			− 4″			− 1″

[1] For perturbation u use factor T.

(10) *Hygiea.*

PERTURBATIONS $n\delta z$, ν, u, FOR 1873, SEPT. 20.4491, BER. M. T.—Continued.

$\log (\vartheta-\vartheta_0)$ rad.	9.8162				
$\log (\vartheta-\vartheta_0)^2$	9.692				
$\log$ T	1.3426				
$\log \frac{a}{\cos i} \sin 1''$	5.183				
	$n\delta z$		ν		u
$\Sigma k \sin (\chi+K)$	$+$ 569″	$\Sigma k \cos (\chi+K)$	$-$ 2134″	$\Sigma k \sin (\chi+K)$	$+$ 207″
$\Sigma k' \cos (\chi+K')$	$+$ 355	$\Sigma k' \sin (\chi+K')$	$-$ 461″	$\Sigma k' \cos (\chi+K')$	$+$ 6
$\Sigma k'' \sin (\chi+K'')$	$-$ 4	$\Sigma k'' \cos (\chi+K'')$	$-$ 1″		
$\log \Sigma k' \cos (\chi+K')$	2.5502	$\log \Sigma k' \sin (\chi+K')$	2.664_n	$\log \Sigma k' \cos (\chi+K')$	0.778
$\log (\vartheta-\vartheta_0) \Sigma k' \cos (\chi+K')$	2.396	$\log (\vartheta-\vartheta_0) \Sigma k' \sin (\chi+K')$	2.510_n	$\log \mathrm{T}.\ \Sigma k' \cos (\chi+K')$	2.121
$(\vartheta-\vartheta_0) \Sigma k' \cos (\chi+K')$	$+$ 249″	$(\vartheta-\vartheta_0) \Sigma k' \sin (\chi+K')$	$-$ 324″	$\mathrm{T}.\ \Sigma k' \cos (\chi+K')$	$+$ 132″
$\log \Sigma k'' \sin (\chi+K'')$	0.602_n	ν	$-$ 2458″	u	$+$ 339″
$\log (\vartheta-\vartheta_0)^2 \Sigma k'' \sin (\chi+K'')$	0.294_n	$\log \nu$ (secs)	3.3906_n	$\log u$	2.530
$(\vartheta-\vartheta_0)^2 \Sigma$	$-$ 2″	$\log \nu$ (rad)	8.0762_n	$\log d\beta$	7.713
		$\log (1+\nu)$	9.99480	$\log \cos a$	8.798_n
				$\log \cos b$	9.619_n
$n\delta z-[n\delta z]$	$+$ 816″			$\log \cos c$	9.958
	$+0^\circ.2267$				
$(8.3192)\ [n'\delta z']$	$+0^\circ.0058$			$\log \Delta x$	6.511_n
$[n\delta z]_1$	$-3^\circ.7989$			$\log \Delta y$	7.332_n
$n\delta z$	$-$ 3.5664			$\log \Delta z$	7.671
$-0.00014\ n\delta z$	$+$ 5				
$n\delta z$	$-\ 3^\circ.566$			Δx	-0.00032
				Δy	-0.00215
				Δz	$+0.00469$

The computation of the geocentric place on page 26 is analogous to the usual method for two body motion, the fundamental equations being (1), (2), (3). A complete set of formulae and an example of the computation is also given in Memoirs of the National Academy of Sciences, Vol. X, Seventh Memoir, p. 215.

CONSTANTS FOR THE EQUATOR.

	A′ yearly var.	B′ yearly var.	C′ yearly var.	log sin a log cos a	log sin b log cos b	log sin c log cos c
1850.0	320°.833+0°.01399	229°.182+0°.01404	238°.657+0°.01310	9.99914 8.799_n	9.95884 9.619_n	9.62355 9.958
1900.0	321.532+0.01399	229.885+0.01405	239.312+0.01308	9.99914 8.797_n	9.95868 9.619_n	9.62423 9.958
1950.0	322.232+0.01399	230.587+0.01406	239.965+0.01306	9.99915 8.795_n	9.95853 9.620_n	9.62490 9.958

(10) *Hygiea.*

Comparison, Observation—Computation, 1873, Sept. 20.4491, Ber. M. T.

	1873	x	+3.0709
Ber. M. T.	Sept. 20.4491	X	−1.00281
c_2+n_2t	104°.0817	Δx	−0.00032
$n\delta z$	− 3.5660	ξ	+2.0678
$M=c_2+n_2t+n\delta z$	100.5157		
		y	−0.89314
$dM°$	− 0°.4843	Y	+0.03260
dM'	− 29'.06	Δy	−0.00215
$d\varphi'$	+ 3'.15	η	−0.86269
$\frac{dv}{d\varphi}$	+ 1.8124		
$\left(\frac{Ddv}{d\varphi}\right)dM'$	+ 68		
		z	−0.19677
$\Delta\varphi.\frac{d\varphi}{20}$	+ 8	Z	+0.01415
		Δz	+0.00469
$\Sigma.d\varphi'$	+ 5'.73	ζ	−0.17793
$d(v-M)/dM$	− 0.0674		
[1] $\frac{1}{2}D_m.dM°$	+ 8		
$\Sigma.dM'$	+ 1'.94	log ρ cos δ cos α	0.31551
$v-M$	+ 12° 0'.14	cos α	9.96515
v_1-M_1	+ 12° 7'.81	sin α	9.58550$_n$
	+ 12°.1302	log ρ cos δ sin α	9.93586$_n$
$\bar{f}=v_1$	112°.6459	log tg α	9.62035$_n$
		α	337° 21' 14''
log cos $\bar{f}$	9.58550$_n$		22^h 29^m 24^s.9
log e_1 cos $\bar{f}$	8.63170$_n$	Red to True α	+1.5
log $(1+e_1 \cos\bar{f})$	9.98099	True α	22^h 29^m 26^s.4
log $\bar{r}$	0.51092	Obs. α (A. N. 2029)	22^h 29^m 07^s.1
log $(1+\nu)$	9.99480		
log r	0.50572		
		log ρ cos δ	0.35036
A'	321°.1548	cos δ	9.99864
B'	229.5058	sin δ	8.89852$_n$
C'	238.9584	log ρ sin δ	9.25025$_n$
		log tg δ	8.89989$_n$
$A'+\bar{f}$	73.8007	δ	−4° 32' 26''
$B'+\bar{f}$	342.1517	Red to True δ	+6''
$C'+\bar{f}$	351.6043	True δ	−4° 32' 20''
		Obs. δ (A. N. 2029)	−4° 33' 27''
log sin a	9.99914		
log sin $(A'+\bar{f})$	9.98240		
log x	0.48726	log ρ	0.35172
log sin b	9.95877		
log sin $(B'+\bar{f})$	9.48643$_n$		
log y	9.95092$_n$	$(O-C)$	
		$\Delta\alpha$ cos δ	−19^s.3
log sin c	9.62387	$\Delta\delta$	−1' 7''
log sin $(C'+\bar{f})$	9.16438$_n$		
log z	9.29397$_n$		

Given a series of observations well distributed around the orbit and extending over as long an interval as is available, the elements can be corrected by the method of least squares.

For this purpose the formulae by Bauschinger[2] are convenient. The equations of condition are set up for the residuals in the plane of the orbit and perpendicular to the plane, as seen from the earth. This resolution of the residuals is convenient because it keeps the same resolution into components as is used in the theory of Hansen.

It is to be noticed that the elements to be used in computing the differential coefficients are the finally adopted constant elements referred to the equator by the proper transformation. The value of r to be used is

$$r=\bar{r}(1+\nu)$$

except in the equation

$$\sin\bar{\varepsilon}=\frac{\bar{r}}{a_2\sqrt{1-e_1^2}}\sin\bar{f} \qquad \text{(Hansen's notation)}$$

[1] Tafel zur Berechnung der wahren Anomalie, Veröffentlichungen des Rechen-Instituts der Königlichen Sternwarte zu Berlin No. 1.

[2] Über das Problem der Bahnverbesserung, Veröffentlichungen des Königlichen Astronomischen Rechen-Instituts zu Berlin, No. 23, Berlin, 1903.

The use of $\bar{\varepsilon}, \bar{f}, r$ and constant elements is equivalent to the use of osculating elements for the given date of observation.

(10) *Hygiea*

Unit of $k=1''$.

$\chi = i\frac{\varepsilon}{2}+j\vartheta$		$n\delta z-[n\delta z]$		ν		u	
i	j	log k	K	log k	K	log k	K
			°		°		°
0	0			1. 604	180. 00	0. 89	270. 00
0	2			2. 8570	251. 434	1. 118	132. 16
0	4			2. 3364	130. 493	8. 25	270
0	6			1. 800	13. 76		
1	1	2. 6771	37. 936	2. 1397	218. 075	1. 057	125. 05
1	3	2. 8627	281. 578	2. 4135	102. 300	1. 161	351. 26
1	5	2. 4238	165. 01	1. 965	345. 16	0. 930	232. 34
−1	1	2. 022	24. 92	0. 55	343. 56	1. 119	273. 46
−1	3	1. 628	93. 53	1. 543	98. 41	0. 981	159. 10
2	0	[1. 545][1] 1. 320	[7. 53][1] 12. 74	0. 711	193. 49	2. 097	17. 99
2	2	3. 5546	77. 048	3. 2776	257. 026	1. 777	169. 24
2	4	2. 8719	321. 053	2. 6054	140. 320	1. 412	30. 12
2	6	2. 389	204. 49	2. 1033	24. 100	1. 034	271. 45
2	8	1. 64	84. 2	1. 62	266. 70		
−2	2	1. 970	57. 96	1. 27	31. 0	1. 824	302. 86
−2	4	0. 602	90. 00	0. 80	127. 21		
3	1			0. 90	214. 77	0. 826	163. 95
3	3	2. 100	297. 46	1. 95	115. 89	0. 446	31. 44
3	5	1. 841	178. 72	1. 583	358. 20	0. 171	248. 34
3	7	1. 12	58. 68	0. 34	219. 62		
−3	1			0. 42	34. 68	0. 673	262. 68
4	0					0. 00	135. 7
4	2	2. 0170	257. 208			0. 270	23. 15
4	4	1. 589	146. 42	0. 97	335. 39	9. 91	263. 7
4	6	1. 14	36. 5	0. 66	213. 39	9. 73	107. 40
5	5	1. 038	14. 0	1. 062	194. 04		
5	7	0. 88	255. 7	0. 94	75. 93		

$(\vartheta-\vartheta_0)$ or T		log k'	K'	log k'	K'	log k'	K'
			°		°		°
0	0			0. 799	270. 00	9. 690	180. 00
0	2			1. 021	68. 77		
0	4			0. 86	313. 16		
2	0	2. 9862	186. 00	2. 6850	186. 047	0. 957	301. 14
2	2	0. 18	94	0. 12	81. 23		
2	4	0. 88	326. 4	0. 60	326. 42		
−2	2	0. 60	66. 20	0. 11	247. 37		
4	0	1. 414	6. 85				
4	2	0. 68	86. 9	0. 580	87. 00		
4	4	0. 11	333. 42	0. 09	326		

$(\vartheta-\vartheta_0)^2$		log k''	K''	log k''	K''
			°		°
2	0	0. 58	189. 70	0. 26	6. 26
4	0	9. 91	14. 10	9. 6	194

$$n\delta z-[n\delta z]=\Sigma k \sin(\chi+K) +(\vartheta-\vartheta_0)\Sigma k' \cos(\chi+K') +(\vartheta-\vartheta_0)^2\Sigma k'' \sin(\chi+K'')$$

$$\nu=\Sigma k \cos(\chi+K) +(\vartheta-\vartheta_0)\Sigma k' \sin(\chi+K') +(\vartheta-\vartheta_0)^2\Sigma k'' \cos(\chi+K'')$$

$$u=\Sigma k \sin(\chi+K) +T\Sigma k' \cos(\chi+K')$$

Where T is expressed in Julian years from date of osculation.

$\chi=i\frac{\varepsilon}{2}+j\vartheta$ where in ε the multiples of 2π must be retained.

$\vartheta_0=221.811$

COMPARISON OF THE REVISED WITH V. ZEIPEL'S ORIGINAL TABLES.

It was originally planned to conclude the example with a least squares solution of the orbit on the basis of the observations used by v. Zeipel for the same purpose, and to test conclusively the relative value of the revised and v. Zeipel's original tables by representing recent observations with both sets of elements and tables.

In the course of the computation doubt arose regarding the accuracy of some of the observations selected by v. Zeipel, which led us to reject them and substitute other observa-

[1] In the determination of the constant c use quantities in brackets.

tions. This substitution produced an unfavorable distribution of the observed places in the orbit and the resulting least squares solution was not satisfactory.

In the meantime, pending a resumption of the least squares solution on the basis of a more favorable distribution of observed places,[1] the following conclusions may be drawn regarding the revised and v. Zeipel's original tables:

1. v. Zeipel's tables have been slightly improved by the correction of some numerical errors.

2. A moderate further improvement has been accomplished by an extension of the tables in so far as seemed practicable without a more exhaustive and unwarranted study of the practical convergence of the auxiliary series, by including certain terms of higher order and degree.

With reference to the correction of the orbit and the representation of observations by a least squares solution, it should be observed that

(1) A symmetrical distribution of the observed positions in the orbit is essential to counteract the effect of neglected perturbations of higher order and degree and of major planets other than *Jupiter*. For the *Hecuba* Group, in general, the mean motions of the minor planets may be nearly commensurable with those of *Saturn*, *Mars*, or the *Earth* in the ratios 3/2, 3/1, or 3/5.

(2) However accurate the initial osculating elements may be, comparatively large residuals may remain on account of neglected perturbations.

TABLE A (XXXV).

Logarithmic. $n\delta z - [n\delta z]$ Unit=1″

	Sin	w^{-3}	w^{-2}	w^{-1}	w^0	w	w^2
$\eta\eta'$	$\frac{1}{2}\varepsilon+\vartheta$				4.1570	4.8741_n	
	$\frac{1}{2}\varepsilon+\vartheta+\Delta$				2.7684_n	3.3827	3.7172_n
η^2	$\frac{1}{2}\varepsilon+\vartheta+\Delta$				4.0056_n	4.7686	
η'^2	$\frac{1}{2}\varepsilon+\vartheta+\Delta$				4.0766_n	4.8295	
j^2	$\frac{1}{2}\varepsilon+\vartheta+\Delta$				4.1365	4.8738_n	
$\eta\eta'$	$\frac{1}{2}\varepsilon+\vartheta+2\Delta$				3.3345	4.5162_n	
η'	$\frac{1}{2}\varepsilon+3\vartheta+2\Delta$				4.2240_n	4.9611	5.6685_n
η	$\frac{1}{2}\varepsilon+3\vartheta+3\Delta$				4.0671	4.8483_n	5.5636
η'^2	$\frac{1}{2}\varepsilon+5\vartheta+3\Delta$				5.0926_n	6.0018	
$\eta\eta'$	$\frac{1}{2}\varepsilon+5\vartheta+4\Delta$				5.2325	6.1714_n	
η^2	$\frac{1}{2}\varepsilon+5\vartheta+5\Delta$				4.7675_n	5.7344	
j^2	$\frac{1}{2}\varepsilon+5\vartheta+4\Delta-\Sigma$				3.8050_n	4.7998	
η'	$-\frac{1}{2}\varepsilon+\vartheta$				3.3112	3.8350_n	4.1355
η	$-\frac{1}{2}\varepsilon+\vartheta+\Delta$				3.2065_n	3.7910	4.0833_n
η'^2	$-\frac{1}{2}\varepsilon+3\vartheta+\Delta$				3.5338	4.6236_n	
$\eta\eta'$	$-\frac{1}{2}\varepsilon+3\vartheta+2\Delta$				4.0879	5.0382	
η^2	$-\frac{1}{2}\varepsilon+3\vartheta+3\Delta$				3.6012_n	4.5318_n	
j^2	$-\frac{1}{2}\varepsilon+3\vartheta+2\Delta-\Sigma$				3.2074	4.1925_n	
η	ε		9.868_n	0.5689	2.922	3.4600_n	3.3670
η'	$\varepsilon+\Delta$		9.482	0.2533_n	2.673_n	3.2959	3.1772_n
$\eta\eta'$	$\varepsilon+2\vartheta+\Delta$	0.746_n	1.384	3.2927_n	4.14906	4.6990_n	
	$\varepsilon+2\vartheta+2\Delta$		9.788_n	2.47560	3.10847_n	3.4540	3.3960_n
η^2	$\varepsilon+2\vartheta+2\Delta$	0.645	1.342_n	2.305_n	3.6179_n	4.4018	
η'^2	$\varepsilon+2\vartheta+2\Delta$	0.326	1.119_n	2.935_n	3.3017_n	4.39206	
j^2	$\varepsilon+2\vartheta+2\Delta$			3.4276_n	4.23764	4.76933_n	
$\eta\eta'$	$\varepsilon+2\vartheta+3\Delta$	0.28_n	1.102	3.1738	3.5449_n	3.8446_n	
$\eta\eta'^2$	$\varepsilon+4\vartheta+2\Delta$			3.6004	4.27485		
η'	$\varepsilon+4\vartheta+3\Delta$	9.057	0.692_n	3.10161	3.9302_n	4.52415	4.78162_n
$\eta^2\eta'$	$\varepsilon+4\vartheta+3\Delta$			4.0519_n	3.7975		
η'^3	$\varepsilon+4\vartheta+3\Delta$			4.1385_n	4.6961		
$j^2\eta'$	$\varepsilon+4\vartheta+3\Delta$			4.2431_n	5.1290		
η	$\varepsilon+4\vartheta+4\Delta$	9.500_n	0.522	2.9351_n	3.8035	4.41616_n	4.63017
η^3	$\varepsilon+4\vartheta+4\Delta$			3.7714	4.2108_n		
$\eta\eta'^2$	$\varepsilon+4\vartheta+4\Delta$			4.4165	5.0931_n		
$j^2\eta$	$\varepsilon+4\vartheta+4\Delta$			4.1524	5.0661_n		
$\eta^2\eta'$	$\varepsilon+4\vartheta+5\Delta$			4.0588_n	4.8136		

[1] Since 1913, when the revision of the tables was concluded, Miss Glancy has continued the problem of (10) *Hygiea* independently at the Observatorio Nacional, Córdoba, with the following highly satisfactory results, which substantiate further the increased accuracy of the revised tables (1) The original osculating elements and the revised tables resulted in a greatly improved representation of the selected observations (1849–1881) over the representation obtained with the original tables. (2) After the correction of the original osculating elements by least squares solution (*a*) on the basis of v. Zeipel's tables and residuals, (*b*) on the basis of the residuals resulting from the revised tables, the representation of the selected observations was equally satisfactory; but 3 later observations, taken in 1910, 1914, and 1917, are represented far better by the revised tables and corresponding elements than by the original tables and corresponding elements. (cf. Astronomical Journal, Vol. 32, p. 27, No. 748, January 1919) A. O. Leuschner.

Logarithmic. TABLE A (XXXV)—Continued. Unit=1″

	Sin	w^{-3}	w^{-2}	w^{-1}	w^{0}	w	w^{2}
$j^2\eta$	$\varepsilon+4\vartheta+3J-\Sigma$			3.2322_n	4.2342		
$j^2\ \eta'$	$\varepsilon+4\vartheta+4J-\Sigma$			2.741_n	3.0962		
η'^2	$\varepsilon+6\vartheta+4J$	0.28	0.64_n	3.8027	4.77998_n	5.52852	
$\eta\ \eta'$	$\varepsilon+6\vartheta+5J$	0.596_n	1.070	3.9374_n	4.91342	5.70347_n	
η^2	$\varepsilon+6\vartheta+6J$	0.255	0.8_n	3.4684	4.50125_n	5.27451	
j^2	$\varepsilon+6\vartheta+5J-\Sigma$	8.8	9.3_n	2.415	3.4823_n	4.2931	
η'^3	$\varepsilon+8\vartheta+5J$			4.5564	5.4999_n		
$\eta\ \eta'^2$	$\varepsilon+8\vartheta+6J$			4.8668_n	5.8416		
$\eta^2\eta'$	$\varepsilon+8\vartheta+7J$			4.6990	5.7030_n		
η^3	$\varepsilon+8\vartheta+8J$			4.0531_n	5.0844		
$j^2\ \eta'$	$\varepsilon+8\vartheta+6J-\Sigma$			3.5829	4.6352_n		
$j^2\eta$	$\varepsilon+8\vartheta+7J-\Sigma$			3.3768_n	4.4540		
η'^2	$-\ \varepsilon+2\vartheta$	0.606	1.422_n	3.2132	3.6657_n	3.9260	
$\eta\ \eta'$	$-\ \varepsilon+2\vartheta+\ J$	0.791_n	1.690	3.3777_n	3.8866	4.72168	
η^2	$-\ \varepsilon+2\vartheta+2J$	0.418	1.365_n	2.894	3.4616_n	3.8078	
j^2	$-\ \varepsilon+2\vartheta+\ J-\Sigma$	9.34	0.28_n	2.938	3.4714_n	3.7862	
η'^3	$-\ \varepsilon+4\vartheta+\ J$			3.5208	4.07255		
$\eta\ \eta'^2$	$-\ \varepsilon+4\vartheta+2J$			3.4965_n	4.59582		
$\eta^2\eta'$	$-\ \varepsilon+4\vartheta+3J$			3.2416	4.5467_n		
η^3	$-\ \varepsilon+4\vartheta+4J$			2.430_n	3.9848		
$j^2\ \eta'$	$-\ \varepsilon+4\vartheta+2J-\Sigma$			3.5496	4.19852_n		
$j^2\eta$	$-\ \varepsilon+4\vartheta+3J-\Sigma$			3.3247_n	4.05994		
$\eta\ \eta'$	$\frac{3}{2}\varepsilon+3\vartheta+2J$				3.6731	4.0029_n	
	$\frac{3}{2}\varepsilon+3\vartheta+3J$				2.3528	3.2475_n	3.9005
η^2	$\frac{3}{2}\varepsilon+3\vartheta+3J$				3.6181_n	4.2122	
j^2	$\frac{3}{2}\varepsilon+3\vartheta+3J$				3.4072_n	4.4000	
$\eta\ \eta'$	$\frac{3}{2}\varepsilon+3\vartheta+4J$				3.5244	4.4012_n	
η'	$\frac{3}{2}\varepsilon+5\vartheta+4J$				3.3533	4.4231_n	5.2725
η	$\frac{3}{2}\varepsilon+5\vartheta+5J$				3.1780_n	4.2730	5.1359_n
η'^2	$\frac{3}{2}\varepsilon+7\vartheta+5J$				4.2775	5.4708_n	
$\eta\ \eta'$	$\frac{3}{2}\varepsilon+7\vartheta+6J$				4.4051_n	5.6177	
η^2	$\frac{3}{2}\varepsilon+7\vartheta+7J$				3.9296	5.1605_n	
η	$2\varepsilon+2\vartheta+2J$		9.486	2.1744_n	2.708	2.889_n	2.599_n
η'	$2\varepsilon+2\vartheta+3J$				1.916_n	2.501	2.516_n
$\eta\ \eta'$	$2\varepsilon+4\vartheta+3J$	8.8_n	0.561	2.789_n	3.5813	4.1074_n	
	$2\varepsilon+4\vartheta+4J$		8.90_n	9.599	1.711	2.5795_n	3.1726
η^2	$2\varepsilon+4\vartheta+4J$	9.2	0.34_n	2.618	3.4962_n	4.0890	
η'	$2\varepsilon+6\vartheta+5J$		9.819_n	0.5840	2.7821	3.7794_n	4.51865
η	$2\varepsilon+6\vartheta+6J$		9.653	0.4645_n	2.5979_n	3.6265	4.38424_n
	$\frac{5}{2}\varepsilon+5\vartheta+5J$				1.2340	2.1166_n	2.7076
η'	$\frac{5}{2}\varepsilon+7\vartheta+6J$				2.3679	3.3518_n	4.0587
η	$\frac{5}{2}\varepsilon+7\vartheta+7J$				2.1758_n	3.1926	3.9204_n
	$(\vartheta-\vartheta_0)$ cos						
η	ε	0.1021_n	0.728	2.8978_n	3.4504	3.7168_n	
η^3	ε	1.377_n	2.346	3.8211_n	4.6762		
$\eta\ \eta'^2$	ε	1.941_n	2.815	4.4076_n	5.1971		
$j^2\eta$	ε	1.364	2.220_n	4.4076	5.1971_n	5.7086	
η'	$\varepsilon+\quad J$	9.658	0.774_n	2.7836	3.3840_n	3.6946	
$\eta^2\ \eta'$	$\varepsilon+\quad J$	1.863	2.755_n	4.2546	5.0814_n		
η'^3	$\varepsilon+\quad J$	1.844	2.642_n	4.1953	4.9770_n		
$j^2\ \eta'$	$\varepsilon+\quad J$	1.170_n	2.049	4.3715_n	5.1770	5.6975_n	
$\eta\ \eta'^2$	$\varepsilon+\quad 2J$	1.742_n	2.574	4.0203_n	4.8466		
$j^2\ \eta'$	$\varepsilon+\quad \Sigma$	0.716	1.65_n	4.0809	4.8829_n	5.4008	
$j^2\eta$	$\varepsilon+\quad J+\Sigma$	1.00_n	1.89	4.3427_n	5.0837	5.5553_n	
$\eta^2\eta'$	$-\ \varepsilon+\quad J$	1.562	2.455_n	3.9535	4.7803_n		
η^2	2ε	9.801	0.43_n	2.5842	3.1493_n	3.4158	
$\eta\ \eta'$	$2\varepsilon+\quad J$	9.357_n	0.473	2.4548_n	3.0830	3.3936_n	
	$(\vartheta-\vartheta_0)^2$ sin						
η	ε		9.56_n	0.42			
η'	$\varepsilon+\quad J$		9.43	0.32_n			

$$n\delta z-[n\delta z]=\Sigma w^s\eta^p\eta'^q j^{2t}C_1 \sin \text{Arg.}+(\vartheta-\vartheta_0)\Sigma w^s\eta^p\eta'^q j^{2t}C_2 \cos \text{Arg.}+(\vartheta-\vartheta_0)^2\Sigma w^s\eta^p\eta'^q j^{2t}C_3 \sin \text{Arg.}$$

where C_1, C_2, C_3 represent the respective coefficients.

TABLE B (XXXVIII).

Logarithmic. $\Phi(\vartheta)$ Unit=1 radian.

	Cos	w^{-6}	w^{-5}	w^{-4}	w^{-3}	w^{-2}	w^{-1}	w^0	w	w^2
				1.5	3.909_n	4.960	6.6748_n	7.2764	7.540_n	7.31
η^2			2.0	4.644_n	5.160	6.150	8.048_n	8.838	8.655_n	8.100_n
η'^2			1.9	3.41_n	4.75_n	6.509	8.2077_n	8.994	8.919_n	
j^2				2.83_n	5.146	6.299_n	7.994	8.740_n	8.656	
$\eta\,\eta'$	Δ		2.34_n	4.446	4.57	6.728_n	8.4022	9.1999_n	9.0854	8.079
$\eta\,\eta'^2$	2ϑ	1.6	2.6_n	5.744	6.535_n	8.3811	9.1031_n	9.0128		
η'	$2\vartheta+\Delta$		0.8_n	3.068	5.2988	7.2212_n	7.3772	8.0372	8.764_n	8.668
$\eta^2\eta'$	$2\vartheta+\Delta$	2.32_n	3.30	5.886_n	6.718	8.5059_n	9.2804	9.2017_n		
η'^3	$2\vartheta+\Delta$			5.301_n	6.149	8.2302_n	9.0154	8.938_n		
$j^2\,\eta'$	$2\vartheta+\Delta$					8.5592	9.3245_n	9.2428		
η^3	$2\vartheta+2\Delta$	2.48	3.40_n	5.422	6.292_n	7.476	8.664_n	8.636		
η	$2\vartheta+2\Delta$		1.22	2.94_n	5.1206_n	7.6416	7.9638_n	7.083_n	8.645	8.582_n
$\eta\,\eta'^2$	$2\vartheta+2\Delta$	1.9	3.0_n	5.442	6.328_n	8.0915_n	8.630_n	8.742		
$j^2\eta$	$2\vartheta+2\Delta$					8.5904_n	9.3489	9.8024_n	9.6532	
$\eta^2\eta'$	$2\vartheta+3\Delta$	2.04_n	3.00	4.98_n	5.89	8.0326	8.1973_n	7.69		
$j^2\eta$	$2\vartheta+\Delta-\Sigma$			4.51	5.42_n	8.1011	8.873_n	8.792		
$j^2\,\eta'$	$2\vartheta+2\Delta-\Sigma$			4.04_n	5.00	6.89_n	8.182	8.158_n		
η'^2	$4\vartheta+2\Delta$		2.66_n	2.7	6.1031	8.4188_n	8.5297	6.0	7.90_n	
$\eta\,\eta'$	$4\vartheta+3\Delta$		2.72	4.369	6.2526_n	8.5594	8.7988_n	7.94_n	8.287	8.210
η^2	$4\vartheta+4\Delta$		2.20_n	4.624_n	5.824	8.0924_n	8.4333	7.24	7.74_n	8.044_n
j^2	$4\vartheta+3\Delta-\Sigma$		1.5_n	2.45	4.68	7.1747_n	7.301	8.111	8.127_n	
η'^3	$6\vartheta+3\Delta$			5.301_n	6.149	9.1294_n	9.7728	9.6609_n		
$\eta\,\eta'^2$	$6\vartheta+4\Delta$			5.92	6.74_n	9.4432	0.14644_n	0.05077		
$\eta^2\eta'$	$6\vartheta+5\Delta$	2.0_n	3.0	5.93_n	6.79	9.2774_n	0.03298	9.9494_n		
η^3	$6\vartheta+6\Delta$	2.0	3.0_n	5.420	6.292_n	8.634	9.4351_n	9.3608		
$j^2\,\eta'$	$6\vartheta+4\Delta-\Sigma$			4.04_n	5.00	8.272_n	9.1028	9.0334_n		
$j^2\eta$	$6\vartheta+5\Delta-\Sigma$			4.51	5.42_n	8.0554	8.926_n	8.864		
	$(\vartheta-\vartheta_0)$ sin									
$\eta\,\eta'$	Δ			2.60_n	4.71	5.94_n	6.507_n	6.606		
η'	$2\vartheta+\Delta$		1.36	2.48	4.49	5.255_n	5.51	5.25_n		
η	$2\vartheta+2\Delta$		1.82_n	2.42	4.64_n	5.350	5.51_n	5.16		
η'^2	$4\vartheta+2\Delta$		2.34	3.00	5.392	6.179_n	6.528_n	6.665		
$\eta\,\eta'$	$4\vartheta+3\Delta$		2.89_n	3.46	5.702_n	6.467	6.851	6.979_n		
η^2	$4\vartheta+4\Delta$		2.66	3.459_n	5.357	6.127_n	6.530_n	6.653		
	$(\vartheta-\vartheta_0)^2$ cos									
η^2				2.08_n	2.08		5.546_n	5.546		
η'^2				2.54	2.54_n		5.396_n	5.396		
$\eta\,\eta'$	Δ			2.5_n	2.5		5.776	5.776_n		
		m'^3	m'^3	m'^3, m'^2	m'^3, m'^2	m'^2, m'	m'^2, m'	m'^2, m'	m'^2, m'	m'^2, m

$\Phi(\vartheta)=\Sigma w^s.\eta^p.\eta'^q.j^{2t}.C_1$ cos Arg. $+(\vartheta-\vartheta_0)\Sigma w^s.\eta^p.\eta'^q.j^{2t}.C_2$ sin Arg. $+(\vartheta-\vartheta_0)^2\Sigma w^s.\eta^p.\eta'^q.j^{2t}.C_3$ cos Arg.
where C_1, C_2, C_3 represent the respective coefficients.

TABLE C (XLIII).

Logarithmic. v Unit=1″

	Cos	w^{-3}	w^{-2}	w^{-1}	w^{0}	w	w^{2}
			8.72	9.88_n	1.6349	2.1070_n	2.2333
η^2		9.80	0.212_n		2.759	3.4922_n	
η'^2		8.9	9.23		2.937	3.6295_n	
j^2					2.937_n	3.6295	
$\eta\eta'$	Δ	9.66_n	9.78		3.1136_n	3.8440	
$\eta\eta'^2$	2ϑ	0.556_n	1.204	3.2111_n	3.7970		
η'	$2\vartheta+\Delta$		0.504_n	2.3472	2.456_n	2.686_n	3.4735
$\eta^2\eta'$	$2\vartheta+\Delta$	0.997	1.711_n	3.6559	4.3103_n		
η'^3	$2\vartheta+\Delta$	0.438	1.220_n	3.3654	4.0763_n		
$j^2\eta'$	$2\vartheta+\Delta$			3.6975_n	4.3810		
η	$2\vartheta+2\Delta$		0.438	2.952_n	3.2529	3.0689_n	3.3979_n
η^3	$2\vartheta+2\Delta$	0.732_n	1.497	3.2410_n	4.0643		
$\eta\eta'^2$	$2\vartheta+2\Delta$	0.772_n	1.589	3.4136	4.0723		
$j^2\eta$	$2\vartheta+2\Delta$			3.9048	4.5649_n	4.9303	
$\eta^2\eta'$	$2\vartheta+3\Delta$	0.505	1.344_n	3.4757_n	2.783		
$j^2\eta$	$2\vartheta+\Delta-\Sigma$	9.33_n	0.15	2.938_n	3.5830		
$j^2\eta'$	$2\vartheta+2\Delta-\Sigma$	9.20	0.10_n	2.0251	3.2961_n		
η'^2	$4\vartheta+2\Delta$	8.9	1.2819_n	3.5514	3.6173_n	3.8147	
$\eta\eta'$	$4\vartheta+3\Delta$	9.75_n	1.5024	3.7885_n	4.1394	4.3110_n	
η^2	$4\vartheta+4\Delta$	9.98	1.1342_n	3.4007	3.9091_n	4.1480	
j^2	$4\vartheta+3\Delta-\Sigma$		9.64_n	2.305	2.542_n	2.749_n	
η'^3	$6\vartheta+3\Delta$	0.438	1.220_n	4.2675	4.7993_n		
$\eta\eta'^2$	$6\vartheta+4\Delta$	1.125_n	1.862	4.6479_n	5.2324		
$\eta^2\eta'$	$6\vartheta+5\Delta$	1.198	1.947_n	4.5397	5.1768_n		
η^3	$6\vartheta+6\Delta$	0.732_n	1.508	3.9457_n	4.6328		
$j^2\eta'$	$6\vartheta+4\Delta-\Sigma$	9.20	0.10_n	3.4099	4.1710_n		
$j^2\eta$	$6\vartheta+5\Delta-\Sigma$	9.70_n	0.56	3.2601_n	4.0542		
$\eta\eta'$	$\frac{1}{2}\varepsilon+\vartheta$				3.4878_n	4.1106	
	$\frac{1}{2}\varepsilon+\vartheta+\Delta$			8.3_n	2.2106	2.7179_n	2.919
j^2	$\frac{1}{2}\varepsilon+\vartheta+\Delta$				3.5709_n	4.2261	
η^2	$\frac{1}{2}\varepsilon+\vartheta+\Delta$				3.4507	4.1296_n	
η'^2	$\frac{1}{2}\varepsilon+\vartheta+\Delta$				3.5100	4.1837_n	
$\eta\eta'$	$\frac{1}{2}\varepsilon+\vartheta+2\Delta$				2.579_n	3.9270	
η'	$\frac{1}{2}\varepsilon+3\vartheta+2\Delta$			0.08	3.6873	4.1471_n	4.7839
η	$\frac{1}{2}\varepsilon+3\vartheta+3\Delta$			9.5	3.5727_n	4.1511	4.7545_n
η'^2	$\frac{1}{2}\varepsilon+5\vartheta+3\Delta$				4.5568	5.1414_n	
$\eta\eta'$	$\frac{1}{2}\varepsilon+5\vartheta+4\Delta$				4.7261_n	5.4067	
η^2	$\frac{1}{2}\varepsilon+5\vartheta+5\Delta$				4.2862	5.0418_n	
j^2	$\frac{1}{2}\varepsilon+5\vartheta+4\Delta-\Sigma$				3.2570	4.0005_n	
η'	$-\frac{1}{2}\varepsilon+\vartheta$			1.086_n	2.7090	3.3467_n	3.7098
η	$-\frac{1}{2}\varepsilon+\vartheta+\Delta$			0.88	2.1967_n	3.0952	3.5836_n
η'^2	$-\frac{1}{2}\varepsilon+3\vartheta+\Delta$				2.514	4.1049_n	
$\eta\eta'$	$-\frac{1}{2}\varepsilon+3\vartheta+2\Delta$				4.0853	3.9122	
η^2	$-\frac{1}{2}\varepsilon+3\vartheta+3\Delta$				3.8341_n	3.8118	
j^2	$-\frac{1}{2}\varepsilon+3\vartheta+2\Delta-\Sigma$				2.416	3.6926_n	
η	ε		9.62	0.58_n	2.143_n	2.682	2.9151_n
η'	$\varepsilon+\Delta$		9.04_n	9.9	2.061	2.666_n	2.9477
$\eta\eta'$	$\varepsilon+2\vartheta+\Delta$	0.444	1.1661_n	3.0588	3.8035_n	4.2554	
	$\varepsilon+2\vartheta+2\Delta$		9.487	2.1744_n	2.7280	2.972_n	2.976
η^2	$\varepsilon+2\vartheta+2\Delta$	0.344_n	1.1143	2.692_n	3.5334	4.0772_n	
η'^2	$\varepsilon+2\vartheta+2\Delta$	0.025_n	0.828	2.634	3.0726	4.0416_n	
j^2	$\varepsilon+2\vartheta+2\Delta$			3.1265	3.8806_n	4.3473	
$\eta\eta'$	$\varepsilon+2\vartheta+3\Delta$	9.98	0.811_n	2.873_n	3.1697	3.5856	
$\eta\eta'^2$	$\varepsilon+4\vartheta+2\Delta$	1.105	1.89_n	2.864	4.3477_n		
η'	$\varepsilon+4\vartheta+3\Delta$	8.8_n	0.398	2.8000_n	3.5327	4.0065_n	4.3207
$\eta^2\eta'$	$\varepsilon+4\vartheta+3\Delta$	1.260_n	2.083	3.0931	4.4160		
η'^3	$\varepsilon+4\vartheta+3\Delta$			3.8375	4.0446_n		
$j^2\eta'$	$\varepsilon+4\vartheta+3\Delta$	0.267	1.15_n	3.9421	4.6972_n		
η	$\varepsilon+4\vartheta+4\Delta$	9.19	0.248_n	2.6356	3.4317_n	3.9469	4.2558_n
η^3	$\varepsilon+4\vartheta+4\Delta$	0.774	1.66_n	3.0934_n	3.7866_n		
$\eta\eta'^2$	$\varepsilon+4\vartheta+4\Delta$			4.1154_n	4.5547		
$j^2\eta$	$\varepsilon+4\vartheta+4\Delta$	0.455_n	1.32	3.8518_n	4.6436		
$\eta^2\eta'$	$\varepsilon+4\vartheta+5\Delta$			3.7579	4.3244_n		
$j^2\eta$	$\varepsilon+4\vartheta+3\Delta-\Sigma$			3.0030	3.8869_n		

TABLE C (XLIII)—Continued.

Logarithmic. ν Unit=1″

	Cos	w^{-3}	w^{-2}	w^{-1}	w^{0}	w	w^{2}
$j^2\ \eta'$	$\varepsilon+4\vartheta+4J-\Sigma$			2.4425	1.85_n		
η'^2	$\varepsilon+6\vartheta+4J$	9.98_n	0.480	3.5016_n	4.3723	4.9952_n	
$\eta\ \eta'$	$\varepsilon+6\vartheta+5J$	0.296	0.823_n	3.6369	4.5582_n	5.2093	
η^2	$\varepsilon+6\vartheta+6J$	9.95_n	0.538	3.1685_n	4.1334	4.8131_n	
j^2	$\varepsilon+6\vartheta+5J-\Sigma$	8.5_n	9.15	2.114_n	3.0881	3.7886_n	
η'^3	$\varepsilon+8\vartheta+5J$			4.2554_n	4.9349		
$\eta\ \eta'^2$	$\varepsilon+8\vartheta+6J$	1.320	2.152_n	4.5657	5.3010_n		
$\eta^2\eta'$	$\varepsilon+8\vartheta+7J$	1.228_n	2.093	4.3995_n	5.1827		
η^3	$\varepsilon+8\vartheta+8J$	0.648	1.54_n	3.7543	4.5812_n		
$j^2\ \eta'$	$\varepsilon+8\vartheta+6J-\Sigma$			3.2818_n	4.1442		
$j^2\eta$	$\varepsilon+8\vartheta+7J-\Sigma$			3.0763	3.9759_n		
η'^2	$-\ \varepsilon+2\vartheta$	0.305	1.1007_n	2.912	3.4958_n	3.8151	
$\eta\ \eta'$	$-\ \varepsilon+2\vartheta+\ J$	0.490_n	1.3330	3.0166_n	3.7273	4.3119	
η^2	$-\ \varepsilon+2\vartheta+2J$	0.117	0.982_n	2.288_n	3.2375_n	3.7892	
j^2	$-\ \varepsilon+2\vartheta+\ J-\Sigma$	9.04	9.96_n	2.636	3.2817_n	3.6568	
η'^3	$-\ \varepsilon+4\vartheta+\ J$			3.2197	3.9650_n		
$\eta\ \eta'^2$	$-\ \varepsilon+4\vartheta+2J$	1.146_n	1.89	3.0204	4.2441		
$\eta^2\eta'$	$-\ \varepsilon+4\vartheta+3J$	1.005	1.78_n	3.5247_n	4.0012_n		
η^3	$-\ \varepsilon+4\vartheta+4J$	0.290_n	1.15	3.1793	2.982		
$j^2\ \eta'$	$-\ \varepsilon+4\vartheta+2J-\Sigma$			3.2486	4.0585_n		
$j^2\eta$	$-\ \varepsilon+4\vartheta+3J-\Sigma$	9.98_n	0.8	2.957_n	3.8580		
η	$\frac{3}{2}\varepsilon+\ \vartheta+\ J$			9.0	2.3363	3.0704_n	3.5111
η'	$\frac{3}{2}\varepsilon+\ \vartheta+2J$			9.5	1.500	2.3585	3.1842_n
$\eta\ \eta'$	$\frac{3}{2}\varepsilon+3\vartheta+2J$				2.779	3.7820_n	
	$\frac{3}{2}\varepsilon+3\vartheta+3J$			9.28	2.1614_n	3.0257	3.6491_n
η^2	$\frac{3}{2}\varepsilon+3\vartheta+3J$				1.32	2.966	
η'^2	$\frac{3}{2}\varepsilon+3\vartheta+3J$				3.3450	4.1111_n	
j^2	$\frac{3}{2}\varepsilon+3\vartheta+3J$				3.2309	4.1965_n	
$\eta\ \eta'$	$\frac{3}{2}\varepsilon+3\vartheta+4J$				3.2991_n	4.1520	
η'	$\frac{3}{2}\varepsilon+5\vartheta+4J$			1.017	3.1617_n	4.1967	5.0160_n
η	$\frac{3}{2}\varepsilon+5\vartheta+5J$			0.88_n	2.9688	4.0380_n	4.8781
η'^2	$\frac{3}{2}\varepsilon+7\vartheta+5J$				4.0855_n	5.2422	
$\eta\ \eta'$	$\frac{3}{2}\varepsilon+7\vartheta+6J$				4.1991	5.3823_n	
η^2	$\frac{3}{2}\varepsilon+7\vartheta+7J$				3.7114_n	4.9188	
j^2	$\frac{3}{2}\varepsilon+7\vartheta+6J-\Sigma$				2.615_n	3.8317	
$\eta\ \eta'$	$-\frac{3}{2}\varepsilon+\ \vartheta$				3.2411	3.7872_n	
η^2	$-\frac{3}{2}\varepsilon+\ \vartheta+\ J$				2.819_n	3.4476	
j^2	$-\frac{3}{2}\varepsilon+\ \vartheta-\quad\Sigma$				2.9181_n	3.4813	
η^2	2ε				2.364_n	3.0737	
$\eta\ \eta'$	$2\varepsilon+\quad J$				2.624	3.3489_n	
η'^2	$2\varepsilon+\quad 2J$				2.207_n	2.978	
j^2	$2\varepsilon+\quad J+\Sigma$				2.620_n	3.2765	
η	$2\varepsilon+2\vartheta+2J$			9.8_n	1.63	2.362_n	2.873
η'	$2\varepsilon+2\vartheta+3J$			9.5	1.796	2.303_n	2.1007
$\eta\ \eta'$	$2\varepsilon+4\vartheta+3J$				1.92_n	2.700	
	$2\varepsilon+4\vartheta+4J$		8.7	8.8	1.5802_n	2.4158	2.9867_n
η^2	$2\varepsilon+4\vartheta+4J$				2.330	3.1764_n	
η'^2	$2\varepsilon+4\vartheta+4J$				3.1079	3.9008_n	
j^2	$2\varepsilon+4\vartheta+4J$				2.736	3.6809_n	
$\eta\ \eta'$	$2\varepsilon+4\vartheta+5J$				2.9881_n	3.8425	
η'	$2\varepsilon+6\vartheta+5J$		9.64	0.53	2.652_n	3.6204	4.3279_n
η	$2\varepsilon+6\vartheta+6J$		9.48_n	0.36_n	2.4419	3.4512_n	4.1892
η'^2	$2\varepsilon+8\vartheta+6J$				3.6135_n	4.6784	
$\eta\ \eta'$	$2\varepsilon+8\vartheta+7J$				3.7124	4.8075_n	
η^2	$2\varepsilon+8\vartheta+8J$				3.2109_n	4.3338	
j^2	$2\varepsilon+8\vartheta+7J-\Sigma$				2.068_n	3.2092	
	$\frac{5}{2}\varepsilon+5\vartheta+5J$			9.3_n	1.140_n	2.0056	2.5727_n
η'	$\frac{5}{2}\varepsilon+7\vartheta+6J$			0.5_n	2.2749_n	3.2377	3.9184_n
η	$\frac{5}{2}\varepsilon+7\vartheta+7J$			0.3	2.0542	3.0565_n	3.7710
	$\frac{7}{2}\varepsilon+7\vartheta+7J$			8.1	0.43_n	1.346	1.959_n
	$(\vartheta-\vartheta_0)$ sin						
$\eta\ \eta'$	J	9.66	0.810_n	2.7559	3.3840_n	3.6946	
η'	$2\vartheta+\ J$		9.79_n	0.54			
η	$2\vartheta+2J$		9.92	0.63_n			

TABLE C (XLIII)—Continued.

Logarithmic. ν Unit = 1″

	$(\vartheta-\vartheta_0)$ sin	w^{-3}	w^{-2}	w^{-1}	w^0	w	w^2
η	ε	9.801_n	0.425	2.5970_n	3.1493	3.4158_n	
η^3	ε	1.075_n	2.045	3.5201_n	4.3751		
$\eta\ \eta'^2$	ε	1.640_n	2.514	4.1066_n	4.8961		
$j^2\eta$	ε	1.063	1.916_n	4.1066	4.8961_n	5.4076	
η'	$\varepsilon+\quad\varDelta$	9.36	0.471_n	2.4824	3.0830_n	3.3936	
$\eta^2\eta'$	$\varepsilon+\quad\varDelta$	1.565	2.456_n	3.9671	4.7890_n		
η'^3	$\varepsilon+\quad\varDelta$	1.543	2.341_n	3.8942	4.6760_n		
$j^2\ \eta'$	$\varepsilon+\quad\varDelta$	0.87_n	1.75	4.0705_n	4.8759	5.3965_n	
$\eta\ \eta'^2$	$\varepsilon+\quad 2\varDelta$	1.441_n	2.273	3.7192_n	4.5456		
$j^2\ \eta'$	$\varepsilon+\quad\Sigma$	0.42	1.36_n	3.7799	4.5819_n	5.0998	
$j^2\eta$	$\varepsilon+\quad\varDelta+\Sigma$	0.695_n	1.585	4.0417_n	4.7827	5.2543_n	
η	$\varepsilon+4\vartheta+4\varDelta$		9.59_n	0.45			
η'	$\varepsilon+4\vartheta+3\varDelta$		9.46	0.34_n			
η	$2\varepsilon+2\vartheta+2\varDelta$		9.45	0.11_n			
η'	$2\varepsilon+2\vartheta+3\varDelta$		9.32_n	0.04			
$\eta^2\eta'$	$-\ \varepsilon+\quad\varDelta$	1.255_n	2.149	3.6240_n	4.4615		
	$(\vartheta-\vartheta_0)^2$ cos						
η	ε		9.25	0.117_n			
η'	$\varepsilon+\quad\varDelta$		9.12_n	0.02			
		m'^2	m'^2	m'^2, m'	m'	m'	m'

$\nu=\Sigma w^s\eta^p\eta'^q j^{2t}C_1$ cos Arg. $+(\vartheta-\vartheta_0)\Sigma w^s\eta^p\eta'^q j^{2t}C_2$ sin Arg. $+(\vartheta-\vartheta_0)^2\Sigma w^s\eta^p\eta'^q j^{2t}\ C_3$ cos Arg.
where C_1, C_2, C_3 represent the respective coefficients.

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Table D (LIV).

Logarithmic. $\Sigma\, U_{p\cdot q}\eta^p\eta'^q \sin \text{Arg}$ Unit=1″.

	Sin	w^{-1}	w^0	w
η	$-\Delta-\Pi'$		3.0621_n	3.7258
η'	$-\Pi'$		2.8235	3.5528_n
	$2\theta+\Delta-\Pi'$		2.2831	2.8483_n
η	$4\theta+3\Delta-\Pi'$	1.705	3.1591_n	3.9166_n
η'	$4\theta+2\Delta-\Pi'$		3.2462	3.8608
η	$\frac{1}{2}\varepsilon+\theta-\Pi'$		3.2112_n	3.8544
η'	$\frac{1}{2}\varepsilon+\theta+\Delta-\Pi'$		2.5875	3.4153_n
	$\frac{1}{2}\varepsilon+3\theta+2\Delta-\Pi'$		2.2787	2.6304_n
η'	$\frac{1}{2}\varepsilon+5\theta+3\Delta-\Pi'$		3.3155	3.5865_n
η	$\frac{1}{2}\varepsilon+5\theta+4\Delta-\Pi'$		3.0779_n	3.3972
η	$-\frac{1}{2}\varepsilon-\theta-2\Delta-\Pi'$		3.1158_n	3.7378
η'	$-\frac{1}{2}\varepsilon-\theta-\Delta-\Pi'$		3.1493	3.7544_n
	$-\frac{1}{2}\varepsilon+\theta-\Pi'$		2.3242	3.0060_n
η'	$-\frac{1}{2}\varepsilon+3\theta+\Delta-\Pi'$		3.3863	4.1833_n
η	$-\frac{1}{2}\varepsilon+3\theta+2\Delta-\Pi'$		3.3532_n	4.1452
η	$\varepsilon+2\theta+\Delta-\Pi'$	2.6364	3.3704_n	3.8423
η'	$\varepsilon+2\theta+2\Delta-\Pi'$	1.423_n	2.706	3.4014_n
	$\varepsilon+4\theta+3\Delta-\Pi'$	1.4042_n	2.1720	2.6339_n
η'	$\varepsilon+6\theta+4\Delta-\Pi'$	2.3306_n	3.1922	3.7582_n
η	$\varepsilon+6\theta+5\Delta-\Pi'$	2.1137	3.0138_n	3.6101
η	$-\varepsilon-2\theta-3\Delta-\Pi'$	2.7175	3.4858_n	3.9484
η'	$-\varepsilon-2\theta-2\Delta-\Pi'$	2.7756_n	3.5070	3.9456_n
	$-\varepsilon-\Delta-\Pi'$		1.6810	2.2463_n
η'	$-\varepsilon+2\theta-\Pi'$	2.8125	3.4427_n	3.7846
η	$-\varepsilon+2\theta+\Delta-\Pi'$	2.9121_n	3.4958	3.8338_n
η	$\frac{3}{2}\varepsilon+3\theta+2\Delta-\Pi'$		2.6058	3.5312_n
η'	$\frac{3}{2}\varepsilon+3\theta+3\Delta-\Pi'$		1.760	1.82_n
	$\frac{3}{2}\varepsilon+5\theta+4\Delta-\Pi'$		1.7510_n	2.8113
η'	$\frac{3}{2}\varepsilon+7\theta+5\Delta-\Pi'$		2.9120_n	4.0813
η	$-\frac{3}{2}\varepsilon-3\theta-4\Delta-\Pi'$		2.8673	3.8458_n
η'	$-\frac{3}{2}\varepsilon-3\theta-3\Delta-\Pi'$		2.9620_n	3.9124
	$-\frac{3}{2}\varepsilon-\theta-2\Delta-\Pi'$		2.0569_n	2.7932
η'	$-\frac{3}{2}\varepsilon+\theta-\Delta-\Pi'$		2.9275_n	3.4708
η	$-\frac{3}{2}\varepsilon+\theta-\Pi'$		2.9702	3.5487_n
η	$2\varepsilon+4\theta+3\Delta-\Pi'$		1.640	2.731_n
η'	$2\varepsilon+4\theta+4\Delta-\Pi'$		1.617	2.340_n
	$2\varepsilon+6\theta+5\Delta-\Pi'$		1.206_n	2.2110
η	$-2\varepsilon-4\theta-5\Delta-\Pi'$		2.4012	3.3634_n
η'	$-2\varepsilon-4\theta-4\Delta-\Pi'$		2.5241_n	3.4544
	$-2\varepsilon-2\theta-3\Delta-\Pi'$		1.5290_n	2.3210
η'	$-2\varepsilon-2\Delta-\Pi'$		2.3174_n	3.0558
η	$-2\varepsilon-\Delta-\Pi'$		2.3514	3.0737_n
		m'		

$$\frac{u}{\iota\cos i}=\Sigma\, U_{p\cdot q}\eta^p\eta'^q \sin \text{Arg.}+n_2 t\left\{K_1(\cos\varepsilon-\epsilon_1)+K_2\sin\varepsilon\right\}+c_1(\cos\varepsilon-\epsilon_1)+c_2\sin\varepsilon.$$

TABLE E_I (LV_I).

Logarithmic. K_1 Unit=1″.

	Cos	w^0	w	w^2
η'^2	$\Delta-\Pi'$	2.9180_n	3.7732	
	$\Delta+\Pi'$	1.9821	2.5473_n	2.8138
η^2	$\Delta+\Pi'$	2.8035	3.7182_n	
η'^2	$\Delta+\Pi'$	3.5175	4.3017_n	
j^2	$\Delta+\Pi'$	3.1764_n	3.9772	
$\eta\ \eta'$	$2\Delta+\Pi'$	3.4580_n	4.2668	
		m'		

$$K_1=\Sigma w^s\eta^p\eta'^q j^{2t} \cos \text{Arg}.$$

TABLE E_2 (LV_{II}).

Logarithmic. K_2 Unit=1″.

	Sin	w^0	w	w^2
η'^2	$\Delta-\Pi'$	2.9180	3.7732_n	
$\eta\ \eta'$	Π'	3.7799	4.5819_n	
	$\Delta+\Pi'$	1.9821_n	2.5473	2.8138_n
η^2	$\Delta+\Pi'$	3.7744_n	4.5420	
η'^2	$\Delta+\Pi'$	3.5175_n	4.3017	
$\eta\ \eta'$	$2\Delta+\Pi'$	3.4580	4.2668_n	
j^2	$\Delta+\Pi'$	3.1764	3.9772_n	
		m'		

$$K_2=\Sigma w^s\eta^p\ \eta'^q j^{2t} \sin \text{Arg}.$$

$$\frac{u}{\iota \cos i}=\Sigma\, U_{p\cdot q}\eta^p\eta'^q \sin \text{Arg}.+nt\Big\{K_1(\cos \epsilon-e)+K_2 \sin \epsilon\Big\}+c_1(\cos \epsilon-e)+c_2 \sin \epsilon$$

TABLE F (LVI)

Logarithmic. $w-w_0$ Unit=1 radian.

	Cos	w^{-3}	w^{-2}	w^{-1}	w^0	w	w^2
			4.360	5.1966_n	5.7767		
	Γ			4.766	6.6599	7.3732_n	7.7492
	2Γ			4.416	7.1194	7.7572_n	8.0553
	3Γ			4.412	6.8442	7.5458_n	7.9060
	4Γ			4.484	6.5883	7.3450_n	7.7602
	5Γ				6.3437	7.1490_n	7.6136
	7Γ				5.875	6.7632_n	7.3134
η_0	$-5\Gamma+2\theta_0+2\Delta_0$				6.5090	6.6325_n	7.4746_n
	$-4\Gamma+2\theta_0+2\Delta_0$			4.161_n	6.169	7.0658	7.8698_n
	$-3\Gamma+2\theta_0+2\Delta_0$			3.19	6.8821_n	7.6078	7.9975_n
	$-2\Gamma+2\theta_0+2\Delta_0$			3.52	7.0986_n	7.6970	7.9394_n
	$-\Gamma+2\theta_0+2\Delta_0$			5.1420	6.359	7.0722_n	7.4480
	$2\theta_0+2\Delta_0$		4.379	7.6355_n	8.2144	8.4125_n	
	$\Gamma+2\theta_0+2\Delta_0$			4.856_n	8.0894_n	8.9548	9.5668_n
	$2\Gamma+2\theta_0+2\Delta_0$			4.92_n	7.8150_n	8.6561	9.2006_n
	$3\Gamma+2\theta_0+2\Delta_0$			5.5174_n	7.6056_n	8.4650	9.0111_n
	$4\Gamma+2\theta_0+2\Delta_0$			5.4248_n	7.4128_n	8.2958	8.8561_n
	$5\Gamma+2\theta_0+2\Delta_0$				7.2254_n	8.1426	8.7346_n
	$7\Gamma+2\theta_0+2\Delta_0$				6.8746_n	7.8484	8.4936_n
η'	$-5\Gamma+2\theta_0+\Delta_0$				6.8776_n	7.5604	7.8425_n
	$-4\Gamma+2\theta_0+\Delta_0$			4.582	6.8815_n	7.4536	7.5238_n
	$-3\Gamma+2\theta_0+\Delta_0$			4.674	6.6271_n	6.7816	7.3174
	$-2\Gamma+2\theta_0+\Delta_0$			4.99	6.7985	7.4732_n	7.7966
	$-\Gamma+2\theta_0+\Delta_0$			5.4623_n			
	$2\theta_0+\Delta_0$		4.605_n	7.1987	7.8314_n	8.1061	
	$\Gamma+2\theta_0+\Delta_0$			5.0056	8.2964	9.1086_n	9.6833
	$2\Gamma+2\theta_0+\Delta_0$			4.38	8.0434	8.8316_n	9.3296
	$3\Gamma+2\theta_0+\Delta_0$			5.6251	7.8458	8.6564_n	9.1558
	$4\Gamma+2\theta_0+\Delta_0$			5.5812	7.6603	8.5030_n	9.0248
	$5\Gamma+2\theta_0+\Delta_0$				7.4778	8.3544_n	8.9050
	$7\Gamma+2\theta_0+\Delta_0$				7.1130	8.0545_n	8.6668
η_0^2		4.664	4.71	5.83			
	Γ				7.8102	8.6250_n	
	2Γ				7.7520_n	8.1242	
	3Γ				7.6172_n	6.6043_n	
	4Γ				7.7135_n	8.2308	
η_0^2	$-4\Gamma+4\theta_0+4\Delta_0$				7.1862	7.9072_n	
	$-3\Gamma+4\theta_0+4\Delta_0$				7.1804	7.8679_n	
	$-2\Gamma+4\theta_0+4\Delta_0$				6.817	7.456_n	
	$-\Gamma+4\theta_0+4\Delta_0$				8.4680_n	8.8822	
	$4\theta_0+4\Delta_0$	4.666	5.807_n	8.0913	8.8270_n	9.2073	
	$\Gamma+4\theta_0+4\Delta_0$				8.7850	9.8236_n	
	$2\Gamma+4\theta_0+4\Delta_0$				8.5144	9.4910_n	
	$3\Gamma+4\theta_0+4\Delta_0$				8.3274	9.3006_n	
	$4\Gamma+4\theta_0+4\Delta_0$				8.1627	9.1494_n	
	$5\Gamma+4\theta_0+4\Delta_0$				8.0050	9.0105_n	
$\eta_0\,\eta'$	$-4\Gamma+4\theta_0+3\Delta_0$				7.354_n	8.1083	
	$-3\Gamma+4\theta_0+3\Delta_0$				7.5708_n	8.2084	
	$-2\Gamma+4\theta_0+3\Delta_0$						
	$-\Gamma+4\theta_0+3\Delta_0$				8.8838	9.0548_n	
	$4\theta_0+3\Delta_0$	4.516_n	6.2084	8.5565_n	9.2180	9.5174_n	
	$\Gamma+4\theta_0+3\Delta_0$				9.2783_n	0.2833	
	$2\Gamma+4\theta_0+3\Delta_0$				9.0241_n	9.9635	
	$3\Gamma+4\theta_0+3\Delta_0$				8.8480_n	9.7850	
	$4\Gamma+4\theta_0+3\Delta_0$				8.6916_n	9.6434	
	$5\Gamma+4\theta_0+3\Delta_0$				8.5401_n	9.5128	
		m'^2	m'^2	m'^2, m'	m'^2, m'	m'	m'

TABLE F (LVI)—Continued.

Logarithmic. $w - w_0$ Unit = 1 radian.

	Cos	w^{-3}	w^{-2}	w^{-1}	w^0	w	w^2
$\eta_0\ \eta'$	$-4\Gamma\ \ +\ \Delta_0$				7.7640	7.8364_n	
	$-3\Gamma\ \ +\ \Delta_0$				7.4203	8.3915	
	$-2\Gamma\ \ +\ \Delta_0$				7.8104_n	8.6268	
	$-\ \Gamma\ \ +\ \Delta_0$				8.0479_n	8.8018	
	$+\ \Delta_0$	4.518_n	5.886_n	5.70_n			
	$\Gamma\ \ +\ \Delta_0$				7.1339	7.8500	
	$2\Gamma\ \ +\ \Delta_0$				7.8421	8.4293_n	
	$3\Gamma\ \ +\ \Delta_0$				7.9669	8.6796_n	
	$4\Gamma\ \ +\ \Delta_0$				7.9760	8.7576_n	
η'^2	$-4\Gamma+4\theta_0+2\Delta_0$				6.9002	7.6938_n	
	$-3\Gamma+4\theta_0+2\Delta_0$				7.1638	7.8502_n	
	$-2\Gamma+4\theta_0+2\Delta_0$						
	$-\ \Gamma+4\theta_0+2\Delta_0$				8.1860_n	8.4016	
	$4\theta_0+2\Delta_0$	3.76	6.0608_n	8.4157	8.9760_n	9.1661	
	$\Gamma+4\theta_0+2\Delta_0$				9.1714	0.1382_n	
	$2\Gamma+4\theta_0+2\Delta_0$				8.9358	9.8333_n	
	$3\Gamma+4\theta_0+2\Delta_0$				8.7718	9.6681_n	
	$4\Gamma+4\theta_0+2\Delta_0$				8.6236	9.5372_n	
η'^2		3.76	5.7516	4.7			
	Γ				7.8677	8.6727_n	
	2Γ				7.8610_n	8.2228	
	3Γ				8.1026_n	8.7296	
	4Γ				8.1538_n	8.8728	
j^2	Γ				7.9418_n	8.7337	
	2Γ				7.9312_n	8.7154	
	3Γ				7.7920_n	8.6154	
	4Γ				7.639_n	8.5001	
j^2	$-4\Gamma+4\theta_0+3\Delta_0-\Sigma_0$				7.446	8.1156_n	
	$-3\Gamma+4\theta_0+3\Delta_0-\Sigma_0$				7.1858	7.8677_n	
	$-2\Gamma+4\theta_0+3\Delta_0-\Sigma_0$						
	$-\ \Gamma+4\theta_0+3\Delta_0-\Sigma_0$				7.6176_n	7.9693	
	$4\theta_0+3\Delta_0-\Sigma_0$		4.804_n	7.168	7.9368_n	8.3724	
	$\Gamma+4\theta_0+3\Delta_0-\Sigma_0$				7.7887	8.8492_n	
	$2\Gamma+4\theta_0+3\Delta_0-\Sigma_0$				7.448	8.4531_n	
	$3\Gamma+4\theta_0+3\Delta_0-\Sigma_0$				7.1976	8.2026_n	
	$4\Gamma+4\theta_0+3\Delta_0-\Sigma_0$				6.978	7.9963_n	
η_0^3	$2\theta_0+2\Delta_0$	5.418_n	6.292	7.4754_n	8.6636		
η_0^3	$6\theta_0+6\Delta_0$	5.418_n	6.292	8.6328_n	9.4351		
$\eta_0^2\eta'$	$2\theta_0+\ \Delta_0$	5.885	6.719_n	8.5059	9.2804_n		
$\eta_0^2\eta'$	$2\theta_0+3\Delta_0$	4.974	5.896_n	8.0326_n	8.1975		
$\eta_0^2\eta'$	$6\theta_0+5\Delta_0$	5.935	6.780_n	9.2774	0.0330_n		
$\eta_0\ \eta'^2$	$2\theta_0$	5.744_n	6.535	8.3811_n	9.1030		
$\eta_0\ \eta'^2$	$2\theta_0+2\Delta_0$	5.441_n	6.327	8.0917	8.6300		
$\eta_0\ \eta'^2$	$6\theta_0+4\Delta_0$	5.919_n	6.744	9.4432_n	0.1464		
η'^3	$2\theta_0+\ \Delta_0$	5.301	6.149_n	8.2302	9.0152_n		
η'^3	$6\theta_0+3\Delta_0$	5.301	6.149_n	9.1294	9.7729_n		
$j^2\eta_0$	$2\theta_0+2\Delta_0$			8.5904	9.3492_n	9.8022	
$j^2\eta_0$	$2\theta_0+\ \Delta_0-\Sigma_0$	4.502_n	5.41	8.1011_n	8.8726		
$j^2\eta_0$	$6\theta_0+5\Delta_0-\Sigma_0$	4.502_n	5.41	8.0554_n	8.9263		
$j^2\ \ \eta'$	$2\theta_0+\ \Delta_0$			8.5592_n	9.3245		
$j^2\ \ \eta'$	$2\theta_0+2\Delta_0-\Sigma_0$	4.057	5.021_n	6.887	8.1804_n		
$j^2\ \ \eta'$	$6\theta_0+4\Delta_0-\Sigma_0$	4.057	5.021_n	8.2718	9.1021_n		

$$w - w_0 = \Sigma C w^s \eta^p \eta'^q j^{2t} \cos \text{Arg}.$$

where C represents the coefficient.

TABLE G (LVII).

Logarithmic. $S \sin \psi + C \cos \psi$ Unit=1″.

	Cos	w^{-3}	w^{-2}	w^{-1}	w^{0}	w	w^{2}
	$\psi-5\Gamma+2\theta_0+2\Delta_0$			8.81	1.082_n	1.5710	1.612_n
	$\psi-4\Gamma+2\theta_0+2\Delta_0$			9.009	1.2314_n	1.5493	0.989_n
	$\psi-3\Gamma+2\theta_0+2\Delta_0$			9.318	0.931	1.604_n	1.916
	$\psi-2\Gamma+2\theta_0+2\Delta_0$			9.207	1.6478	2.1070_n	2.2333
	$\psi-\Gamma+2\theta_0+2\Delta_0$			9.711	1.950	2.3426_n	2.3713
	$\psi+2\theta_0+2\Delta_0$		9.196	2.1712_n	2.5678	2.565_n	
	$\psi+\Gamma+2\theta_0+2\Delta_0$			9.230_n	2.3541_n	3.1493	3.7107_n
	$\psi+2\Gamma+2\theta_0+2\Delta_0$			9.220_n	1.9114_n	2.6867	3.1657_n
	$\psi+3\Gamma+2\theta_0+2\Delta_0$			9.724_n	1.5372_n	2.3831	2.8623_n
	$\psi+4\Gamma+2\theta_0+2\Delta_0$			9.494_n	1.2544_n	2.1315	2.6333_n
	$\psi+5\Gamma+2\theta_0+2\Delta_0$			9.100_n	1.018_n	1.9034	2.4248_n
η_0	$\psi-5\Gamma+4\theta_0+4\Delta_0$			9.771_n	1.042_n	1.868	2.357_n
	$\psi-4\Gamma+4\theta_0+4\Delta_0$			0.064_n	1.723_n	2.3515	2.6814_n
	$\psi-3\Gamma+4\theta_0+4\Delta_0$			0.3185_n	2.1626_n	2.6961	2.9214_n
	$\psi-2\Gamma+4\theta_0+4\Delta_0$			0.497_n	2.7787_n	3.0649	3.0993_n
	$\psi-\Gamma+4\theta_0+4\Delta_0$			1.0286_n	3.2379_n	3.1223	3.9385_n
	$\psi+4\theta_0+4\Delta_0$	9.199	9.04_n	2.6172	3.2511_n	3.4930	
	$\psi+\Gamma+4\theta_0+4\Delta_0$			0.7226	3.1702	4.1580_n	4.9365
	$\psi+2\Gamma+4\theta_0+4\Delta_0$			0.669	2.7877	3.7083_n	4.3605
	$\psi+3\Gamma+4\theta_0+4\Delta_0$			0.9435	2.5117	3.4261_n	4.0450
	$\psi+4\Gamma+4\theta_0+4\Delta_0$			0.5122	2.2732	3.2042_n	
η_0	$\psi-5\Gamma$			9.814_n	1.925	2.634_n	2.984
	$\psi-4\Gamma$			0.0434_n	2.0527	2.6896_n	2.9432
	$\psi-3\Gamma$			0.3541_n	2.145	2.675_n	2.744
	$\psi-2\Gamma$		9.140	0.362_n	2.1351	2.3850_n	2.4864_n
	$\psi-\Gamma$			0.4164_n	2.3504_n	3.0929	3.5397_n
	ψ		9.274_n	0.1436_n			
	$\psi+\Gamma$			0.3102_n	2.497	3.1875_n	3.5978
	$\psi+2\Gamma$		9.137_n	9.918	1.9006_n	1.0453	2.8834
	$\psi+3\Gamma$			9.465	0.812_n	2.5218_n	3.3564
	$\psi+4\Gamma$			9.20_n	1.406_n	1.729	
η'	$\psi-5\Gamma+4\theta_0+3\Delta_0$			9.476	1.327	1.889_n	2.2299
	$\psi-4\Gamma+4\theta_0+3\Delta_0$			9.781	1.447	2.1506_n	2.5419
	$\psi-3\Gamma+4\theta_0+3\Delta_0$			9.811	2.1070	2.6309_n	2.8608
	$\psi-2\Gamma+4\theta_0+3\Delta_0$			0.3489	2.5095	2.9557_n	3.0952
	$\psi-\Gamma+4\theta_0+3\Delta_0$			0.9511	3.3599	2.7758	3.9726
	$\psi+4\theta_0+3\Delta_0$	8.76_n	0.158	2.7932_n	3.3085	3.4526_n	
	$\psi+\Gamma+4\theta_0+3\Delta_0$			9.961_n	3.3609_n	4.3114	5.0691_n
	$\psi+2\Gamma+4\theta_0+3\Delta_0$			0.491_n	2.9943_n	3.8728	4.4922_n
	$\psi+3\Gamma+4\theta_0+3\Delta_0$			1.0464_n	2.7293_n	3.6067	4.1945_n
	$\psi+4\Gamma+4\theta_0+3\Delta_0$			0.678_n	2.4992_n	3.3946	
η'	$\psi-5\Gamma+\Delta_0$			9.848	2.0766_n	2.712	2.9697_n
	$\psi-4\Gamma+\Delta_0$			0.0792	2.1609_n	2.6968	2.7976_n
	$\psi-3\Gamma+\Delta_0$			0.3941	2.157_n	2.491	1.51
	$\psi-2\Gamma+\Delta_0$		9.013_n	0.248	2.0455	2.7898_n	3.2380
	$\psi-\Gamma+\Delta_0$			9.901	2.584	3.2539_n	3.6434
	ψ		9.885_n	0.8518			
	$\psi+\Gamma+\Delta_0$			0.1664	1.836	2.448	3.3029_n
	$\psi+2\Gamma+\Delta_0$		9.009	9.76_n	2.1633	2.6170_n	2.2433
	$\psi+3\Gamma+\Delta_0$			9.38_n	2.1064	2.7194_n	2.9212
	$\psi+4\Gamma+\Delta_0$				1.9892	2.6870_n	
η_0^2	$\psi-5\Gamma+6\theta_0+6\Delta_0$				2.3144	2.9730_n	
	$\psi-4\Gamma+6\theta_0+6\Delta_0$				2.9538	3.3785_n	
	$\psi-3\Gamma+6\theta_0+6\Delta_0$				3.3102	3.5843_n	
	$\psi-2\Gamma+6\theta_0+6\Delta_0$				3.4970	3.8423_n	
	$\psi-\Gamma+6\theta_0+6\Delta_0$				3.9455	3.7269_n	
	$\psi+6\theta_0+6\Delta_0$	9.95_n	1.1109_n	3.1673_n	3.9296	4.3377_n	
	$\psi+\Gamma+6\theta_0+6\Delta_0$				3.9144_n	5.0372	
	$\psi+2\Gamma+6\theta_0+6\Delta_0$				3.5594_n	4.5942	
	$\psi+3\Gamma+6\theta_0+6\Delta_0$				3.3121_n	4.3236	
η_0^2	$\psi-5\Gamma+2\theta_0+2\Delta_0$				2.1657	2.7221_n	
	$\psi-4\Gamma+2\theta_0+2\Delta_0$				2.1255	2.8004_n	
	$\psi-3\Gamma+2\theta_0+2\Delta_0$				2.234	3.1304_n	
	$\psi-2\Gamma+2\theta_0+2\Delta_0$				2.576	3.3804_n	
	$\psi-\Gamma+2\theta_0+2\Delta_0$				3.1995	3.8325_n	
	$\psi+2\theta_0+2\Delta_0$	0.344_n	1.017	2.689_n	3.4822	3.9938_n	
	$\psi+\Gamma+2\theta_0+2\Delta_0$				2.2480	3.2839_n	
	$\psi+2\Gamma+2\theta_0+2\Delta_0$		9.45		3.1612	3.8424_n	

TABLE G (LVII)—Continued.

$S \sin \psi + C \cos \psi$

Logarithmic. Unit=1″.

	Cos	w^{-3}	w^{-2}	w^{-1}	w^0	w	w^2
η_0^2	$\psi-5\Gamma-2\theta_0-2\Delta_0$				2.700$_n$	3.5481	
	$\psi-4\Gamma-2\theta_0-2\Delta_0$				2.817$_n$	3.6251	
	$\psi-3\Gamma-2\theta_0-2\Delta_0$				2.9247$_n$	3.6905	
	$\psi-2\Gamma-2\theta_0-2\Delta_0$		9.59$_n$		3.0241$_n$	3.7470	
	$\psi-\Gamma-2\theta_0-2\Delta_0$				3.1364$_n$	3.8346	
	$\psi\quad-2\theta_0-2\Delta_0$	0.117	0.95$_n$	2.297$_n$	2.7856$_n$	3.6614	
	$\psi+\Gamma-2\theta_0-2\Delta_0$				2.8942$_n$	3.5604	
	$\psi+2\Gamma-2\theta_0-2\Delta_0$				2.297$_n$	3.1129	
$\eta_0\eta'$	$\psi-5\Gamma+6\theta_0+5\Delta_0$				2.4885$_n$	3.1691	
	$\psi-4\Gamma+6\theta_0+5\Delta_0$				2.976$_n$	3.5560	
	$\psi-3\Gamma+6\theta_0+5\Delta_0$				3.6541$_n$	3.8829	
	$\psi-2\Gamma+6\theta_0+5\Delta_0$				3.9514$_n$	4.1632	
	$\psi-\Gamma+6\theta_0+5\Delta_0$				4.3903$_n$	4.0037$_n$	
	$\psi\quad+6\theta_0+5\Delta_0$	0.295	1.366	3.6364	4.3301$_n$	4.6662	
	$\psi+\Gamma+6\theta_0+5\Delta_0$				4.4005	5.4966$_n$	
	$\psi+2\Gamma+6\theta_0+5\Delta_0$				4.0582	5.0612$_n$	
	$\psi+3\Gamma+6\theta_0+5\Delta_0$				3.8204	4.8027$_n$	
$\eta_0\eta'$	$\psi-5\Gamma+2\theta_0+\Delta_0$				2.426$_n$	3.0684	
	$\psi-4\Gamma+2\theta_0+\Delta_0$				2.399$_n$	3.0310	
	$\psi-3\Gamma+2\theta_0+\Delta_0$				2.410$_n$	3.1305	
	$\psi-2\Gamma+2\theta_0+\Delta_0$				2.701$_n$	3.4602	
	$\psi-\Gamma+2\theta_0+\Delta_0$				3.2842$_n$	3.8558	
	$\psi\quad+2\theta_0+\Delta_0$	0.444	1.188$_n$	3.0569	3.7266$_n$	4.1122	
	$\psi+\Gamma+2\theta_0+\Delta_0$				2.8541	3.5823$_n$	
	$\psi+2\Gamma+2\theta_0+\Delta_0$				3.2191$_n$	3.7635	
$\eta_0\eta'$	$\psi-5\Gamma-2\theta_0-\Delta_0$				3.1551	3.9530$_n$	
	$\psi-4\Gamma-2\theta_0-\Delta_0$				3.2454	3.9948$_n$	
	$\psi-3\Gamma-2\theta_0-\Delta_0$				3.3100	4.0023$_n$	
	$\psi-2\Gamma-2\theta_0-\Delta_0$		9.93		3.3277	3.9401$_n$	
	$\psi-\Gamma-2\theta_0-\Delta_0$				3.1976	3.4598$_n$	
	$\psi\quad-2\theta_0-\Delta_0$	0.490$_n$	1.324	3.0145$_n$	3.7326	4.2787	
	$\psi+\Gamma-2\theta_0-\Delta_0$				3.3632	3.9402$_n$	
	$\psi+2\Gamma-2\theta_0-\Delta_0$				2.7792	3.5224$_n$	
$\eta_0\eta'$	$\psi-5\Gamma+2\theta_0+3\Delta_0$				2.2738$_n$	2.847	
	$\psi-4\Gamma+2\theta_0+3\Delta_0$				2.116$_n$	3.0290	
	$\psi-3\Gamma+2\theta_0+3\Delta_0$				2.5858$_n$	3.3787	
	$\psi-2\Gamma+2\theta_0+3\Delta_0$				2.809$_n$	3.5429	
	$\psi-\Gamma+2\theta_0+3\Delta_0$				2.650$_n$	3.7297	
	$\psi\quad+2\theta_0+3\Delta_0$	9.98	0.60$_n$	2.873$_n$	2.685	3.7980	
	$\psi+\Gamma+2\theta_0+3\Delta_0$				3.5126$_n$	4.2856	
	$\psi+2\Gamma+2\theta_0+3\Delta_0$		9.46$_n$		3.3438$_n$	4.1208	
η'^2	$\psi-5\Gamma+6\theta_0+4\Delta_0$				1.9950	2.7422$_n$	
	$\psi-4\Gamma+6\theta_0+4\Delta_0$				2.6112	3.1949$_n$	
	$\psi-3\Gamma+6\theta_0+4\Delta_0$				3.0556	3.5583$_n$	
	$\psi-2\Gamma+6\theta_0+4\Delta_0$				3.7934	3.7947$_n$	
	$\psi-\Gamma+6\theta_0+4\Delta_0$				4.2260	4.4064	
	$\psi\quad+6\theta_0+4\Delta_0$	9.98$_n$	0.76$_n$	3.5017$_n$	4.1098	4.3552$_n$	
	$\psi+\Gamma+6\theta_0+4\Delta_0$				4.2852$_n$	5.3521	
	$\psi+2\Gamma+6\theta_0+4\Delta_0$				3.9567$_n$	4.9249	
η'^2	$\psi-5\Gamma+2\theta_0+2\Delta_0$				2.5018	3.0963$_n$	
	$\psi-4\Gamma+2\theta_0+2\Delta_0$				2.453	3.0935$_n$	
	$\psi-3\Gamma+2\theta_0+2\Delta_0$				2.4799	3.2779$_n$	
	$\psi-2\Gamma+2\theta_0+2\Delta_0$				2.9375	3.6294$_n$	
	$\psi-\Gamma+2\theta_0+2\Delta_0$				3.2833	3.8982$_n$	
	$\psi\quad+2\theta_0+2\Delta_0$	0.025$_n$	0.60	2.634	3.2781	4.0439$_n$	
	$\psi+\Gamma+2\theta_0+2\Delta_0$				3.5607	4.2381$_n$	
	$\psi+2\Gamma+2\theta_0+2\Delta_0$				3.4629	4.1704$_n$	
η'^2	$\psi-5\Gamma-2\theta_0$				3.0090$_n$	3.7477	
	$\psi-4\Gamma-2\theta_0$				3.0676$_n$	3.7445	
	$\psi-3\Gamma-2\theta_0$				3.0764$_n$	3.6664	
	$\psi-2\Gamma-2\theta_0$				2.958$_n$	3.3121	
	$\psi-\Gamma-2\theta_0$				3.1140	4.0201$_n$	
	$\psi\quad-2\theta_0$	0.305	1.127$_n$	2.912	3.5491$_n$	3.9085	
	$\psi+\Gamma-2\theta_0$				3.0396$_n$	3.6320	
	$\psi+2\Gamma-2\theta_0$				2.4706$_n$	3.2330	

TABLE G (LVII)—Continued.

Logarithmic. $S \sin \psi + C \cos \psi$ Unit=1″.

	Cos	w^{-3}	w^{-2}	w^{-1}	w^0	w	w^2
j^2	$\psi-5\Gamma+6\theta_0+5\Delta_0-\Sigma_0$				2.006	2.7505_n	
	$\psi-4\Gamma+6\theta_0+5\Delta_0-\Sigma_0$				2.335	2.981_n	
	$\psi-3\Gamma+6\theta_0+5\Delta_0-\Sigma_0$				2.544	3.1436_n	
	$\psi-2\Gamma+6\theta_0+5\Delta_0-\Sigma_0$				2.718	3.2445_n	
	$\psi-\Gamma+6\theta_0+5\Delta_0-\Sigma_0$				2.970	2.9116_n	
	$\psi+6\theta_0+5\Delta_0-\Sigma_0$	8.6_n	9.7	2.114_n	2.923	3.4067_n	
	$\psi+\Gamma+6\theta_0+5\Delta_0-\Sigma_0$				2.7948_n	3.9420	
	$\psi+2\Gamma+6\theta_0+5\Delta_0-\Sigma_0$				2.3824_n	3.4488	
j^2	$\psi-5\Gamma+2\theta_0+2\Delta_0$				9.6	2.387	
	$\psi-4\Gamma+2\theta_0+2\Delta_0$				1.916_n	2.911	
	$\psi-3\Gamma+2\theta_0+2\Delta_0$				2.5178_n	3.3047	
	$\psi-2\Gamma+2\theta_0+2\Delta_0$				2.938_n	3.6294	
	$\psi-\Gamma+2\theta_0+2\Delta_0$				3.3406_n	3.9330	
	$\psi+2\theta_0+2\Delta_0$		0.5910	3.1266	3.8021_n	4.1894	
	$\psi+\Gamma+2\theta_0+2\Delta_0$				3.4070	4.3178_n	
	$\psi+2\Gamma+2\theta_0+2\Delta_0$				3.0472	3.9308_n	
j^2	$\psi-5\Gamma-2\theta_0-\Delta_0+\Sigma_0$				0.732_n	1.085	
	$\psi-4\Gamma-2\theta_0-\Delta_0+\Sigma_0$				0.35	1.895_n	
	$\psi-3\Gamma-2\theta_0-\Delta_0+\Sigma_0$				1.463	2.5146_n	
	$\psi-2\Gamma-2\theta_0-\Delta_0+\Sigma_0$				2.064	3.0255_n	
	$\psi-\Gamma-2\theta_0-\Delta_0+\Sigma_0$				2.6816	3.6280_n	
	$\psi-2\theta_0-\Delta_0+\Sigma_0$	9.04	0.11_n	2.636	3.3284_n	3.7399	
	$\psi+\Gamma-2\theta_0-\Delta_0+\Sigma_0$				3.0572_n	3.6430	
	$\psi+2\Gamma-2\theta_0-\Delta_0+\Sigma_0$				2.9121_n	3.5491	
η_0^3	$\psi+4\theta_0+4\Delta_0$	0.775	1.65_n	3.1052_n	3.0342_n		
	$\psi-4\theta_0-4\Delta_0$	0.29_n	1.10	3.1888	3.6104_n		
	$\psi+8\theta_0+8\Delta_0$	0.65	1.54_n	3,7520	4.5812_n		
$\eta_0^2\eta'$	$\psi+4\theta_0+5\Delta_0$			3.7577	4.3244_n		
	$\psi+4\theta_0+3\Delta_0$	1.260_n	2.081	3.1240	4.1388		
	$\psi-4\theta_0-3\Delta_0$	1.005	1.77_n	3.5356_n	3.3560		
	$\psi+8\theta_0+7\Delta_0$	1.228_n	2.093	4.3980_n	5.1827		
$\eta_0\eta'^2$	$\psi+4\theta_0+4\Delta_0$			4.1155_n	4.5547		
	$\psi+4\theta_0+2\Delta_0$	1.106	1.88_n	2.831	4.1803_n		
	$\psi-4\theta_0-2\Delta_0$	1.146_n	1.88	3.0422	4.0180		
	$\psi+8\theta_0+6\Delta_0$	1.321	2.152_n	4.5658	5.3010_n		
η'^3	$\psi+4\theta_0+3\Delta_0$			3.8375	4.0446_n		
	$\psi-4\theta_0-\Delta_0$			3.2197	3.9650_n		
	$\psi+8\theta_0+5\Delta_0$			4.2553_n	4.9349		
$j^2\ \eta_0$	$\psi+4\theta_0+3\Delta_0-\Sigma_0$			3.0024	3.8634_n		
	$\psi-4\theta_0-3\Delta_0+\Sigma_0$	9.98_n	0.8	2.956_n	3.8331		
	$\psi+8\theta+7\Delta_0-\Sigma_0$			3.0757	3.9759_n		
	$\psi+4\theta_0+4\Delta_0$	0.46_n	1.32	3.8514_n	4.6436		
$j^2\ \eta'$	$\psi+4\theta_0+4\Delta_0-\Sigma_0$			2.442	1.846_n		
	$\psi-4\theta_0-2\Delta_0+\Sigma_0$			3.2486	4.0585_n		
	$\psi+8\theta_0+6\Delta_0-\Sigma_0$			3.2818_n	4.1441		
	$\psi+4\theta_0+3\Delta_0$	0.27	1.15_n	3.9421	4.6972_n		

$$S \sin \psi + C \cos \psi = \Sigma C w^s \eta^p \eta'^q j^{2t} \cos \text{Arg}.$$

where C represents the coefficient.

II. TABLES FOR THE DETERMINATION OF THE PERTURBATIONS OF THE HECUBA GROUP OF MINOR PLANETS.

DEVELOPMENT OF THE DIFFERENTIAL EQUATIONS FOR W AND FOR THE THIRD COORDINATE.

It would be futile to attempt to give a brief but comprehensive outline of the fundamental developments in the theory of Bohlin-v. Zeipel which would assist the reader to an understanding of the construction of the tables. In broad outlines, the problem is the integration of Hansen's differential equations for $n\delta z$, ν, and $\frac{u}{\cos i}$, by means of the method developed by Bohlin and according to the modifications introduced by v. Zeipel for purposes of numerical computation. The first division of the problem is the development of functions of the partial derivatives of the perturbative function; the second division of the problem is the integration of the Hansen equations in the form of infinite series.

For the theory the reader is referred to the original works of Hansen[1], Bohlin[2], and v. Zeipel[3]. As indicated in the introduction to the first section, unless otherwise stated, the references to Bohlin refer to the French edition and are designated by B; references to v. Zeipel are designated by Z. Although duplication of material which can be found in either reference is to be avoided, our experience in attempting to reproduce v. Zeipel's tables led us to fill in certain gaps which are troublesome to the reader and the computer.

The first section of v. Zeipel's theory is concerned with an independent development of Hansen's differential equations for $n\delta z$ and ν and a repetition of the differential equation for $\frac{u}{\cos i}$, and the introduction of Bohlin's argument θ. In passing, it is well to emphasize two facts: First, the variables ε and f used throughout the theory are *analogous* to Hansen's $\bar{\varepsilon}$ and $\bar{f}$; the dash is unnecessary, for the physically real values do not appear. Second, the constant elements a, e, π, c, Ω, i are neither osculating nor mean elements; they are defined in the section on constants of integration.

The perturbative function and its partial derivatives are developed in Fourier's series, in which the arguments depend upon the relative positions of the disturbed and disturbing bodies and in which the coefficients are infinite series in ascending powers of the eccentricities and the inclination of the orbits. The coefficients in the latter are elliptic integrals depending upon the ratio of the semi-major axes.

Since these elliptic integrals are functions of the ratio of the semi-major axes, or of the mean daily motions, they can be developed in Taylor's series, in which the given function and its successive partial derivatives are expressed for exact commensurability and the series proceeds according to a small quantity w, defined by $w=1-2\mu$, where μ is the ratio of *Jupiter's* mean motion to that of the planet and where μ differs but little from $\frac{1}{2}$. These elliptic integrals enter the coefficients in all of the subsequent trigonometric series. Hence all the coefficients are series in w. With some exceptions the terms in w°, w, and w^2 have been used. The development of all functions in powers of w is the essential principle underlying the group method of determining perturbations.

The following pages contain the tables which are, in general, parallel to those of v. Zeipel. At the end of sections 2, 3, 4, 5 there are brief written comparisons. To facilitate comparisons

[1] Auseinandersetzung einer zweckmässigen Methode zur Berechnung der absoluten Störungen der kleinen Planeten.

[2] Formeln und Tafeln zur gruppenweisen Berechnung der allgemeinen Störungen benachbarter Planeten. Nova Acta Reg. Soc. Sc. Upsaliensis, Ser. III, Band XVII, 1896.

Sur le Développement des Perturbations Planétaires. Application aux Petites Planètes. Stockholm, 1902.

[3] Angenäherte Jupiterstörungen für die Hecuba-Gruppe. St. Pétersbourg, 1902.

with v. Zeipel's tables, those numerical quantities which are in disagreement are inclosed in brackets. There are also certain mathematical developments useful to the reader. These relations are sometimes taken from v. Zeipel and sometimes supplement his text.

Certain simple functions of the elliptic integrals $\gamma_i^{m \cdot n}$, defined by Z 19, eqs. (73), (74), (75), are tabulated in Table I (cf. Z 23).

Tables II–IV w^2 (cf. Z 26–32), giving the partial derivatives of the perturbative function, are computed according to Z 24, eq. (77), by means of Table I and B 184, Tables XVI–XVIII and B (Ger.) 182, Tables XII–XIV.

The elimination of *Jupiter's* mean anomaly from the argument gives Z 25, eq. (78), in which the coefficients are derived from Table II–IV w^2 by the formulae given in B 61. These coefficients are tabulated in Tables V–VII w^2 (cf. Z 33–39).

Logarithmic. TABLE I[1]. Unit=1″.

n	0	1	2	3	4	5	6	7	8	9	10
$\omega_0^{1.n}$	2. 1464086	{1. 196164}	1. 655608	1. 380429	1. 124592	0. 880042	0. 642905	0. 411123	0. 183046	9. 95811	9. 73559
$\omega_1^{1.n}$	1. 333861	1. 334774	1. 068171	0. 817690	0. 576825	0. 34328	0. 11256	9. 88623	9. 66263	9. 44121	9. 22161
$\omega_2^{1.n}$	0. 892830	0. 934567	0. 688725	0. 45110	0. 21906	9. 99110	9. 76625	9. 54386	9. 32346	9. 10470	8. 88734
$\omega_3^{1.n}$	0. 54795	0. 60716	0. 37239	0. 14249	9. 91620	9. 69269	9. 47141	9. 25194	9. 03398	8. 81730	8. 60172
$\omega_4^{1.n}$	0. 24722	0. 31599	0. 08801	9. 86323	9. 64095	9. 4207	9. 2021	8. 9849	8. 7688		
$\omega_5^{1.n}$	9. 97185	0. 04660	9. 82322	9. 60211	9. 38289	9. 1650	8. 9485				
$\omega_0^{3.n}$	{1. 898161}	2. 283141	2. 153770	2. 006171	1. 847875	1. 68250	1. 51210	1. 33802	1. 16081	0. 98134	0. 79996
$\omega_1^{3.n}$	2. 221528	2. 382049	2. 227850	2. 065161	1. 896645	1. 72387	1. 54784	1. 36930	1. 18855	1. 00614	0. 82229
$\omega_2^{3.n}$	2. 226015	2. 367601	2. 201284	2. 03009	1. 85527	1. 67762	1. 49771	1. 31598			
$\omega_3^{3.n}$	2. 17409	2. 30577	2. 13230	1. 95568	1. 77661	1. 5954	1. 4127				
$\omega_0^{5.n}$	2. 726380	3. 001680	2. 941123	2. 856672	2. 75505	2. 64058	2. 51617	2. 38392			
$\omega_1^{5.n}$	3. 22034	3. 44169	3. 34392	3. 23252	3. 11061						
$\overline{\omega}_0^{1.n}$	2. 086910$_n$	{1. 696947$_n$}	2. 025704$_n$	1. 862100$_n$	1. 694089$_n$	1. 522172$_n$	1. 347043$_n$	1. 16938$_n$	0. 98938$_n$	0. 80767$_n$	0. 62442$_n$
$\overline{\omega}_1^{1.n}$	1. 806020$_n$	1. 906225$_n$	1. 716180$_n$	1. 529203$_n$	1. 342993$_n$	1. 15669$_n$	0. 96997$_n$	0. 78270$_n$	0. 59483$_n$	0. 40635$_n$	0. 21732$_n$
$\overline{\omega}_2^{1.n}$	1. 603899$_n$	1. 704402$_n$	1. 508098$_n$	1. 31435$_n$	1. 12169$_n$	0. 92956$_n$	0. 73762$_n$	0. 54570$_n$	0. 35368$_n$	0. 16150$_n$	9. 96912$_n$
$\overline{\omega}_3^{1.n}$	1. 41336$_n$	1. 51356$_n$	1. 31543$_n$	1. 11886$_n$	0. 92322$_n$	0. 72817$_n$	0. 53345$_n$	0. 33892$_n$	0. 1144$_n$	9. 9500$_n$	
$\overline{\omega}_4^{1.n}$	1. 22659$_n$	1. 32679$_n$	1. 12769$_n$	0. 92971$_n$	0. 73250$_n$	0. 53579$_n$	0. 33942$_n$	0. 14337$_n$	9. 9476$_n$		
$\overline{\omega}_0^{3.n}$	{2. 51524$_n$}	2. 84832$_n$	2. 78195$_n$	2. 69286$_n$	2. 58759$_n$	2. 47019$_n$	2. 34339$_n$	2. 20917$_n$	2. 06865$_n$	1. 92310$_n$	1. 77326$_n$
$\overline{\omega}_1^{3.n}$	2. 95077$_n$	3. 15383$_n$	3. 01132$_n$	2. 91803$_n$	2. 78638$_n$	2. 64800$_n$	2. 50413$_n$	2. 35570$_n$	2. 20324$_n$	2. 01749$_n$	1. 88881$_n$
$\overline{\omega}_2^{3.n}$	3. 10392$_n$	3. 27800$_n$	3. 14309$_n$	3. 00185$_n$	2. 85545$_n$	2. 70474$_n$	2. 55037$_n$	2. 39290$_n$			
$\overline{\omega}_0^{5.n}$	3. 51583$_n$	3. 80371$_n$	3. 76747$_n$	3. 71207$_n$	3. 64099$_n$	3. 55696$_n$	3. 46222$_n$	3. 35851$_n$			
$\overline{\omega}_1^{5.n}$	4. 13087$_n$	4. 37108$_n$	4. 29565$_n$	4. 20810$_n$	4. 11042$_n$						
$\overline{\overline{\omega}}_0^{1.n}$	1. 34814	{1. 79250}	1. 94314	1. 92018	1. 85954	1. 77438	1. 67197	1. 55683	1. 43166	1. 29874	1. 15943
$\overline{\overline{\omega}}_1^{1.n}$	1. 93221	2. 12908	2. 01556	1. 89337	1. 76366	1. 62754	1. 48597	1. 33971	1. 18943	1. 03563	0. 87878
$\overline{\overline{\omega}}_2^{1.n}$	1. 99420	2. 14965	2. 00180	1. 85113	1. 69761	1. 54142	1. 38270	1. 22170	1. 05858	0. 89355	
$\overline{\overline{\omega}}_3^{1.n}$	1. 96578	2. 10476	1. 94185	1. 77768	1. 61213	1. 44520	1. 27690	1. 10730			
$\overline{\overline{\overline{\omega}}}_0^{3.n}$	{2. 81277}	3. 11481	3. 09258	3. 05198	2. 99474	2. 92310	2. 83919	2. 74499	2. 64179	2. 53103	2. 41372
$\overline{\overline{\overline{\omega}}}_1^{3.n}$	3. 38698	3. 62237	3. 54448	3. 45591	3. 35805	3. 25216	3. 13926	3. 02025			
$\overline{\overline{\overline{\omega}}}_2^{3.n}$	3. 68273	3. 88445	3. 77729								

[1] For $n=0$ the table gives log $\frac{1}{2}\,\omega$. For explanation of brackets, { }, see [ ], Z 14, eq. (63). (v.Zeipel) $\overline{\omega}_i^{m.n} = -\overline{\omega}_i^{s.n}$ (Bohlin).

Logarithmic.

TABLE II.

Unit = 1''.

n	0	1	2	3	4	5	6	7	8	9	10
$P_{0\cdot0}[n.-n]$	——	1.196164	1.956638	1.857550	1.726652	1.579012	1.421056	1.256221	1.08614	0.91235	0.73559
$P_{1\cdot0}[n+1.-n]$	1.634891_n	1.936834_n	1.846322_n	1.720780_n	1.576825_n	1.42156_n	1.25869_n	1.09035_n	0.91790_n	0.74224_n	0.56403_n
$P_{1\cdot0}[n-1.-n]$	1.634891	——	2.310470_n	2.497527_n	2.531520_n	2.506584_n	2.441893_n	2.353544_n	2.24835_n	2.13091_n	2.00396_n
$P_{0\cdot1}[n.-n+1]$	——	1.635804	1.640475_n	2.019701_n	2.112868_n	2.113048_n	2.065516_n	1.988837_n	1.89209_n	1.78110_n	1.65922_n
$P_{0\cdot1}[n.-n-1]$	——	2.025593	2.832690	2.880695	2.859122	2.798595	2.713086	2.610276	2.49144	2.36884	2.23544
$P_{2\cdot0}[n+2.-n]$	2.023856	2.225475	2.096678	1.95119	1.79511	1.63176	1.46311	1.29047	1.11470	0.93639	0.75603
$P_{2\cdot0}[n.-n]$	——	2.261969	2.117706	2.10627	2.04755	1.96143	1.85717	1.73972	1.61291	1.47809	1.33716
$P_{2\cdot0}[n-2.-n]$	2.023856_n	2.199192_n	——	2.659837	2.948238	3.069862	3.113630	3.110973	3.07683	3.02020	2.94643
$P_{1\cdot1}[n+1.-n+1]$	2.283141_n	2.492130_n	2.272769_n	1.95625_n	1.45482_n	0.76106	1.33715	1.42913	1.41871	1.36100	1.27562
$P_{1\cdot1}[n-1.-n+1]$	2.283141	——	1.396759	2.572122	2.881369	3.017744	3.071278	3.075436	3.04626	2.99338	2.92258
$P_{1\cdot1}[n+1.-n-1]$	2.283141_n	2.817171_n	2.874471_n	2.858127_n	2.800971_n	2.71782_n	2.61666_n	2.50228_n	2.37777_n	2.24526_n	2.10626_n
$P_{1\cdot1}[n-1.-n-1]$	2.283141	——	3.206887_n	3.531457_n	3.673689_n	3.730739_n	3.737245_n	3.710125_n	3.65862_n	3.58899_n	3.50313_n
$P_{0\cdot2}[n.-n+2]$	——	2.177073	2.038481	1.952501	2.071811	2.196710	2.269672	2.295743	2.28582	2.24911	2.19176
$P_{0\cdot2}[n.-n]$	——	2.404066	2.698322_n	3.116000_n	3.292184_n	3.367379_n	3.385271_n	3.366005_n	3.32023_n	3.25499_n	3.17458_n
$P_{0\cdot2}[n.-n-2]$	——	2.722215	3.551521	3.707664	3.773067	3.785150	3.761920	3.713611	3.64622	3.56422	3.47045
$P_{0\cdot0}[n+1.-n+1]+\sigma$	1.898161	2.584171	2.630891	2.608231	2.546845	2.46065	2.35720	2.24111	2.11505	1.98134	1.84135
$P_{0\cdot0}[n-1.-n-1]-\sigma$	1.898161_n	——	2.153770	2.307201	2.324996	2.28456	2.21107	2.11617	2.00591	1.88443	1.75420
$P_{0\cdot0}[n+1.-n-1]+\delta$	1.898161_n	2.584171_n	2.630891_n	2.608231_n	2.546845_n	2.46065_n	2.35720_n	2.24111_n	2.11505_n	1.98134_n	1.84135_n
$P_{0\cdot0}[n-1.-n+1]-\delta$	1.898161	——	2.153770_n	2.307201_n	2.324996_n	2.28456_n	2.21107_n	2.11617_n	2.00591_n	1.88443_n	1.75420_n
$P_{3\cdot0}[n+1.-n]$	2.25167_n										
$P_{3\cdot0}[n-1.-n]$	2.25167		2.44348_n		2.30607_n						
$P_{3\cdot0}[n-3.-n]$					3.02278_n		3.55125_n		3.63307_n		
$P_{2\cdot1}[n.-n+1]$		2.95011									
$P_{2\cdot1}[n-2.-n+1]$		2.82358_n		2.50563_n		3.54376_n					
$P_{2\cdot1}[n.-n-1]$		3.21039		3.42436							
$P_{2\cdot1}[n-2.-n-1]$				3.71288		4.30087		4.47100			
$P_{1\cdot2}[n-1.-n+2]$	2.91212		2.68308_n								
$P_{1\cdot2}[n+1.-n]$	2.93746_n										
$P_{1\cdot2}[n-1.-n]$	2.93746		2.55874		4.06225						
$P_{1\cdot2}[n+1.-n-2]$	2.91212_n		3.71917_n								
$P_{1\cdot2}[n-1.-n-2]$			3.94779_n		4.59492_n		4.78966_n				
$P_{0\cdot3}[n.-n+1]$		2.95804									
$P_{0\cdot3}[n.-n-1]$		3.06432		3.99638_n							
$P_{0\cdot3}[n.-n-3]$		3.36346		4.44360		4.65468					
$P_{1\cdot0}[n.-n+1]+\sigma$		3.09682_n									
$P_{1\cdot0}[n.-n-1]-\sigma$		2.93746_n		3.11584_n							
$P_{1\cdot0}[n-2.-n-1]-\sigma$				2.92477_n		3.24597_n		3.24515_n			

$P_{1\cdot0}[n.-n-1]+\delta$		3.09682		3.49574							
$P'_{1\cdot0}[n.-n+1]-\delta$		2.93746									
$P'_{1\cdot0}[n-2.-n+1]-\delta$		2.93746_n		2.92477		3.24597					
$P_{0\cdot1}[n+1.-n]+\sigma$	2.52256										
$P'_{0\cdot1}[n-1.-n]-\sigma$	2.52256_n		1.72430		2.57107_n						
$P'_{0\cdot1}[n-1.-n-2]-\sigma$			3.24621		3.58597		3.59437				
$P'_{0\cdot1}[n+1.-n]+\delta$	2.52256_n										
$P'_{0\cdot1}[n+1.-n-2]+\delta$	2.81256_n		3.72333_n								
$P'_{0\cdot1}[n-1.-n+2]-\delta$	2.81256		2.79444_n								
$P'_{0\cdot1}[n-1.-n]-\delta$	2.52256		3.07659_n		3.47840_n						

Logarithmic. **Table II w.** Unit=1″.

n	0	1	2	3	4	5	6	7	8	9	10
$P_{0\cdot0}[n.-n]$	———	1. 696947$_n$	2. 326734$_n$	2. 339221$_n$	2. 296149$_n$	2. 221142$_n$	2. 125194$_n$	2. 01448$_n$	1. 89247$_n$	1. 76191$_n$	1. 62442$_n$
$P_{1\cdot0}[n+1.-n]$	2. 107050	2. 508285	2. 494331	2. 432293	2. 342993	2. 23587	2. 11610	1. 98682	1. 85010	1. 70738	1. 55974
$P_{1\cdot0}[n-1.-n]$	2. 107050$_n$	———	2. 722988	3. 003816	3. 120166	3. 160145	3. 154557	3. 11841	3. 05995	2. 98480	2. 89638
$P_{0\cdot1}[n.-n+1]$	———	2. 207255$_n$	0. 613819	2. 368909	2. 620202	2. 717783	2. 744353	2. 72873	2. 68442	2. 61961	2. 53911
$P_{0\cdot1}[n.-n-1]$	———	2. 556580$_n$	3. 228771$_n$	3. 377812$_n$	3. 438885$_n$	3. 448053$_n$	3. 422723$_n$	3. 37281$_n$	3. 30420$_n$	3. 22121$_n$	3. 12662$_n$
$P_{2\cdot0}[n+2.-n]$	2. 65255$_n$	2. 92884$_n$	2. 85948$_n$	2. 76460$_n$	2. 65301$_n$	2. 52955$_n$	2. 39716$_n$	2. 25778$_n$	2. 11278$_n$	1. 96311$_n$	1. 80954$_n$
$P_{2\cdot0}[n.-n]$	———	2. 92348$_n$	3. 05366$_n$	3. 10049$_n$	3. 08583$_n$	3. 03584$_n$	2. 96268$_n$	2. 87290$_n$	2. 77085$_n$	2. 65893$_n$	2. 53921$_n$
$P_{2\cdot0}[n-2.-n]$	2. 65255	2. 81627	———	3. 20793$_n$	3. 55914$_n$	3. 74023$_n$	3. 83827$_n$	3. 88477$_n$	3. 89537$_n$	3. 87961$_n$	3. 84335$_n$
$P_{1\cdot1}[n+1.-n+1]$	2. 84832	3. 16261	3. 03557	2. 81950	2. 49006	1. 80523	1. 93390$_n$	2. 21271$_n$	2. 28216$_n$	2. 27834$_n$	
$P_{1\cdot1}[n-1.-n+1]$	2. 84832$_n$	———	2. 54847	2. 78202$_n$	3. 36491$_n$	3. 61642$_n$	3. 74911$_n$	3. 81610$_n$	3. 84000$_n$	3. 83352$_n$	
$P_{1\cdot1}[n+1.-n-1]$	2. 84832	3. 43829	3. 55417	3. 59183	3. 58361	3. 54488	3. 48425	3. 40707	3. 31690	3. 21626	
$P_{1\cdot1}[n-1.-n-1]$	2. 84832$_n$	———	3. 66096	4. 06072	4. 27377	4. 39417	4. 45708	4. 48041	4. 47449	4. 44629	
$P_{0\cdot2}[n.-n+2]$	———	2. 79945$_n$	2. 75578$_n$	2. 66099$_n$	2. 71530$_n$	2. 84736$_n$	2. 96308$_n$	3. 03888$_n$	3. 07708$_n$	3. 08493$_n$	
$P_{0\cdot2}[n.-n]$	———	3. 04572$_n$	2. 17570	3. 46537	3. 80207	3. 97510	4. 06697	4. 10839	4. 11475	4. 09540	
$P_{0\cdot2}[n.-n-2]$	———	3. 28370$_n$	3. 97539$_n$	4. 22145$_n$	4. 36389$_n$	4. 44247$_n$	4. 47743$_n$	4. 48070$_n$	4. 45961$_n$	4. 41955$_n$	
$P_{0\cdot0}[n+1.-n+1]+\sigma$	2. 51524$_n$	3. 14934$_n$	3. 25907$_n$	3. 29492$_n$	3. 28656$_n$	3. 24834$_n$	3. 18849$_n$	3. 11226$_n$	3. 02289$_n$	2. 92310$_n$	2. 81465$_n$
$P_{0\cdot0}[n-1.-n-1]-\sigma$	2. 51524	———	2. 78195$_n$	2. 99389$_n$	3. 06471$_n$	3. 07225$_n$	3. 04236$_n$	2. 98732$_n$	2. 91375$_n$	2. 82619$_n$	2. 72750$_n$
$P_{0\cdot0}[n+1.-n-1]+\delta$	2. 51524	3. 14934	3. 25907	3. 29492	3. 28656	3. 24834	3. 18849	3. 11226	3. 02289	2. 92310	2. 81465
$P_{0\cdot0}[n-1.-n+1]-\delta$	2. 51524$_n$	———	2. 78195	2. 99389	3. 06471	3. 07225	3. 04236	2. 98732	2. 91375	2. 82619	2. 72750
$P_{3\cdot0}[n+1.-n]$	3. 04286										
$P_{3\cdot0}[n-1.-n]$	3. 04286$_n$		3. 36582		3. 63249						
$P_{3\cdot0}[n-3.-n]$					3. 67411		4. 29317		4. 52091		
$P_{2\cdot1}[n.-n+1]$		3. 71643$_n$									
$P_{2\cdot1}[n-2.-n+1]$		3. 55282		2. 58404$_n$		4. 12637					
$P_{2\cdot1}[n.-n-1]$		3. 93243$_n$		4. 34583$_n$							
$P_{2\cdot1}[n-2.-n-1]$				4. 29563$_n$		4. 98494$_n$		5. 25186$_n$			
$P_{1\cdot2}[n-1.-n+2]$	3. 54910$_n$		3. 45486								
$P_{1\cdot2}[n+1.-n]$	3. 62947										
$P_{1\cdot2}[n-1.-n]$	3. 62947$_n$		3. 59594		4. 54375$_n$						
$P_{1\cdot2}[n+1.-n-2]$	3. 54910		4. 43013								
$P_{1\cdot2}[n-1.-n-2]$			4. 44517		5. 21067		5. 51720				
$P_{0\cdot3}[n.-n+1]$		3. 66472$_n$									
$P_{0\cdot3}[n.-n-1]$		3. 77533$_n$		4. 34178							
$P_{0\cdot3}[n.-n-3]$		3. 95620$_n$		4. 97543$_n$		5. 32048$_n$					
$P_{1\cdot0}[n.-n+1]+\sigma$		3. 75365									
$P_{1\cdot0}[n.-n-1]-\sigma$		3. 62946		3. 89906							
$P_{1\cdot0}[n-2.-n-1]-\sigma$				3. 66409		4. 06165		4. 13352			

$P_{1\cdot 0}[n.-n-1]+\partial$		3.75365_n		4.22531_n							
$P_{1\cdot 0}[n.-n+1]-\partial$		3.62946_n									
$P_{1\cdot 0}[n-2.-n+1]-\partial$		3.62946		3.66409_n		4.06165_n					
$P_{0\cdot 1}[n+1.-n]+\sigma$	3.25180_n										
$P_{0\cdot 1}[n-1.-n]-\sigma$	3.25180		2.99523_n		2.98848						
$P_{0\cdot 1}[n-1.-n-2]-\sigma$			3.91658_n		4.34713_n		4.43848_n				
$P_{0\cdot 1}[n+1.-n]+\partial$	3.25180										
$P_{0\cdot 1}[n+1.-n-2]+\partial$	3.49076		4.39370								
$P_{0\cdot 1}[n-1.-n+2]-\partial$	3.49076_n		3.53278								
$P_{0\cdot 1}[n-1.-n]-\partial$	3.25180_n		3.76576		4.24543						

TABLE II $w.^2$

Unit = 1″.

$P_{0\cdot 0}[n.-n]$	—	1.79250	2.24417	2.39730	2.46160	2.47335	2.45012	2.40193	2.33475	2.25298	2.15943
$P_{1\cdot 0}[n+1.-n]$	2.23324_n	2.73114_n	2.79371_n	2.79646_n	2.76366_n	2.70672_n	2.63210_n	2.54383_n	2.44470_n	2.33666_n	
$P_{1\cdot 0}[n-1.-n]$	2.23324	—	2.74649_n	3.11775_n	3.31910_n	3.43434_n	3.49490_n	3.51720_n	3.51090_n	3.48266_n	
$P_{0\cdot 1}[n.-n+1]$	—	2.43011	2.37864	1.47552_n	2.60654_n	2.88392_n	3.01794_n	3.08198_n	3.10228_n	3.09238_n	
$P_{0\cdot 1}[n.-n-1]$	—	2.71373	3.21558	3.47213	3.62606	3.71456	3.75768	3.76766	3.75213	3.71674	
$P_{2\cdot 0}[n.-n]$	—		3.52227								
$P_{2\cdot 0}[n-2.-n]$			—		3.80784	4.04609	4.20009				
$P_{1\cdot 1}[n-1.-n+1]$		—		3.00694_n							
$P_{1\cdot 1}[n+1.-n-1]$		3.73949_n									
$P_{1\cdot 1}[n-1.-n-1]$		—		4.22410_n	4.50219_n	4.68741_n					
$P_{0\cdot 2}[n.-n]$	—		3.44001								
$P_{0\cdot 2}[n.-n-2]$			4.02969	4.35355	4.57425						
$P_{0\cdot 0}[n+1.-n+1]+\sigma$		3.41584									
$P_{0\cdot 0}[n-1.-n-1]-\sigma$		—		3.35301							
$P_{0\cdot 0}[n+1.-n-1]+\partial$		3.41584_n									
$P_{0\cdot 0}[n-1.-n+1]-\partial$		—		3.35301_n							
$P_{1\cdot 0}[n.-n+1]+\sigma$		4.13333_n									
$P_{1\cdot 0}[n.-n-1]+\partial$		4.13333									
$P_{1\cdot 0}[n.-n+1]-\partial$		4.04093									
$P_{1\cdot 0}[n-2.-n+1]-\partial$		4.04093_n		4.09610							
$P_{0\cdot 1}[n+1.-n]+\sigma$	3.68801										
$P_{0\cdot 1}[n-1.-n]-\sigma$	3.68801_n										
$P_{0\cdot 1}[n+1.-n]+\partial$	3.68801_n										
$P_{0\cdot 1}[n-1.-n+2]-\partial$			3.97690_n								
$P_{0\cdot 1}[n-1.-n]-\partial$	3.68801										

Logarithmic. TABLE III. Unit=1''.

n	0	1	2	3	4	5	6	7	8	9	10
$Q_{0\cdot0}[n.-n]$	1. 634891	1. 770420	2. 056512	1. 930336	1. 784183	1. 626670	1. 461775	1. 291778	1. 11765	0. 91075	0. 76140
$Q_{1\cdot0}[n+1.-n]$	2. 172568$_n$	2. 297774$_n$	2. 122754$_n$	1. 946074$_n$	1. 767441$_n$	1. 58698$_n$	1. 40490$_n$	1. 22142$_n$	1. 03671$_n$	0. 85090$_n$	0. 66417$_n$
$Q_{1\cdot0}[n-1.-n]$	2. 172568$_n$	2. 500222$_n$	2. 769564$_n$	2. 777719$_n$	2. 736598$_n$	2. 664600$_n$	2. 571599$_n$	2. 463518$_n$	2. 34380$_n$	2. 21522$_n$	2. 07946$_n$
$Q_{0\cdot1}[n.-n+1]$	2. 283141	2. 297774	1. 273400	1. 914007$_n$	2. 093349$_n$	2. 116256$_n$	2. 076961$_n$	2. 003548$_n$	1. 90802$_n$	1. 79730$_n$	1. 67521$_n$
$Q_{0\cdot1}[n.-n-1]$	2. 283141	2. 637762	2. 968456	2. 973184	2. 929137	2. 854882	2. 760132	2. 650672	2. 52983	2. 40035	2. 26382
$Q_{2\cdot0}[n+2.-n]$	2. 453562	2. 566336	2. 380403	2. 19478	2. 00881	1. 82226	1. 63507	1. 44723	1. 25876	1. 06969	0. 88007
$Q_{2\cdot0}[n.-n]$	2. 754592	3. 031368	2. 893804	2. 769075	2. 632508	2. 48787	2. 33729	2. 18203	2. 02324	1. 86136	1. 69674
$Q_{2\cdot0}[n-2.-n]$	2. 453562	2. 916121	3. 162607	3. 293194	3. 356762	3. 372490	3. 353968	3. 310214	3. 24697	3. 16860	3. 07783
$Q_{1\cdot1}[n+1.-n+1]$	2. 937465$_n$	2. 970967$_n$	2. 681434$_n$	2. 35193$_n$	1. 93963$_n$	1. 22851$_n$	1. 18462	1. 43523	1. 46243	1. 41794	1. 33737
$Q_{1\cdot1}[n-1.-n+1]$	2. 937465$_n$	3. 124618$_n$	2. 744514$_n$	2. 427082	2. 956557	3. 110594	3. 160757	3. 157977	3. 12177	3. 06253	2. 98615
$Q_{1\cdot1}[n+1.-n-1]$	2. 937465$_n$	3. 237886$_n$	3. 187950$_n$	3. 108818$_n$	3. 010146$_n$	2. 89742$_n$	2. 77408$_n$	2. 64243$_n$	2. 50409$_n$	2. 36024$_n$	2. 21178$_n$
$Q_{1\cdot1}[n-1.-n-1]$	2. 937465$_n$	3. 414623$_n$	3. 721093$_n$	3. 840450$_n$	3. 893162$_n$	3. 900319$_n$	3. 875164$_n$	3. 826219$_n$	3. 75883$_n$	3. 67708$_n$	3. 58369$_n$
$Q_{0\cdot2}[n.-n+2]$	2. 912119	2. 900312	2. 571441	2. 299258	2. 217983	2. 254404	2. 298011	2. 314149	2. 30055	2. 26230	2. 20417
$Q_{0\cdot2}[n.-n]$	2. 937465	3. 150133	2. 137444	3. 010528$_n$	3. 275873$_n$	3. 374162$_n$	3. 399983$_n$	3. 383558$_n$	3. 33861$_n$	3. 27329$_n$	3. 19243$_n$
$Q_{0\cdot2}[n.-n-2]$	2. 912119	3. 372021	3. 724815	3. 821212	3. 856471	3. 850676	3. 815716	3. 759151	3. 68567	3. 59900	3. 50152
$Q_{0\cdot0}[n+1.-n+1]+\sigma$	2. 615092	2. 937464	2. 883904	2. 804889	2. 707475	2. 59632	2. 47451	2. 34446	2. 20738	2. 06175	1. 91742
$Q_{0\cdot0}[n-1.-n-1]-\sigma$	2. 615092	2. 937464	2. 883904	2. 804889	2. 707475	2. 59632	2. 47454	2. 34446	2. 20738	2. 06475	1. 91742
$Q_{0\cdot0}[n+1.-n-1]+\delta$	2. 615092$_n$	2. 937464$_n$	2. 883904$_n$	2. 804889$_n$	2. 707475$_n$	2. 59632$_n$	2. 47454$_n$	2. 34446$_n$	2. 20738$_n$	2. 06175$_n$	1. 91742$_n$
$Q_{0\cdot0}[n-1.-n+1]-\delta$	2. 615092$_n$	2. 937464$_n$	2. 883904$_n$	2. 804889$_n$	2. 707475$_n$	2. 59632$_n$	2. 47454$_n$	2. 34446$_n$	2. 20738$_n$	2. 06175$_n$	1. 91742$_n$
$Q_{3\cdot0}[n+1.-n]$	3. 16940$_n$										
$Q_{3\cdot0}[n-1.-n]$	3. 16940$_n$				3. 31760$_n$						
$Q_{3\cdot0}[n-3.-n]$					3. 78025$_n$				3. 95271$_n$		
$Q_{2\cdot1}[n.-n+1]$		[3. 83982]									
$Q_{2\cdot1}[n-2.-n+1]$		3. 67126				3. 74216$_n$					
$Q_{2\cdot1}[n.-n-1]$				4. 00864							
$Q_{2\cdot1}[n-2.-n-1]$				4. 38857				4. 67817			
$Q_{1\cdot2}[n-1.-n+2]$			3. 57901$_n$								
$Q_{1\cdot2}[n+1.-n]$	3. 74233$_n$										
$Q_{1\cdot2}[n-1.-n]$	3. 74233$_n$				4. 13481						
$Q_{1\cdot2}[n+1.-n-2]$			4. 06877$_n$								
$Q_{1\cdot2}[n-1.-n-2]$			4. 51776$_n$				4. 93640$_n$				
$Q_{0\cdot3}[n.-n+1]$		3. 77983									
$Q_{0\cdot3}[n.-n-1]$				3. 88524$_n$							
$Q_{0\cdot3}[n.-n-3]$		4. 05121				4. 73013					
$Q_{1\cdot0}[n.-n+1]+\sigma$		3. 86075$_n$									
$Q_{1\cdot0}[n.-n-1]-\sigma$				3. 54750$_n$							
$Q_{1\cdot0}[n-2.-n-1]-\sigma$				3. 78391$_n$				3. 57149$_n$			

$Q_{1\cdot 0}[n.-n-1]+\delta$				3. 86667							
$Q_{1\cdot 0}[n.-n+1]-\delta$		3. 74232									
$Q_{1\cdot 0}[n-2.-n+1]-\delta$		3. 74232				3. 71396					
$Q_{0\cdot 1}[n+1.-n]+\sigma$	3. 37022										
$Q_{0\cdot 1}[n-1.-n]-\sigma$	3. 37022				2.55964_n						
$Q_{0\cdot 1}[n-1.-n-2]-\sigma$			4. 02594				3. 87205				
$Q_{0\cdot 1}[n+1.-n]+\delta$	3.37022_n										
$Q_{0\cdot 1}[n+1.-n-2]+\delta$			4.02594_n								
$Q_{0\cdot 1}[n-1.-n+2]-\delta$			3.65248_n								
$Q_{0\cdot 1}[n-1.-n]-\delta$	3.37022_n				3.89184_n						

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Logarithmic. TABLE III *w*. Unit = 1″.

n	0	1	2	3	4	5	6	7	8	9	10
$Q_{0\cdot0}[n.-n]$	2. 107050$_n$	2. 324123$_n$	2. 500012$_n$	2. 456410$_n$	2. 383497$_n$	2. 290224$_n$	2. 182060$_n$	2. 06264$_n$	1. 93418$_n$	1. 79862$_n$	1. 65717$_n$
$Q_{1\cdot0}[n+1.-n]$	2. 761374	2. 948526	2. 828604	2. 701746	2. 568459	2. 42951	2. 28563	2. 13748	1. 98561	1. 83045	1. 67243
$Q_{1\cdot0}[n-1.-n]$	2. 761374	3. 117300	3. 287547	3. 346232	3. 362633	3. 346294	3. 304908	3. 24416	3. 16781	3. 07910	2. 98013
$Q_{0\cdot1}[n.-n+1]$	2. 848322$_n$	2. 948526$_n$	2. 553490$_n$	1. 83791	2. 55053	2. 70884	2. 75385	2. 74519	2. 70312	2. 63898	2. 55831
$Q_{0\cdot1}[n.-n-1]$	2. 848322$_n$	3. 238533$_n$	3. 460532$_n$	3. 526791$_n$	3. 545790$_n$	3. 530227$_n$	3. 488898$_n$	3. 42791$_n$	3. 35124$_n$	3. 26216$_n$	3. 16280$_n$
$Q_{2\cdot0}[n+2.-n]$	3. 19118$_n$	3. 35005$_n$	3. 20584$_n$	3. 05864$_n$	2. 90832$_n$	2. 75501$_n$	2. 59890$_n$	2. 44023$_n$	2. 27924$_n$	2. 11612$_n$	
$Q_{2\cdot0}[n.-n]$	3. 49221$_n$	3. 78130$_n$	3. 72230$_n$	3. 64840$_n$	3. 55797$_n$	3. 45496$_n$	3. 34206$_n$	3. 22114$_n$	3. 09370$_n$	2. 96069$_n$	
$Q_{2\cdot0}[n-2.-n]$	3. 19118$_n$	3. 62597$_n$	3. 79235$_n$	3. 92707$_n$	4. 02294$_n$	4. 08050$_n$	4. 10554$_n$	4. 10412$_n$	4. 08099$_n$	4. 04028$_n$	
$Q_{1\cdot1}[n+1.-n+1]$	3. 62947	3. 72967	3. 50687	3. 25188	2. 94397	2. 52009	1. 34395	2. 13049$_n$	2. 30111$_n$	2. 32725$_n$	
$Q_{1\cdot1}[n-1.-n+1]$	3. 62947	3. 85386	3. 66684	3. 13763	3. 29292$_n$	3. 67977$_n$	3. 83164$_n$	3. 89857$_n$	3. 91787$_n$	3. 90586$_n$	
$Q_{1\cdot1}[n+1.-n-1]$	3. 62947	3. 95032	3. 93470	3. 89345	3. 83267	3. 75650	3. 66791	3. 56910	3. 46175	3. 34713	
$Q_{1\cdot1}[n-1.-n-1]$	3. 62947	4. 09283	4. 30437	4. 44724	4. 54300	4. 59781	4. 61956	4. 61498	4. 58903	4. 54583	
$Q_{0\cdot2}[n.-n+2]$	3. 54909$_n$	3. 61500$_n$	3. 35799$_n$	3. 10979$_n$	2. 96526$_n$	2. 95859$_n$	3. 01478$_n$	3. 06840$_n$	3. 09806$_n$	3. 10227$_n$	
$Q_{0\cdot2}[n.-n]$	3. 62947$_n$	3. 88133$_n$	3. 61213$_n$	2. 74115	3. 72579	3. 96812	4. 07974	4. 12792	4. 13642	4. 11736	
$Q_{0\cdot2}[n.-n-2]$	3. 54909$_n$	4. 01770$_n$	4. 26538$_n$	4. 40391$_n$	4. 49178$_n$	4. 53874$_n$	4. 55360$_n$	4. 54322$_n$	4. 51236$_n$	4. 46502$_n$	
$Q_{0\cdot0}[n+1.-n+1]+\sigma$	3. 32494$_n$	3. 62946$_n$	3. 60373$_n$	3. 55967$_n$	3. 49933$_n$	3. 42500$_n$	3. 33884$_n$	3. 24270$_n$	3. 13782$_n$	3. 02565$_n$	2. 90711$_n$
$Q_{0\cdot0}[n-1.-n-1]-\sigma$	3. 32494$_n$	3. 62946$_n$	3. 60373$_n$	3. 55967$_n$	3. 49933$_n$	3. 42500$_n$	3. 33884$_n$	3. 24270$_n$	3. 13782$_n$	3. 02565$_n$	2. 90711$_n$
$Q_{0\cdot0}[n+1.-n-1]+\delta$	3. 32494	3. 62946	3. 60373	3. 55967	3. 49933	3. 42500	3. 33884	3. 24270	3. 13782	3. 02565	2. 90711
$Q_{0\cdot0}[n-1.-n+1]-\delta$	3. 32494	3. 62946	3. 60373	3. 55967	3. 49933	3. 42500	3. 33884	3. 24270	3. 13782	3. 02565	2. 90711
$Q_{3\cdot0}[n+1.-n]$	4. 03175										
$Q_{3\cdot0}[n-1.-n]$	4. 03175				4. 32136						
$Q_{3\cdot0}[n-3.-n]$					4. 50550				4. 80030		
$Q_{2\cdot1}[n.-n+1]$		[4. 68736$_n$]									
$Q_{2\cdot1}[n-2.-n+1]$		4. 49191$_n$				4. 24224					
$Q_{2\cdot1}[n.-n-1]$				4. 89559$_n$							
$Q_{2\cdot1}[n-2.-n-1]$				5. 07295$_n$				5. 48264$_n$			
$Q_{1\cdot2}[n-1.-n+2]$			4. 42446								
$Q_{1\cdot2}[n+1.-n]$	4. 54525										
$Q_{1\cdot2}[n-1.-n]$	4. 54525				4. 41262						
$Q_{1\cdot2}[n+1.-n-2]$			4. 85406								
$Q_{1\cdot2}[n-1.-n-2]$			5. 16184				5. 69280				
$Q_{0\cdot3}[n.-n+1]$		4. 58186$_n$									
$Q_{0\cdot3}[n.-n-1]$				2. 59704							
$Q_{0\cdot3}[n.-n-3]$		4. 74024$_n$				5. 43190$_n$					
$Q_{1\cdot0}[n.-n+1]+\sigma$		4. 63965									
$Q_{1\cdot0}[n.-n-1]-\sigma$				4. 40186							
$Q_{1\cdot0}[n-2.-n-1]-\sigma$				4. 59921				[4. 49128] [1]			

$Q_{1\cdot 0}[n.-n-1]+\delta$				4.67205_n							
$Q_{1\cdot 0}[n.-n+1]-\delta$		4.54525_n									
$Q_{1\cdot 0}[n-2.-n+1]-\delta$		4.54525_n				4.57721_n					
$Q_{0\cdot 1}[n+1.-n]+\sigma$	4.19088_n										
$Q_{0\cdot 1}[n-1.-n]-\sigma$	4.19088_n				3.22170_n						
$Q_{0\cdot 1}[n-1.-n-2]-\sigma$			4.79432_n				4.75273_n				
$Q_{0\cdot 1}[n+1.-n]+\delta$	4.19088										
$Q_{0\cdot 1}[n+1.-n-2]+\delta$			4.79432								
$Q_{0\cdot 1}[n-1.-n+2]-\delta$			4.47930								
$Q_{0\cdot 1}[n-1.-n]-\delta$	4.19088				4.71754						

[1] In v. Zeipel's table this quantity is misplaced. It occurs in the column $n=5$.

Logarithmic. TABLE IIIu^2. Unit = 1″.

n	0	1	2	3	4	5	6	7	8	9	10
$Q_{0\cdot0}[n.-n]$	2. 23324	2. 52014	2. 58292	2. 60872	2. 60803	2. 58234	2. 53549	2. 47134	2. 39284	2. 30268	2. 20271
$Q_{1\cdot0}[n+1.-n]$	3. 05368$_n$	3. 28708$_n$	3. 21286$_n$	3. 13038$_n$	3. 03946$_n$	2. 94059$_n$	2. 83441$_n$	2. 72166$_n$	2. 60305$_n$	2. 47917$_n$	
$Q_{1\cdot0}[n-1.-n]$	3. 05368$_n$	3. 41485$_n$	3. 50016$_n$	3. 57826$_n$	3. 63742$_n$	3. 67159$_n$	3. 68131$_n$	3. 66944$_n$	3. 63892$_n$	3. 59276$_n$	
$Q_{0\cdot1}[n.-n+1]$	3. 11484	3. 28708	3. 09682	2. 73074	2. 08444$_n$	2. 81745$_n$	3. 01401$_n$	3. 09669$_n$	3. 12342$_n$	3. 11549$_n$	
$Q_{0\cdot1}[n.-n-1]$	3. 11484	3. 51345	3. 63465	3. 73328	3. 80402	3. 84435	3. 85750	3. 84754	3. 81809	3. 77253	
$Q_{2\cdot0}[n.-n]$			4. 21525								
$Q_{2\cdot0}[n-2.-n]$			4. 15081			4. 45147	4. 51320				
$Q_{1\cdot1}[n-1.-n+1]$				4. 00343$_n$							
$Q_{1\cdot1}[n+1.-n-1]$		4. 36877$_n$									
$Q_{1\cdot1}[n-1.-n-1]$		4. 47566$_n$			4. 86044$_n$	4. 95144$_n$					
$Q_{0\cdot2}[n.-n+2]$	3. 90847										
$Q_{0\cdot2}[n.-n]$			4. 22059								
$Q_{0\cdot2}[n.-n-2]$	3. 90847			4. 66360	4. 78473						
$Q_{0\cdot0}[n+1.-n+1]+\sigma$		4. 04093									
$Q_{0\cdot0}[n-1.-n-1]-\sigma$		4. 04093									
$Q_{0\cdot0}[n+1.-n-1]+\delta$		4. 04093$_n$									
$Q_{0\cdot0}[n-1.-n+1]-\delta$		[4. 04093$_n$]		4. 00956$_n$							
$Q_{1\cdot0}[n.-n+1]+\sigma$		5. 14238$_n$									
$Q_{1\cdot0}[n.-n-1]+\delta$											
$Q_{1\cdot0}[n.-n+1]-\delta$		5. 06752									
$Q_{1\cdot0}[n-2.-n+1]-\delta$		5. 06752									
$Q_{0\cdot1}[n+1.-n]+\sigma$	4. 72557										
$Q_{0\cdot1}[n-1.-n]-\sigma$	4. 72557										
$Q_{0\cdot1}[n+1.-n]+\delta$	4. 72557$_n$										
$Q_{0\cdot1}[n-1.-n+2]-\delta$			5. 01861$_n$								
$Q_{0\cdot1}[n-1.-n]-\delta$	[4. 72557$_n$]										

Logarithmic. TABLE IV. Unit=1″.

	n	0	1	2	3	4	5	6
	$R_{0\cdot0}[n.-n+1]+\pi'$	1.898161_n	2.283141_n	2.153770_n	2.006171_n	1.847875_n	1.68250_n	1.51210_n
	$R_{0\cdot0}[n.-n-1]-\pi'$	1.898161	2.283141	2.153770	2.006171	1.847875	1.68250	1.51210
	$R_{1\cdot0}[n+1.-n+1]+\pi'$	2.522558	2.683079	2.528880	2.366191	2.197675	2.02490	1.84887
	$R_{1\cdot0}[n-1.-n+1]+\pi'$	2.522558	2.937464	2.958041	2.924776	2.858077	2.76886	2.66352
	$R_{1\cdot0}[n+1.-n-1]-\pi'$	2.522558_n	2.683079_n	2.528880_n	2.366191_n	2.197675_n	2.02490_n	1.84887_n
	$R_{1\cdot0}[n-1.-n-1]-\pi'$	2.522558_n	2.937464_n	2.958044_n	2.924776_n	2.858077_n	2.76886_n	2.66352_n
	$R_{0\cdot1}[n.-n+2]+\pi'$	2.812563_n	3.024413_n	2.794447_n	2.523495_n	2.197675_n	1.76164_n	0.74650_n
	$R_{0\cdot1}[n.-n]+\pi'$	2.522558_n	3.024413_n	3.076598_n	3.058902_n	3.001314_n	2.91803_n	2.81684_n
	$R_{0\cdot1}[n.-n]-\pi'$	2.522558	2.462558	1.724281	1.856833_n	2.093957_n	2.12968_n	2.09513_n
	$R_{0\cdot1}[n.-n-2]-\pi'$	2.812563	3.261391	3.246209	3.190606	3.108845	3.00884	2.89541
	$R_{2\cdot0}[n.-n+1]+\pi'$		3.49105_n					
	$R_{2\cdot0}[n-2.-n+1]+\pi'$		3.36728_n					
	$R_{1\cdot1}[n-1.-n+2]+\pi'$			3.70912				
	$R_{1\cdot1}[n+1.-n]+\pi'$	3.30370						
	$R_{1\cdot1}[n-1.-n]+\pi'$	3.30370						
	$R_{1\cdot1}[n+1.-n]-\pi'$	3.30370_n						
	$R_{1\cdot1}[n-1.-n]-\pi'$	3.30370_n						
	$R_{0\cdot2}[n.-n+1]+\pi'$		3.81842_n					
	$R_{0\cdot2}[n.-n+1]-\pi'$		3.21895					
	$R_{0\cdot0}[n-1.-n]-\sigma+\pi'$	2.72638_n						
	$R_{0\cdot0}[n+1.-n]+\delta+\pi'$	2.72638						
	$R_{0\cdot0}[n-1.-n+2]-\delta+\pi'$			2.94112				
	$R_{0\cdot0}[n+1.-n]+\sigma-\pi'$	2.72638						
	$R_{0\cdot0}[n-1.-n]-\delta-\pi'$	2.72638_n						
Factor v.	$R_{0\cdot0}[n.-n+1]+\pi'$	2.51524	2.84832	2.78195	2.69286	2.58759	2.47019	
	$R_{0\cdot0}[n.-n-1]-\pi'$	2.51524_n	2.84832_n	2.78195_n	2.69286_n	2.58759_n	2.47019_n	
	$R_{1\cdot0}[n+1.-n+1]+\pi'$	3.25180_n	3.45486_n	3.34235_n	3.21906_n	3.08741_n	2.94903_n	
	$R_{1\cdot0}[n-1.-n+1]+\pi'$	3.25180_n	3.62946_n	3.66471_n	3.66409_n	3.63529_n	3.58453_n	
	$R_{1\cdot0}[n+1.-n-1]-\pi'$	3.25180	3.45468	3.34235	3.21906	3.08741	2.94903	
	$R_{1\cdot0}[n-1.-n-1]-\pi'$	3.25180	3.62946	3.66471	3.66409	3.63529	3.58453	
	$R_{0\cdot1}[n.-n+2]+\pi'$	3.49076	3.69598	3.53278	3.33224	3.08741	2.77380	
	$R_{0\cdot1}[n.-n]+\pi'$	3.25180	3.69598	3.76576	3.78484	3.76831	3.72575	
	$R_{0\cdot1}[n.-n]-\pi'$	3.25180_n	3.33141_n	2.99523_n	2.24789_n	2.51136	2.76863	
	$R_{0\cdot1}[n.-n-2]-\pi'$	3.49076_n	3.89134_n	3.91658_n	3.90661_n	3.87001_n	3.81284_n	
	$R_{2\cdot0}[n.-n+1]+\pi'$		4.36208					
	$R_{2\cdot0}[n-2.-n+1]+\pi'$		4.18801					
	$R_{1\cdot1}[n-1.-n+2]+\pi'$			4.52584_n				
	$R_{1\cdot1}[n+1.-n]+\pi'$	4.13780_n						
	$R_{1\cdot1}[n-1.-n]+\pi'$	4.13780_n						
	$R_{1\cdot1}[n+1.-n]-\pi'$	4.13780						
	$R_{1\cdot1}[n-1.-n]-\pi'$	4.13780						
	$R_{0\cdot2}[n.-n+1]+\pi'$		4.60272					
	$R_{0\cdot2}[n.-n+1]-\pi'$		4.07416_n					
	$R_{0\cdot0}[n-1.-n]-\sigma+\pi'$	3.51583						
	$R_{0\cdot0}[n+1.-n]+\delta+\pi'$	3.51583_n						
	$R_{0\cdot0}[n-1.-n+2]-\delta+\pi'$			3.76747_n				
	$R_{0\cdot0}[n+1.-n]+\sigma-\pi'$	3.51583_n						
	$R_{0\cdot0}[n-1.-n]-\delta-\pi'$	3.51583						
Factor v^2.	$R_{0\cdot0}[n.-n+1]+\pi'$		3.1148_n		3.0520_n			
	$R_{0\cdot0}[n.-n-1]-\pi'$		3.1148		3.0520		2.9231	
	$R_{1\cdot0}[n+1.-n+1]+\pi'$							
	$R_{1\cdot0}[n-1.-n+1]+\pi'$				4.0961			
	$R_{1\cdot0}[n+1.-n-1]-\pi'$		3.9234_n					
	$R_{1\cdot0}[n-1.-n-1]-\pi'$		4.0409_n				4.0774_n	
	$R_{0\cdot1}[n.-n+2]+\pi'$	3.8736_n						
	$R_{0\cdot1}[n.-n]+\pi'$			4.1593_n				
	$R_{0\cdot1}[n.-n]-\pi'$			3.6562				
	$R_{0\cdot1}[n.-n-2]-\pi'$	3.8736_n				4.3090		

TABLE V.

Unit=1″.

n	0	1	2	3	4	5	6	7	8	9	10
$P_{0\cdot0}(n.-n)$	———	+ 15.7096	+ 90.4978	+ 72.0360	+ 53.2917	+ 37.9325	+ 26.3667	+ 18.0394	+ 12.194	+ 8.172	+ 5.440
$P_{1\cdot0}(n+1.-n)$	− 43.141	− 78.609	+ 20.300	+ 55.479	+ 68.840	+ 68.434	+ 60.958	+ 50.825	+ 40.497	+ 31.252	+ 23.535
$P_{1\cdot0}(n-1.-n)$	+ 43.141	− 7.855	− 294.893	− 422.487	− 448.971	− 415.891	− 355.726	− 288.845	− 225.93	− 171.96	− 128.11
$P_{0\cdot1}(n.-n+1)$	———	+ 43.232	− 43.699	− 104.641	− 129.678	− 129.732	− 116.283	− 97.462	− 78.00	− 60.41	− 45.63
$P_{0\cdot1}(n.-n-1)$	———	+ 106.070	+ 680.283	+ 759.792	+ 722.973	+ 628.920	+ 516.529	+ 407.639	+ 312.20	+ 233.80	+ 171.96
$P_{2\cdot0}(n+2.-n)$	+ 105.647	+ 126.796	+ 99.984	+ 91.548	+ 93.487	+ 95.377	+ 93.271	+ 86.916	+ 77.46	+ 66.53	+ 55.38
$P_{2\cdot0}(n.-n)$	———	+ 222.102	− 93.564	− 427.143	− 710.886	− 882.229	− 940.779	− 912.941	− 829.60	− 718.87	− 600.53
$P_{2\cdot0}(n-2.-n)$	− 105.647	− 156.231	+ 249.644	+ 1009.605	+ 1679.002	+ 2095.709	+ 2247.590	+ 2191.601	+ 1999.68	+ 1738.68	+ 1456.57
$P_{1\cdot1}(n+1.-n+1)$	− 191.93	− 310.55	− 209.26	− 195.06	− 223.02	− 253.70	− 268.97	− 265.52	− 246.77	− 218.68	− 186.46
$P_{1\cdot1}(n-1.-n+1)$	+ 191.93	———	+ 46.78	+ 478.00	+ 955.49	+ 1301.17	+ 1469.07	+ 1482.08	+ 1385.39	+ 1226.51	+ 1042.02
$P_{1\cdot1}(n+1.-n-1)$	− 191.93	− 550.33	+ 271.44	+ 798.27	+ 1175.06	+ 1364.58	+ 1394.14	+ 1312.67	+ 1166.27	+ 993.10	+ 818.08
$P_{1\cdot1}(n-1.-n-1)$	+ 191.93	− 106.07	− 2630.66	− 4919.41	− 6524.67	− 7266.23	− 7268.47	− 6760.67	− 5961.29	− 5050.44	− 4145.64
$P_{0\cdot2}(n.-n+2)$	———	+ 150.35	+ 109.26	+ 89.64	+ 117.98	+ 157.29	+ 186.07	+ 197.58	+ 193.12	+ 177.46	+ 155.51
$P_{0\cdot2}(n.-n)$	———	+ 253.55	− 499.25	− 1306.17	− 1959.67	− 2330.12	− 2428.12	− 2322.76	− 2090.40	− 1798.83	− 1494.79
$P_{0\cdot2}(n.-n-2)$	———	+ 527.49	+ 5360.58	+ 5101.10	+ 5930.17	+ 6097.47	+ 5779.90	+ 5171.43	+ 4428.10	+ 3666.25	+ 2954.27
$P_{0\cdot0}(n+1.-n+1)+\sigma$	+ 79.10	+ 383.86	+ 427.46	+ 405.72	+ 352.24	+ 288.84	+ 227.62	+ 174.22	+ 130.33	+ 95.79	+ 69.40
$P_{0\cdot0}(n-1.-n-1)-\sigma$	− 79.10	———	+ 142.49	+ 202.86	+ 211.35	+ 192.56	+ 162.58	+ 130.67	+ 101.37	+ 76.64	+ 56.78
$P_{0\cdot0}(n+1.-n-1)+\delta$	− 79.10	− 383.86	− 427.46	− 405.72	− 352.24	− 288.84	− 227.62	− 174.22	− 130.33	− 95.79	− 69.40
$P_{0\cdot0}(n-1.-n+1)-\delta$	+ 79.10	———	− 142.49	− 202.86	− 211.35	− 192.56	− 162.58	− 130.67	− 101.37	− 76.64	− 56.78
$P_{3\cdot0}(n+1.-n)$	− 179										
$P_{3\cdot0}(n-1.-n)$	+ 179		− 194		+ 2857						
$P_{3\cdot0}(n-3.-n)$					− 3585		− 8819		−11254		
$P_{2\cdot1}(n.-n+1)$		+ 892									
$P_{2\cdot1}(n-2.-n+1)$		− 666		− 746		− 5840					
$P_{2\cdot1}(n.-n-1)$		+ 2174		− 5739							
$P_{2\cdot1}(n-2.-n-1)$				+13482		+38961		+53361			
$P_{1\cdot2}(n-1.-n+2)$	+ 817		− 482								
$P_{1\cdot2}(n+1.-n)$	− 866										
$P_{1\cdot2}(n-1.-n)$	+ 866		+ 861		+15460						
$P_{1\cdot2}(n+1.-n-2)$	− 817		+ 1883								
$P_{1\cdot2}(n-1.-n-2)$			−15988		−57139		−84730				
$P_{0\cdot3}(n.-n+1)$		+ 908									
$P_{0\cdot3}(n.-n-1)$		+ 1160		− 9917							
$P_{0\cdot3}(n.-n-3)$		+ 2309		+27771		+45152					
$P_{1\cdot0}(n.-n+1)+\sigma$		− 1250									
$P_{1\cdot0}(n.-n-1)-\sigma$		− 866		− 900							
$P_{1\cdot0}(n-2.-n-1)-\sigma$				− 1247		− 2340		− 2281			

$P_{1\cdot 0}(n.-n-1)+\delta$		+ 1634		+ 3942							
$P_{1\cdot 0}(n.-n+1)-\delta$		+ 866									
$P_{1\cdot 0}(n-2.-n+1)-\delta$		− 866		+ 1044		+ 2147					
$P_{0\cdot 1}(n+1.-n)+\sigma$	+ 333										
$P_{0\cdot 1}(n-1.-n)-\sigma$	− 333		+ 53		− 372						
$P_{0\cdot 1}(n-1.-n-2)-\sigma$			+ 1763		+ 3855		+ 3930				
$P_{0\cdot 1}(n+1.-n)+\delta$	− 333										
$P_{0\cdot 1}(n+1.-n-2)+\delta$	− 649		− 5288								
$P_{0\cdot 1}(n-1.-n+2)-\delta$	+ 649		− 623								
$P_{0\cdot 1}(n-1.-n)-\delta$	+ 333		− 1193		− 3009						

TABLE Vw. Unit=1''.

n	0	1	2	3	4	5	6	7	8	9	10
$P_{0\cdot0}(n.-n)$	———	− 49.77	− 212.19	− 218.38	− 197.76	− 166.40	− 133.41	− 103.39	− 78.07	− 57.80	− 42.11
$P_{1\cdot0}(n+1.-n)$	+ 127.95	+ 297.43	+ 99.93	− 57.00	− 175.24	− 243.85	− 269.59	− 264.86	− 241.46	− 209.11	− 174.28
$P_{1\cdot0}(n-1.-n)$	− 127.95	+ 24.88	+ 740.62	+ 1336.40	+ 1714.29	+ 1861.91	+ 1827.67	+ 1675.31	+ 1460.30	+ 1225.69	+ 998.30
$P_{0\cdot1}(n.-n+1)$	———	− 161.16	+ 4.11	+ 233.84	+ 417.06	+ 522.14	+ 555.08	+ 535.46	+ 483.52	+ 416.49	+ 346.03
$P_{0\cdot1}(n.-n-1)$	———	− 360.23	− 1693.44	− 2386.8	− 2747.2	− 2805.8	− 2646.8	− 2359.4	− 2014.6	− 1664.2	− 1338.5
$P_{2\cdot0}(n+2.-n)$	− 449, 3	− 693.9	− 517.5	− 421.4	− 404.7	− 428.1	− 458.0	− 474.8	− 470.9	− 447.7	− 409.5
$P_{2\cdot0}(n.-n)$	———	− 987.2	− 703.1	+ 338.4	+ 1769.5	+ 3138.4	+ 4173.4	+ 4777.7	+ 4968.0	+ 4830.2	+ 4464.0
$P_{2\cdot0}(n-2.-n)$	+ 449.3	+ 648.8	− 634.5	− 3373.0	− 6656.7	− 9633.2	− 11773.5	− 12899.7	−13075.8	− 12509.4	−11436.8
$P_{1\cdot1}(n+1.-n+1)$	+ 705.2	+ 1454.2	+ 1087.4	+ 893.8	+ 934.6	+ 1108.1	+ 1301.8	+ 1443.2	+ 1500.9	+ 1476.1	
$P_{1\cdot1}(n-1.-n+1)$	− 705.2	———	+ 351.5	− 839.2	− 2942.5	− 5178.8	− 6999.5	− 8154.2	− 8610.8	− 8481.8	
$P_{1\cdot1}(n+1.-n-1)$	+ 705.2	+ 2383.2	+ 1042.2	− 866.6	− 3034.2	− 4910.7	− 6214.1	− 6884.8	− 6991.5	− 6675.7	
$P_{1\cdot1}(n-1.-n-1)$	− 705.2	+ 360.2	+ 7121.2	+16274.2	+ 25650.6	+ 33200.8	+ 37910.8	+ 39666.4	+38884.5	+ 36265.0	
$P_{0\cdot2}(n.-n+2)$	———	− 630.2	− 569.9	− 458.1	− 519.2	− 703.7	− 918.5	− 1093.6	− 1194.2	− 1216.0	
$P_{0\cdot2}(n.-n)$	———	− 1111.0	+ 149.9	+ 2919.9	+ 6339.7	+ 9412.8	+ 11667.3	+ 12834.8	+13024.3	+ 12456.5	
$P_{0\cdot2}(n.-n-2)$	———	− 1921.8	− 9449.0	−16651.5	− 23114.8	− 27699.4	− 30021.4	− 30248.4	−28814.6	− 26275.8	
$P_{0\cdot0}(n+1.-n+1)+\sigma$	− 327.52	− 1410.40	− 1815.81	− 1972.04	− 1934.46	− 1771.50	− 1543.43	− 1295.0	− 1054.1	− 837.7	− 652.6
$P_{0\cdot0}(n-1.-n-1)-\sigma$	+ 327.52	———	− 605.27	− 986.02	− 1160.67	− 1181.00	− 1102.45	− 971.2	− 819.9	− 670.2	− 534.0
$P_{0\cdot0}(n+1.-n-1)+\delta$	+ 327.52	+ 1410.40	+ 1815.81	+ 1972.04	+ 1934.46	+ 1771.50	+ 1543.43	+ 1295.0	+ 1054.1	+ 837.7	+ 652.6
$P_{0\cdot0}(n-1.-n+1)-\delta$	− 327.52	———	+ 605.27	+ 986.02	+ 1160.67	+ 1181.00	+ 1102.45	+ 971.2	+ 819.9	+ 670.2	+ 534.0
$P_{3\cdot0}(n+1.-n)$	+1104										
$P_{3\cdot0}(n-1.-n)$	−1104		+ 2975		− 6146						
$P_{3\cdot0}(n-3.-n)$					+ 14870		+ 47337		+74635		
$P_{2\cdot1}(n.-n+1)$		− 5205									
$P_{2\cdot1}(n-2.-n+1)$		+ 3571		+ 339		+ 22690					
$P_{2\cdot1}(n.-n-1)$		−10942		+ 2561							
$P_{2\cdot1}(n-2.-n-1)$				−47527		−183621		−318377			
$P_{1\cdot2}(n-1.-n+2)$	−3541		+ 2850								
$P_{1\cdot2}(n+1.-n)$	+4261										
$P_{1\cdot2}(n-1.-n)$	−4261		+ 3794		− 47653						
$P_{1\cdot2}(n+1.-n-2)$	+3541		+ 8025								
$P_{1\cdot2}(n-1.-n-2)$			+46770		+231778		+449089				
$P_{0\cdot3}(n.-n+1)$		− 4621									
$P_{0\cdot3}(n.-n-1)$		− 5961		+21967							
$P_{0\cdot3}(n.-n-3)$		− 9041		−94500		−209160					
$P_{1\cdot0}(n.-n+1)+\sigma$		+ 5671									
$P_{1\cdot0}(n.-n-1)-\sigma$		+ 4260		+ 5954							
$P_{1\cdot0}(n-2.-n-1)-\sigma$				+ 6586		+ 15068		+ 17484			

$P_{1\cdot0}(n.-n-1)+\delta$		$-$ 7081		$-$20744							
$P_{1\cdot0}(n.-n+1)-\delta$		$-$ 4260									
$P_{1\cdot0}(n-2.-n+1)-\delta$		$+$ 4260		$-$ 5600		$-$ 13887					
$P_{0\cdot1}(n+1.-n)+\sigma$	$-$1786										
$P_{0\cdot1}(n-1.-n)-\sigma$	$+$1786		$-$ 989		$+$ 974						
$P_{0\cdot1}(n-1.-n-2)-\sigma$			$-$ 8252		$-$ 22240		$-$ 27446				
$P_{0\cdot1}(n+1.-n)+\delta$	$+$1786										
$P_{0\cdot1}(n+1.-n-2)+\delta$	$+$3096		$+$24757								
$P_{0\cdot1}(n-1.-n+2)-\delta$	$-$3096		$+$ 3410								
$P_{0\cdot1}(n-1.-n)-\delta$	$-$1786		$+$ 5831		$+$ 17597						

TABLE $V\nu^2$.

Unit=1″.

$P_{0\cdot0}(n.-n)$	—	$+$ 62	$+$ 175	$+$ 250	$+$ 290	$+$ 297	$+$ 282	$+$ 252	$+$ 216	$+$ 179	$+$144
$P_{1\cdot0}(n+1.-n)$	$-$ 171	$-$ 507	$-$ 446	$-$ 251	$-$ 1	$+$ 235	$+$ 417	$+$ 533	$+$ 586	$+$ 588	
$P_{1\cdot0}(n-1.-n)$	$+$ 171	$-$ 31	$-$ 734	$-$ 1686	$-$ 2664	$-$ 3462	$-$ 3971	$-$4173	$-$4107	$-$3844	
$P_{0\cdot1}(n.-n+1)$	—	$+$ 269	$+$ 239	$-$ 30	$-$ 404	$-$ 765	$-$ 1042	$-$1208	$-$1266	$-$1237	
$P_{0\cdot1}(n.-n-1)$	—	$+$ 517	$+$ 1643	$+$ 2966	$+$ 4227	$+$ 5183	$+$ 5724	$+$5857	$+$5651	$+$5209	
$P_{2\cdot0}(n.-n)$			$+$ 3217								
$P_{2\cdot0}(n-2.-n)$			$+$ 646		$+$11174	$+$18846	$+$26496				
$P_{1\cdot1}(n-1.-n+1)$		—		$-$ 986							
$P_{1\cdot1}(n+1.-n-1)$		$-$ 4972									
$P_{1\cdot1}(n-1.-n-1)$		$-$ 517		$-$22685	$-$42351	$-$64234					
$P_{0\cdot2}(n.-n+2)$	—										
$P_{0\cdot2}(n.-n)$	—		$+$ 2754								
$P_{0\cdot2}(n.-n-2)$	—		$+$10708	$+$22571	$+$37519						
$P_{0\cdot0}(n-1.-n-1)-\sigma$		—		$+$ 2254		$+$ 3351					
$P_{0\cdot0}(n+1.-n-1)+\delta$		$-$ 2605									
$P_{0\cdot0}(n-1.-n+1)-\delta$		—		$-$ 2254							
$P_{1\cdot0}(n.-n+1)+\sigma$		$-$13593									
$P_{1\cdot0}(n.-n-1)+\delta$		$+$16199									
$P_{1\cdot0}(n.-n+1)-\delta$		$+$10988									
$P_{1\cdot0}(n-2.-n+1)-\delta$		$-$10988		$+$14731							
$P_{0\cdot1}(n+1.-n)+\sigma$	$+$4875										
$P_{0\cdot1}(n-1.-n)-\sigma$	$-$4875										
$P_{0\cdot1}(n+1.-n)+\delta$	$-$4875										
$P_{0\cdot1}(n-1.-n+2)-\delta$			$-$ 9482								
$P_{0\cdot1}(n-1.-n)-\delta$	$+$4875										

TABLE VI.

Unit = 1″.

n	0	1	2	3	4	5	6	7	8	9	10
$Q_{0\cdot0}(n.-n)$	+43.141	+58.941	+113.897	+85.180	+60.839	+42.332	+28.958	+19.758	+13.112	+8.725	+5.773
$Q_{1\cdot0}(n+1.-n)$	−148.79	−169.04	−18.77	+39.45	+63.14	+67.20	+61.47	+51.87	+41.56	+32.17	+24.25
$Q_{1\cdot0}(n-1.-n)$	−148.79	−345.86	−702.15	−727.17	−666.93	−567.79	−459.78	−359.27	−273.14	−203.40	−148.94
$Q_{0\cdot1}(n.-n+1)$	+191.99	+198.51	+18.77	−82.04	−123.98	−130.69	−119.39	−100.82	−80.91	−62.70	−47.34
$Q_{0\cdot1}(n.-n-1)$	+191.99	+434.27	+929.94	+940.12	+849.45	+715.95	+575.62	+447.38	+338.71	+251.39	+183.58
$Q_{2\cdot0}(n+2.-n)$	+284.16	+276.53	+164.39	+119.94	+106.65	+102.12	+97.26	+89.65	+79.51	+68.16	+56.68
$Q_{2\cdot0}(n.-n)$	+568.32	+1001.22	+213.59	−370.68	−787.74	−1015.36	−1085.72	−1047.12	−943.56	−810.74	−671.88
$Q_{2\cdot0}(n-2.-n)$	+284.16	+989.93	+2099.34	+2959.17	+3486.04	+3644.88	+3508.30	+3180.28	+2753.6	+2301.3	+1868.8
$Q_{1\cdot1}(n+1.-n+1)$	−961.86	−935.33	−470.84	−306.90	−272.99	−278.31	−283.17	−275.22	−254.19	−224.64	−191.29
$Q_{1\cdot1}(n-1.-n+1)$	−769.93	−1332.35	−564.67	+349.39	+1090.78	+1551.40	+1746.43	+1741.18	+1606.8	+1405.7	+1181.6
$Q_{1\cdot1}(n+1.-n-1)$	−769.93	−1295.09	−146.61	+595.49	+1099.98	+1358.22	+1420.25	+1350.54	+1205.0	+1027.8	+846.8
$Q_{1\cdot1}(n-1.-n-1)$	−961.86	−3032.17	−6656.21	−8805.72	−9942.80	−10096.98	−9516.42	−8491.72	−7263.0	−6011.1	−4844.0
$Q_{0\cdot2}(n.-n+2)$	+816.81	+794.90	+372.77	+199.19	+165.19	+179.64	+198.61	+206.13	+199.78	+182.94	+160.02
$Q_{0\cdot2}(n.-n)$	+865.89	+1412.97	+137.23	−1024.54	−1887.44	−2366.80	−2511.79	−2418.56	−2180.8	−1876.3	−1557.5
$Q_{0\cdot2}(n.-n-2)$	+816.81	+2355.16	+5306.59	+6625.40	+7185.73	+7090.49	+6542.08	+5743.16	+4849.2	+3971.9	+3173.4
$Q_{0\cdot0}(n+1.-n+1)+\sigma$	+412.18	+865.89	+765.43	+638.10	+509.89	+394.75	+298.22	+221.03	+161.20	+116.08	+82.68
$Q_{0\cdot0}(n-1.-n-1)-\sigma$	+412.18	+865.89	+765.43	+638.10	+509.89	+394.75	+298.22	+221.03	+161.20	+116.08	+82.68
$Q_{0\cdot0}(n+1.-n-1)+\delta$	−412.18	−865.89	−765.43	−638.10	−509.89	−394.75	−298.22	−221.03	−161.20	−116.08	−82.68
$Q_{0\cdot0}(n-1.-n+1)-\delta$	−412.18	−865.89	−765.43	−638.10	−509.89	−394.75	−298.22	−221.03	−161.20	−116.08	−82.68
$Q_{3\cdot0}(n+1.-n)$	−1477										
$Q_{3\cdot0}(n-1.-n)$	−1477				+3919						
$Q_{3\cdot0}(n-3.-n)$					−11748				−17938		
$Q_{2\cdot1}(n.-n+1)$		[+6916]									
$Q_{2\cdot1}(n-2.-n+1)$		+4691				−8364					
$Q_{2\cdot1}(n.-n-1)$				−4841							
$Q_{2\cdot1}(n-2.-n-1)$				+40197				+78052			
$Q_{1\cdot2}(n-1.-n+2)$			−3793								
$Q_{1\cdot2}(n+1.-n)$	−5525										
$Q_{1\cdot2}(n-1.-n)$	−5525				+17415						
$Q_{1\cdot2}(n+1.-n-2)$			−1102								
$Q_{1\cdot2}(n-1.-n-2)$			−43556				−112546				
$Q_{0\cdot3}(n.-n+1)$		+6023									
$Q_{0\cdot3}(n.-n-1)$				−7678							
$Q_{0\cdot3}(n.-n-3)$		+11251				+53719					
$Q_{1\cdot0}(n.-n+1)+\sigma$		−7257									
$Q_{1\cdot0}(n.-n-1)-\sigma$				−2252							
$Q_{1\cdot0}(n-2.-n-1)-\sigma$				−7356				−4612			

$Q_{1\cdot 0}(n.-n-1)+\partial$				+ 8633							
$Q_{1\cdot 0}(n.-n+1)-\partial$		+ 5525									
$Q_{1\cdot 0}(n-2.-n+1)-\partial$		+ 5525				+ 5965					
$Q_{0\cdot 1}(n+1.-n)+\sigma$	+2345										
$Q_{0\cdot 1}(n-1.-n)-\sigma$	+2345				− 363						
$Q_{0\cdot 1}(n-1.-n-2)-\sigma$			+10615				+ 7448				
$Q_{0\cdot 1}(n+1.-n)+\partial$	−2345										
$Q_{0\cdot 1}(n+1.-n-2)+\partial$			−10615								
$Q_{0\cdot 1}(n-1.-n+2)-\partial$			− 4492								
$Q_{0\cdot 1}(n-1.-n)-\partial$	−2345				− 7795						

TABLE VI*w*.

Unit=1″.

n	0	1	2	3	4	5	6	7	8	9	10
$Q_{0\cdot0}(n.-n)$	− 127.95	− 210.93	− 316.24	− 286.03	− 241.82	− 195.08	− 152.08	− 115.52	− 85.94	− 62.90	− 45.41
$Q_{1\cdot0}(n+1.-n)$	+ 577.26	+ 782.77	+ 357.68	+ 74.16	− 113.43	− 218.86	− 263.20	− 267.07	− 247.01	− 215.35	− 180.02
$Q_{1\cdot0}(n-1.-n)$	+ 577.26	+ 1415.55	+ 2255.10	+ 2648.42	+ 2788.44	+ 2707.42	+ 2474.2	+ 2158.8	+ 1815.4	+ 1482.8	+1182.3
$Q_{0\cdot1}(n.-n+1)$	− 705.22	− 888.23	− 357.68	+ 68.85	+ 355.25	+ 511.49	+ 567.4	+ 556.2	+ 504.8	+ 435.5	+ 361.7
$Q_{0\cdot1}(n.-n-1)$	− 705.22	− 1731.94	− 2887.57	− 3363.50	− 3513.91	− 3390.22	− 3082.5	− 2678.6	− 2245.1	− 1828.8	−1454.8
$Q_{2\cdot0}(n+2.-n)$	− 1553.0	− 1821.2	− 1090.6	− 711.5	− 552.9	− 506.4	− 502.3	− 502.8	− 490.7	− 462.9	
$Q_{2\cdot0}(n.-n)$	− 3106.1	− 5779.9	− 3694.8	− 1232.6	+ 1222.5	+ 3245.6	+ 4645.2	+ 5411.6	+ 5633.9	+ 5454.9	
$Q_{2\cdot0}(n-2.-n)$	− 1553.0	− 4907.9	− 8296.6	− 12105	− 15636	− 18197	− 19489	− 19558	− 18624	−17008	
$Q_{1\cdot1}(n+1.-n+1)$	+ 4613.3	+ 5366.2	+ 3033.8	+ 1854.9	+ 1411.8	+ 1354.2	+ 1440.5	+ 1533.4	+ 1566.8	+ 1529.5	
$Q_{1\cdot1}(n-1.-n+1)$	+ 3908.1	+ 7142.7	+ 4822.2	+ 1304.0	− 2495.9	− 5806.9	− 8204.8	− 9585.8	− 10044	− 9792.9	
$Q_{1\cdot1}(n+1.-n-1)$	+ 3908.1	+ 7187.2	+ 4272.7	+ 1097.5	− 1982.2	− 4462.6	− 6133.7	− 7006.8	− 7207.3	− 6919.9	
$Q_{1\cdot1}(n-1.-n-1)$	+ 4613.3	+14115	+ 24486	+ 34732	+ 43698	+ 49782	+ 52434	+ 51923	+ 48920	+44286	
$Q_{0\cdot2}(n.-n+2)$	− 3540.7	− 4121.0	− 2280.3	− 1287.6	− 923.1	− 909.1	− 1034.6	− 1170.6	− 1253.3	− 1265.5	
$Q_{0\cdot2}(n.-n)$	− 4260.6	− 7609.0	− 4093.8	+ 551.5	+ 5318.5	+ 9292.2	+ 12016	+ 13425	+ 13691	+13103	
$Q_{0\cdot2}(n.-n-2)$	− 3540.7	−10416	− 18424	− 25346	− 31030	− 34574	− 35777	− 34932	− 32536	−29175	
$Q_{0\cdot0}(n+1.-n+1)+\sigma$	− 2113.2	− 4260.5	− 4015.4	− 3628.0	− 3157.4	− 2660.8	− 2181.9	− 1748.6	− 1373.5	− 1060.8	− 807.4
$Q_{0\cdot0}(n-1.-n-1)-\sigma$	− 2113.2	− 4260.5	− 4015.4	− 3628.0	− 3157.4	− 2660.8	− 2181.9	− 1748.6	− 1373.5	− 1060.8	− 807.4
$Q_{0\cdot0}(n+1.-n-1)+\delta$	+ 2113.2	+ 4260.5	+ 4015.4	+ 3628.0	+ 3157.4	+ 2660.8	+ 2181.9	+ 1748.6	+ 1373.5	+ 1060.8	+ 807.4
$Q_{0\cdot0}(n-1.-n+1)-\delta$	+ 2113.2	+ 4260.5	+ 4015.4	+ 3628.0	+ 3157.4	+ 2660.8	+ 2181.9	+ 1748.6	+ 1373.5	+ 1060.8	+ 807.4
$Q_{3\cdot0}(n+1.-n)$	+10758										
$Q_{3\cdot0}(n-1.-n)$	+10758				− 2344						
$Q_{3\cdot0}(n-3.-n)$					+ 58043				+124030		
$Q_{2\cdot1}(n.-n+1)$		[−48681]									
$Q_{2\cdot1}(n-2.-n+1)$		−31039				+ 28058					
$Q_{2\cdot1}(n.-n-1)$				− 24814							
$Q_{2\cdot1}(n-2.-n-1)$				−181030				−490100			
$Q_{1\cdot2}(n-1.-n+2)$			+ 26574								
$Q_{1\cdot2}(n+1.-n)$	+35095										
$Q_{1\cdot2}(n-1.-n)$	+35095				− 36497						
$Q_{1\cdot2}(n+1.-n-2)$			+ 34613								
$Q_{1\cdot2}(n-1.-n-2)$			+182000				+636060				
$Q_{0\cdot3}(n.-n+1)$		−38182									
$Q_{0\cdot3}(n.-n-1)$				+ 395							
$Q_{0\cdot3}(n.-n-3)$		−54985				−270330					
$Q_{1\cdot0}(n.-n+1)+\sigma$		+43616									
$Q_{1\cdot0}(n.-n-1)-\sigma$				+ 17970							
$Q_{1\cdot0}(n-2.-n-1)-\sigma$				+ 46993				+ 37989			

$Q_{1\cdot0}(n.-n-1)+\delta$				-54250							
$Q_{1\cdot0}(n.-n+1)-\delta$		-35095									
$Q_{1\cdot0}(n-2.-n+1)-\delta$		-35095				-43097					
$Q_{0\cdot1}(n+1.-n)+\sigma$	-15520										
$Q_{0\cdot1}(n-1.-n)-\sigma$	-15520				-1666						
$Q_{0\cdot1}(n-1.-n-2)-\sigma$			-62277				-56589				
$Q_{0\cdot1}(n+1.-n)+\delta$	$+15520$										
$Q_{0\cdot1}(n+1.-n-2)+\delta$			$+62277$								
$Q_{0\cdot1}(n-1.-n+2)-\delta$			$+30150$								
$Q_{0\cdot1}(n-1.-n)-\delta$	$+15520$				$+52184$						

TABLE VI w^2. Unit=1″

$Q_{0\cdot0}(n.-n)$	$+171.1$	$+331.2$	$+382.8$	$+406.2$	$+405.5$	$+382.2$	$+343.2$	$+296.0$	$+247.1$	$+200.8$	$+159.5$
$Q_{1\cdot0}(n+1.-n)$	-1132	-1771	-1250	-741	-284	$+83$	$+346$	$+509$	$+587$	$+602$	
$Q_{1\cdot0}(n-1.-n)$	-1132	-2765	-3546	-4396	-5150	-5650	-5830	-5707	-5335	-4818	
$Q_{0\cdot1}(n.-n+1)$	$+1303$	$+1937$	$+1250$	$+538$	-121	-657	-1033	-1249	-1329	-1305	
$Q_{0\cdot1}(n.-n-1)$	$+1303$	$+3262$	$+4312$	$+5411$	$+6368$	$+6988$	$+7203$	$+7040$	$+6578$	$+5923$	
$Q_{2\cdot0}(n.-n)$			$+14502$								
$Q_{2\cdot0}(n-2.-n)$			$+17506$			$+41211$	[+48544]				
$Q_{1\cdot1}(n+1.-n-1)$		-20144									
$Q_{1\cdot1}(n-1.-n+1)$				-10618							
$Q_{1\cdot1}(n-1.-n-1)$		-33161			-88436	-110380					
$Q_{0\cdot2}(n.-n+2)$	$+8100$										
$Q_{0\cdot2}(n.-n)$			$+16618$								
$Q_{0\cdot2}(n.-n-2)$	$+8100$			$+46088$	$+60916$						
$Q_{0\cdot0}(n+1.-n+1)+\sigma$		$+10988$									
$Q_{0\cdot0}(n-1.-n-1)-\sigma$		$+10988$				$+8601$					
$Q_{0\cdot0}(n+1.-n-1)+\delta$		-10988									
$Q_{0\cdot0}(n-1.-n+1)-\delta$		[-10988]		-10223							
$Q_{1\cdot0}(n.-n+1)+\sigma$		-138800									
$Q_{1\cdot0}(n.-n+1)-\delta$		$+116820$									
$Q_{1\cdot0}(n-2.-n+1)-\delta$		$+116820$									
$Q_{0\cdot1}(n+1.-n)+\sigma$	$+53159$										
$Q_{0\cdot1}(n-1.-n)-\sigma$	$+53159$										
$Q_{0\cdot1}(n+1.-n)+\delta$	-53159										
$Q_{0\cdot1}(n-1.-n+2)-\delta$			-104380								
$Q_{0\cdot1}(n-1.-n)-\delta$	[-53159]										

TABLE VII.

Unit=1″.

	n	0	1	2	3	4	5	6
	$R_{0\cdot0}(n.-n+1)+\pi'$	− 79.10	− 191.93	− 142.48	− 101.43	− 70.45	− 48.14	− 32.52
	$R_{0\cdot0}(n.-n-1)-\pi'$	+ 79.10	+ 191.93	+ 142.48	+ 101.43	+ 70.45	+ 48.14	+ 32.52
	$R_{1\cdot0}(n+1.-n+1)+\pi'$	+ 372.6	+ 482.0	+ 266.7	+ 130.9	+ 52.0	+ 9.6	− 10.7
	$R_{1\cdot0}(n-1.-n+1)+\pi'$	+ 293.5	+ 865.9	+ 979.2	+ 942.4	+ 826.9	+ 683.6	+542.1
	$R_{1\cdot0}(n+1.-n-1)-\pi'$	− 293.5	− 290.1	− 124.2	− 29.5	+ 18.5	+ 38.5	+ 43.2
	$R_{1\cdot0}(n-1.-n-1)-\pi'$	− 372.6	− 1057.8	− 1121.6	− 1043.8	− 897.4	− 731.7	−574.6
	$R_{0\cdot1}(n.-n+2)+\pi'$	− 649.5	− 1057.8	− 622.9	− 333.8	− 157.6	− 57.8	− 5.6
	$R_{0\cdot1}(n.-n)+\pi'$	− 333.1	− 1057.8	− 1192.9	− 1145.3	− 1003.0	− 828.0	−655.9
	$R_{0\cdot1}(n.-n)-\pi'$	+ 333.1	+ 290.1	+ 53.0	− 71.9	− 124.1	− 134.8	−124.5
	$R_{0\cdot1}(n.-n-2)-\pi'$	+ 649.5	+ 1825.5	+ 1762.8	+ 1551.0	+ 1284.8	+ 1020.6	+786.0
	$R_{2\cdot0}(n.-n+1)+\pi'$		− 3119					
	$R_{2\cdot0}(n-2.-n+1)+\pi'$		− 2330					
	$R_{1\cdot1}(n-1.-n+2)+\pi'$			+ 5118				
	$R_{1\cdot1}(n+1.-n)+\pi'$	+ 2012						
	$R_{1\cdot1}(n-1.-n)+\pi'$	+ 2012						
	$R_{1\cdot1}(n-1.-n)-\pi'$	− 2012						
	$R_{1\cdot1}(n-1.-n)-\pi'$	− 2012						
	$R_{0\cdot2}(n.-n+1)+\pi'$		− 6583					
	$R_{0\cdot2}(n.-n+1)-\pi'$		+ 1656					
	$R_{0\cdot0}(n-1.-n)-\sigma+\pi'$	− 533						
	$R_{0\cdot0}(n+1.-n)+\delta+\pi'$	+ 533						
	$R_{0\cdot0}(n-1.-n+2)-\delta+\pi'$			+ 873				
	$R_{0\cdot0}(n+1.-n)+\sigma-\pi'$	+ 533						
	$R_{0\cdot0}(n-1.-n)-\delta-\pi'$	− 533						
Factor w	$R_{0\cdot0}(n.-n+1)+\pi'$	+ 327.5	+ 705.2	+ 605.3	+ 493.0	+ 386.9	+ 295.2	
	$R_{0\cdot0}(n.-n-1)-\pi'$	− 327.5	− 705.2	− 605.3	− 493.0	− 386.9	− 295.2	
	$R_{1\cdot0}(n+1.-n+1)+\pi'$	− 1950	− 2850	− 1897	− 1163	− 643	− 299	
	$R_{1\cdot0}(n-1.-n+1)+\pi'$	− 1622	− 4260	− 4923	− 5107	− 4898	− 4432	
	$R_{1\cdot0}(n+1.-n-1)-\pi'$	+ 1622	+ 2145	+ 1292	+ 670	+ 256	+ 4	
	$R_{1\cdot0}(n-1.-n-1)-\pi'$	+ 1950	+ 4966	+ 5529	+ 5600	+ 5285	+ 4727	
	$R_{0\cdot1}(n.-n+2)+\pi'$	+ 3096	+ 4966	+ 3410	+ 2149	+ 1223	+ 594	
	$R_{0\cdot1}(n.-n)+\pi'$	+ 1786	+ 4966	+ 5831	+ 6093	+ 5866	+ 5318	
	$R_{0\cdot1}(n.-n)-\pi'$	− 1786	− 2145	− 989	− 177	+ 325	+ 587	
	$R_{0\cdot1}(n.-n-2)-\pi'$	− 3096	− 7786	− 8252	− 8065	− 7413	− 6499	
	$R_{2\cdot0}(n.-n+1)+\pi'$		+23018					
	$R_{2\cdot0}(n-2.-n+1)+\pi'$		+15418					
	$R_{1\cdot1}(n-1.-n+2)+\pi'$			−33562				
	$R_{1\cdot1}(n+1.-n)+\pi'$	−13734						
	$R_{1\cdot1}(n-1.-n)+\pi'$	−13734						
	$R_{1\cdot1}(n+1.-n)-\pi'$	+13734						
	$R_{1\cdot1}(n-1.-n)-\pi'$	+13734						
	$R_{0\cdot2}(n.-n+1)+\pi'$		+40061					
	$R_{0\cdot2}(n.-n+1)-\pi'$		−11862					
	$R_{0\cdot0}(n-1.-n)-\sigma+\pi'$	+ 3280						
	$R_{0\cdot0}(n+1.-n)+\delta+\pi'$	− 3280						
	$R_{0\cdot0}(n-1.-n+2)-\delta+\pi'$			− 5854				
	$R_{0\cdot0}(n+1.-n)+\sigma-\pi'$	− 3280						
	$R_{0\cdot0}(n-1.-n)-\delta-\pi'$	+ 3280						
Factor w^2	$R_{0\cdot0}(n.-n+1)+\pi'$		− 1303		− 1127			
	$R_{0\cdot0}(n.-n-1)-\pi'$		+ 1303		+ 1127		+ 838	
	$R_{1\cdot0}(n+1.-n+1)+\pi'$							
	$R_{1\cdot0}(n-1.-n+1)+\pi'$				+13600			
	$R_{1\cdot0}(n+1.-n-1)-\pi'$		− 7080				−14465	
	$R_{1\cdot0}(n-1.-n-1)-\pi'$		−12290					
	$R_{0\cdot1}(n.-n+2)+\pi'$	− 7475						
	$R_{0\cdot1}(n.-n)+\pi'$			−14430				
	$R_{0\cdot1}(n.-n)-\pi'$			+ 4532				
	$R_{0\cdot1}(n.-n-2)-\pi'$	+ 7475				+20370		

With these tables we compute terms of the first order in the mass in Hansen's differential equations for the function W and the perturbation in the third coordinate. See Z 7, eq. (33) and Z 8, eq. (39). The first order parts of the equations are expressed in Z 41, eqs. (82), (83), in the form of trigonometric series, in which the coefficients are computed from the formulæ given in B 67. These coefficients comprise Tables VIII–XIVw^2 (cf. Z, 42–48).

Table XV (cf. Z 50, eq. (88)) is an auxiliary table of the same type of construction, which is employed in the computation of terms of the second order in the mass in the differential equation for W (cf. Z 53).

Logarithmic — TABLE VIII.[1] — Unit = 1″.

	n	0	1	2	3	4	5	6	7	8	9	10
	$F_{0\cdot0}(n.-n)$	——	1.673285	2.433759	2.334671	2.203773	2.056133	1.898177	1.733342	1.56326	1.38947	1.21271
	$F_{1\cdot0}(n+1.-n)$	1.935920$_n$	2.318698$_n$	0.79226$_n$	2.031587	2.206222	2.234945	2.201667	2.13346	2.04226	1.93508	1.81622
	$F_{1\cdot0}(n-1.-n)$	1.935920	2.056618$_n$	3.071724$_n$	3.175139$_n$	3.180222$_n$	3.135409$_n$	3.060270$_n$	2.964820$_n$	2.85447$_n$	2.73315$_n$	2.60315$_n$
	$F_{0\cdot1}(n.-n+1)$	——	2.112925	2.117596$_n$	2.496822$_n$	2.589989$_n$	2.590169$_n$	2.542637$_n$	2.465958$_n$	2.36921$_n$	2.25822$_n$	2.13634$_n$
	$F_{0\cdot1}(n.-n-1)$	——	2.502714	3.309811	3.357816	3.336243	3.275716	3.190207	3.087397	2.97156	2.84596	2.71256
	$F_{2\cdot0}(n+2.-n)$	2.324886	2.512335	2.336833	2.278726	2.297482	2.326457	2.336014	2.321047	2.28339	2.22709	2.15527
	$F_{2\cdot0}(n.-n)$	——	2.878904	2.107541	2.945378$_n$	3.251084$_n$	3.372758$_n$	3.414280$_n$	3.409205$_n$	3.37282$_n$	3.31422$_n$	3.23878$_n$
	$F_{2\cdot0}(n-2.-n)$	2.324886$_n$	1.51151$_n$	3.351267	3.677441	3.827111	3.890251	3.901422	3.877911	3.82926	3.76195	3.67998
	$F_{1\cdot1}(n+1.-n+1)$	2.584171$_n$	2.913604$_n$	2.717323$_n$	2.660799$_n$	2.727273$_n$	2.800935$_n$	2.841198$_n$	2.846629$_n$	2.82297$_n$	2.77663$_n$	2.71219$_n$
	$F_{1\cdot1}(n-1.-n+1)$	2.584171	2.454799$_n$	2.320098	3.236866	3.511858	3.632828	3.677530	3.675961	3.64277	3.58693	3.51384
	$F_{1\cdot1}(n+1.-n-1)$	2.584171$_n$	3.154992$_n$	2.583997	3.258957	3.466673	3.550454	3.571125	3.552680	3.50693	3.44139	3.36058
	$F_{1\cdot1}(n-1.-n-1)$	2.584171	2.984357$_n$	4.007854$_n$	4.235981$_n$	4.339837$_n$	4.376075$_n$	4.369477$_n$	4.333339$_n$	4.27524$_n$	4.20058$_n$	4.11275$_n$
	$F_{0\cdot2}(n.-n+2)$	——	2.654194	2.515602	2.429622	2.548932	2.673834	2.746793	2.772863	2.76294	2.72623	2.66888
	$F_{0\cdot2}(n.-n)$	——	2.881187	3.175443$_n$	3.593121$_n$	3.769305$_n$	3.844500$_n$	3.862392$_n$	3.843126$_n$	3.79735$_n$	3.73211$_n$	3.65170$_n$
	$F_{0\cdot2}(n.-n-2)$	——	3.199336	4.028642	4.184785	4.250188	4.262271	4.239041	4.190732	4.12334	4.04134	3.94757
	$F_{0\cdot0}(n+1.-n+1)+\sigma$	2.375282	3.061292	3.108012	3.085352	3.023966	2.93777	2.83432	2.71823	2.59217	2.45846	2.31847
	$F_{0\cdot0}(n-1.-n-1)-\sigma$	2.375282$_n$	——	2.630891	2.784322	2.802117	2.76168	2.68819	2.59329	2.48303	2.36155	2.23132
	$F_{0\cdot0}(n+1.-n-1)+\delta$	2.375282$_n$	3.061292$_n$	3.108012$_n$	3.085352$_n$	3.023966$_n$	2.93777$_n$	2.83432$_n$	2.71823$_n$	2.59217$_n$	2.45846$_n$	2.31847$_n$
	$F_{0\cdot0}(n-1.-n+1)-\delta$	2.375282	——	2.630891$_n$	2.784322$_n$	2.802117$_n$	2.76168$_n$	2.68819$_n$	2.59329$_n$	2.48303$_n$	2.36155$_n$	2.23132$_n$
Factor w	$F_{0\cdot0}(n.-n)$	——	2.17407$_n$	2.80385$_n$	2.81634$_n$	2.77327$_n$	2.69826$_n$	2.60231$_n$	2.49160$_n$	2.36959$_n$	2.23903$_n$	2.10154$_n$
	$F_{1\cdot0}(n+1.-n)$	2.40808	2.89260	2.61060	1.30651$_n$	2.57056$_n$	2.77368$_n$	2.84137$_n$	2.84714$_n$	2.81568$_n$	2.75940$_n$	2.68487$_n$
	$F_{1\cdot0}(n-1.-n)$	2.40808$_n$	2.58559	3.47166	3.67505	3.76194	3.78630	3.77099	3.72820	3.66492	3.58609	3.49478
	$F_{0\cdot1}(n.-n+1)$	——	2.68438$_n$	1.09094	2.84603	3.09732	3.19490	3.22147	3.20585	3.16154	3.09673	3.01623
	$F_{0\cdot1}(n.-n-1)$	——	3.03370$_n$	3.70589$_n$	3.85493$_n$	3.91601$_n$	3.92517$_n$	3.89984$_n$	3.84993$_n$	3.78132$_n$	3.69833$_n$	3.60374$_n$
	$F_{2\cdot0}(n+2.-n)$	2.95358$_n$	3.23873$_n$	3.11084$_n$	3.00363$_n$	2.96993$_n$	2.99436$_n$	3.03311$_n$	3.06053$_n$	3.06783$_n$	3.05500$_n$	3.02391$_n$
	$F_{2\cdot0}(n.-n)$	——	3.51482$_n$	3.50053$_n$	2.44665$_n$	3.59612	3.90886	4.05465	4.12446	4.14800	4.14015	4.10902
	$F_{2\cdot0}(n-2.-n)$	2.95358	2.34335	3.79017$_n$	4.20261$_n$	4.42532$_n$	4.55258$_n$	4.62051$_n$	4.64765$_n$	4.64471$_n$	4.61891$_n$	4.57492$_n$
	$F_{1\cdot1}(n+1.-n+1)$	3.14935	3.57940	3.46185	3.35842	3.36644	3.44584	3.52668	3.58147	3.60643	3.60539	
	$F_{1\cdot1}(n-1.-n+1)$	3.14935$_n$	3.08298	3.14736	[3.48489$_n$]	4.00073$_n$	4.23280$_n$	4.35556$_n$	4.41647$_n$	4.43623$_n$	4.42674$_n$	
	$F_{1\cdot1}(n+1.-n-1)$	3.14935	3.78804	3.55943	3.07546$_n$	3.85262$_n$	4.09729$_n$	4.21567$_n$	4.26985$_n$	4.28308$_n$	4.26782$_n$	
	$F_{1\cdot1}(n-1.-n-1)$	3.14935$_n$	3.54815	4.44151	4.75558	4.93430	5.03585	5.08675	5.10174	5.08965	5.05672	
	$F_{0\cdot2}(n.-n+2)$	——	3.27657$_n$	3.23290$_n$	3.13811$_n$	3.19242$_n$	3.32448$_n$	3.44020$_n$	3.51600$_n$	3.55120$_n$	3.56205$_n$	
	$F_{0\cdot2}(n.-n)$	——	3.52284$_n$	2.65282	3.94249	4.27919	4.45222	4.54409	4.58551	4.59187	4.57252	
	$F_{0\cdot2}(n.-n-2)$	——	3.76082$_n$	4.45251$_n$	4.69857$_n$	4.84101$_n$	4.91959$_n$	4.95455$_n$	4.95782$_n$	4.93673$_n$	4.89677$_n$	

	$F_{0\cdot0}(n+1.-n+1)+\sigma$	2. 99236$_n$	3. 62646$_n$	3. 73619$_n$	3. 77204$_n$	3. 76368$_n$	3. 72546$_n$	3. 66561$_n$	3. 58938$_n$	3. 50001$_n$	3. 40022$_n$	3. 29177$_n$
	$F_{0\cdot0}(n-1.-n-1)-\sigma$	2. 99236	——	3. 25907$_n$	3. 47101$_n$	3. 54183$_n$	3. 54937$_n$	3. 51948$_n$	3. 46444$_n$	3. 39087$_n$	3. 30331$_n$	3. 20462$_n$
	$F_{0\cdot0}(n+1.-n-1)+\delta$	2. 99236	3. 62646	3. 73619	3. 77204	3. 76368	3. 72546	3. 66561	3. 58938	3. 50001	3. 40022	3. 29177
	$F_{0\cdot0}(n-1.-n+1)-\delta$	2. 99236$_n$	——	3. 25907	3. 47101	3. 54183	3. 54937	3. 51948	3. 46444	3. 39087	3. 30331	3. 20462
Factor u^2	$F_{0\cdot0}(n.-n)$	——	2. 2697	2. 7214	2. 8744	2. 9388	2. 9505	2. 9272	2. 8790	2. 8119	2. 7301	2. 6366
	$F_{1\cdot0}(n+1.-n)$	2. 5342$_n$	3. 1189$_n$	3. 1164$_n$	2. 9280$_n$	2. 2494$_n$	2. 6910	3. 0131	3. 1434	3. 1968	3. 2065	
	$F_{1\cdot0}(n-1.-n)$	2. 5342	2. 7389$_n$	3. 4676$_n$	3. 7755$_n$	3. 9531$_n$	4. 0555$_n$	4. 1079$_n$	4. 1245$_n$	4. 1140$_n$	4. 0825$_n$	
	$F_{0\cdot1}(n.-n+1)$	——	2. 9072	2. 8557	1. 9557$_n$	3. 0836$_n$	3. 3610$_n$	3. 4950$_n$	3. 5591$_n$	3. 5794$_n$	3. 5695$_n$	
	$F_{0\cdot1}(n.-n-1)$	——	3. 1908	3. 6927	3. 9492	4. 1032	4. 1917	4. 2348	4. 2448	4. 2292	4. 1938	
	$F_{2\cdot0}(n.-n)$	——		4. 0329								
	$F_{2\cdot0}(n-2.-n)$			3. 8756		4. 6503		4. 9726				
	$F_{1\cdot1}(n-1.-n+1$,		3. 3936$_n$		3. 5361$_n$							
	$F_{1\cdot1}(n+1.-n-1)$		4. 1034$_n$									
	$F_{1\cdot1}(n-1.-n-1)$		[3. 7670]		4. 8998$_n$		5. 3223$_n$					
	$F_{0\cdot2}(n.-n+2)$											
	$F_{0\cdot2}(n.-n)$	——		3. 9171								
	$F_{0\cdot2}(n.-n-2)$	——		4. 5068		5. 0514						
	$F_{0\cdot0}(n-1.-n-1)-\sigma$		——		3. 8301		4. 0023					
	$F_{0\cdot0}(n+1.-n-1)+\delta$		3. 8929$_n$									
	$F_{0\cdot0}(n-1.-n+1)-\delta$		——		3. 8301$_n$							

[1] It is convenient to have this table in seconds of arc also.

Logarithmic. **TABLE IX.**[1] Unit=1″.

	n	0	1	2	3	4	5	6	7	8	9	10
	$G_{0\cdot0}(n.-n)$	1. 634891	1. 439686	1. 826715$_n$	1. 770059$_n$	1. 660318$_n$	1. 525471$_n$	1. 376121$_n$	1. 217493$_n$	1. 05211$_n$	0. 88196$_n$	0. 70814$_n$
	$G_{1\cdot0}(n+1.-n)$	2. 023857$_n$	1. 740753$_n$	1. 917856$_n$	1. 927654$_n$	1. 914277$_n$	1. 86965$_n$	1. 79959$_n$	1. 71025$_n$	1. 60585$_n$	1. 48983$_n$	1. 36459$_n$
	$G_{1\cdot0}(n-1.-n)$	2. 283141$_n$	2. 350404$_n$	2. 436203	2. 622307	2. 654871	2. 623379	2. 555976	2. 465550	2. 35871	2. 23993	2. 11185
	$G_{0\cdot1}(n.-n+1)$	2. 283141	2. 049382	2. 025986	2. 104641	2. 131548	2. 109818	2. 053762	1. 973612	1. 87535	1. 76428	1. 64259
	$G_{0\cdot1}(n.-n-1)$	2. 283141	2. 346611	2. 634099$_n$	2. 763025$_n$	2. 775609$_n$	2. 733913$_n$	2. 660317$_n$	2. 565735$_n$	2. 45592$_n$	2. 33487$_n$	2. 20508$_n$
	$G_{2\cdot0}(n+2.-n)$	2. 251669	2. 054473	0. 542701	1. 673240$_n$	1. 872876$_n$	1. 941516$_n$	1. 953253$_n$	1. 930618$_n$	1. 88358$_n$	1. 81832$_n$	1. 73875$_n$
	$G_{2\cdot0}(n.-n)$	2. 754592	2. 710784	2. 728285	2. 864698	2. 965479	3. 013074	3. 018683	2. 993200	2. 94396	2. 87668	2. 79502
	$G_{2\cdot0}(n-2.-n)$	2. 590848	2. 894131	2. 597213$_n$	3. 136437$_n$	3. 325120$_n$	3. 401835$_n$	3. 419045$_n$	3. 398628$_n$	3. 35168$_n$	3. 28538$_n$	3. 20413$_n$
	$G_{1\cdot1}(n+1.-n+1)$	2. 886452$_n$	2. 671647$_n$	2. 059927$_n$	1. 782544	2. 223601	2. 361811	2. 411416	2. 413615	2. 38430	2. 33246	2. 26325
	$G_{1\cdot1}(n-1.-n+1)$	2. 983112$_n$	3. 001797$_n$	2. 886806$_n$	3. 001111$_n$	3. 124901$_n$	3. 196128$_n$	3. 220094$_n$	3. 208491$_n$	3. 16994$_n$	3. 11102$_n$	3. 03610$_n$
	$G_{1\cdot1}(n+1.-n-1)$	2. 761928$_n$	2. 718191$_n$	2. 972738$_n$	3. 072386$_n$	3. 138813$_n$	3. 163746$_n$	3. 154462$_n$	3. 118772$_n$	3. 06222$_n$	2. 98945$_n$	2. 90362$_n$
	$G_{1\cdot1}(n-1.-n-1)$	3. 062127$_n$	3. 315456$_n$	3. 197526	3. 628655	3. 787099	3. 847462	3. 854044	3. 826068	3. 77341	3. 70266	3. 61772
	$G_{0\cdot2}(n.-n+2)$	2. 912119	2. 693921	2. 188198	1. 29898	1. 84986$_n$	2. 13017$_n$	2. 23936$_n$	2. 27652$_n$	2. 27057$_n$	2. 23551$_n$	2. 17899$_n$
	$G_{0\cdot2}(n.-n)$	2. 937465	2. 957065	3. 055278	3. 200797	3. 307904	3. 360489	3. 370043	3. 347712	3. 30104	3. 23589	3. 15597
	$G_{0\cdot2}(n.-n-2)$	2. 912119	3. 114004	3. 258778$_n$	3. 553495$_n$	3. 669745$_n$	3. 707950$_n$	3. 700505$_n$	3. 662731$_n$	3. 60283$_n$	3. 52641$_n$	3. 43700$_n$
	$G_{0\cdot0}(n+1.-n+1)+\sigma$	2. 404818	1. 992001	1. 951744$_n$	2. 238919$_n$	2. 289146$_n$	2. 26227$_n$	2. 19592$_n$	2. 10522$_n$	1. 99764$_n$	1. 87801$_n$	1. 74907$_n$
	$G_{0\cdot0}(n-1.-n-1)-\sigma$	2. 756164	2. 937464	2. 681654	2. 366191	1. 94049	0. 9836	1. 4304$_n$	1. 60533$_n$	1. 61843$_n$	1. 57046$_n$	1. 48964$_n$
	$G_{0\cdot0}(n+1.-n-1)+\delta$	2. 404818$_n$	1. 992001$_n$	1. 951744	2. 238919	2. 289146	2. 26227	2. 19592	2. 10522	1. 99764	1. 87801	1. 74907
	$G_{0\cdot0}(n-1.-n+1)-\delta$	2. 756164$_n$	2. 937464$_n$	2. 681654$_n$	2. 366191$_n$	1. 94049$_n$	0. 9836$_n$	1. 4304	1. 60533	1. 61843	1. 57046	1. 48964
Factor w	$G_{0\cdot0}(n.-n)$	2. 10705$_n$	2. 04685$_n$	2. 03404	2. 17823	2. 18669	2. 13895	2. 05974	1. 96028	1. 84633	1. 72180	1. 58899
	$G_{1\cdot0}(n+1.-n)$	2. 65255	2. 54290	2. 41806	2. 40791	2. 44888	2. 47354	2. 46930	2. 43899	2. 38700	2. 31799	2. 23511
	$G_{1\cdot0}(n-1.-n)$	2. 84832	3. 00240	2. 25278$_n$	2. 98478$_n$	3. 16887$_n$	3. 23317$_n$	3. 23892$_n$	3. 20883$_n$	3. 15392$_n$	3. 08094$_n$	2. 99390$_n$
	$G_{0\cdot1}(n.-n+1)$	2. 84832$_n$	2. 75275$_n$	2. 56336$_n$	2. 60078$_n$	2. 68023$_n$	2. 72655$_n$	2. 73464$_n$	2. 71161$_n$	2. 66487$_n$	2. 59934$_n$	2. 51902$_n$
	$G_{0\cdot1}(n.-n-1)$	2. 84832$_n$	3. 00496$_n$	2. 69838	3. 14924	3. 29676	3. 34661	3. 34462	3. 30967	3. 25141	3. 17597	3. 08714
	$G_{2\cdot0}(n+2.-n)$	3. 04285$_n$	2. 96317$_n$	2. 49592$_n$	8. 9445	2. 28955	2. 51168	2. 60979	2. 65226	2. 65962	2. 64211	
	$G_{2\cdot0}(n.-n)$	3. 49220$_n$	3. 56162$_n$	3. 39880$_n$	3. 41162$_n$	3. 52203$_n$	3. 62520$_n$	3. 69230$_n$	3. 72425$_n$	3. 72689$_n$	3. 70623$_n$	
	$G_{2\cdot0}(n-2.-n)$	3. 30153$_n$	3. 62263$_n$	3. 17426$_n$	3. 36313	3. 81210	4. 00369	4. 09959	4. 14189	4. 14774	4. 12750	
	$G_{1\cdot1}(n+1.-n+1)$	3. 59196	3. 50310	3. 08664	2. 36611	2. 59736$_n$	2. 93011$_n$	3. 07018$_n$	3. 13789$_n$	3. 16324$_n$	3. 15887$_n$	
	$G_{1\cdot1}(n-1.-n+1)$	3. 66401	3. 76125	3. 57678	3. 57435	3. 69858	3. 82142	3. 90454	3. 94867	3. 96061	3. 94722	
	$G_{1\cdot1}(n+1.-n-1)$	3. 50554	3. 57892	3. 52922	3. 58063	3. 68601	3. 77403	3. 82803	3. 85016	3. 84548	3. 81929	
	$G_{1\cdot1}(n-1.-n-1)$	3. 72579	4. 02457	3. 35713	3. 92118$_n$	4. 28687$_n$	4. 45373$_n$	4. 53669$_n$	4. 57015$_n$	4. 56981$_n$	4. 54488$_n$	
	$G_{0\cdot2}(n.-n+2)$	3. 54909$_n$	3. 45647$_n$	3. 05713$_n$	2. 56977$_n$	2. 06145	2. 69745	2. 90437	3. 00723	3. 05504	3. 06688	
	$G_{0\cdot2}(n.-n)$	3. 62947$_n$	3. 73135$_n$	3. 64281$_n$	3. 72335$_n$	3. 86693$_n$	3. 98197$_n$	4. 05380$_n$	4. 08795$_n$	4. 09194$_n$	4. 07225$_n$	
	$G_{0\cdot2}(n.-n-2)$	3. 54909$_n$	3. 81772$_n$	2. 67578	3. 90075	4. 18183	4. 31858	4. 38499	4. 40764	4. 39955	4. 36878	

	$G_{0\cdot0}(n+1.-n+1)+\sigma$	3. 16381$_n$	3. 15827$_n$	2. 58410$_n$	2. 49979	2. 85224	2. 94559	2. 95660	2. 92500	2. 86611	2. 78862	2. 69707
	$G_{0\cdot0}(n-1.-n-1)-\sigma$	3. 44221$_n$	3. 62942$_n$	3. 44791$_n$	3. 21904$_n$	2. 92221$_n$	2. 47531$_n$	1. 36173	2. 28744	2. 42556	2. 44648	2. 41578
	$G_{0\cdot0}(n+1.-n-1)+\delta$	3. 16381	3. 15827	2. 58410	2. 49979$_n$	2. 85224$_n$	2. 94559$_n$	2. 95660$_n$	2. 92500$_n$	2. 86611$_n$	2. 78862$_n$	2. 69707$_n$
	$G_{0\cdot0}(n-1.-n+1)-\delta$	3. 44221	3. 62942	3. 44791	3. 21904	2. 92221	2. 47531	1. 36173$_n$	2. 28744$_n$	2. 42556$_n$	2. 44648$_n$	2. 41578$_n$
Factor n^2	$G_{0\cdot0}(n.-n)$	2. 2332	2. 3164	1. 5024	1. 9685$_n$	2. 2393$_n$	2. 3276$_n$	2. 3436$_n$	2. 3193$_n$	2. 2674$_n$	2. 1964$_n$	2. 1116$_n$
	$G_{1\cdot0}(n+1.-n)$	2. 9825$_n$	3. 0110$_n$	2. 7515$_n$	2. 5962$_n$	2. 5991$_n$	2. 6724$_n$	2. 7396$_n$	2. 7789$_n$	2. 7896$_n$	2. 7761$_n$	
	$G_{1\cdot0}(n-1.-n)$	3. 1148$_n$	3. 3396$_n$	3. 0680$_n$	2. 1166	3. 1618	3. 4063	3. 5186	3. 5672	3. 5769	3. 5573	
	$G_{0\cdot1}(n.-n+1)$	3. 1148	3. 1438	2. 8873	2. 7768	2. 8368	2. 9416	3. 0219	3. 0667	3. 0800	3. 0680	
	$G_{0\cdot1}(n.-n-1)$	3. 1148	3. 3478	3. 0112	2. 7163$_n$	3. 3194$_n$	3. 5286$_n$	3. 6278$_n$	3. 6697$_n$	3. 6743$_n$	3. 6527$_n$	
	$G_{2\cdot0}(n-2.-n)$			3. 9709								
	$G_{1\cdot1}(n-1.-n-1)$		4. 5912									
	$G_{0\cdot2}(n.-n-2)$	3. 9085										
	$G_{0\cdot0}(n-1.-n-1)-\sigma$		4. 0409									

[1] It is convenient to have this table in seconds of arc also.

Logarithmic. TABLE X.[1] Unit = 1′.

	n	0	1	2	3	4	5	6	7	8	9	10
	$H_{0\cdot0}(n.-n)$	1.634891_n	1.955978_n	2.469664_n	2.360312_n	2.223808_n	2.072607_n	1.912179_n	1.745521_n	1.57402_n	1.39915_n	1.22149_n
	$H_{1\cdot0}(n+1.-n)$	2.283141	2.497539	2.132783	1.289495_n	2.007687_n	2.122538_n	2.124651_n	2.075490_n	1.99603_n	1.89695_n	1.78371_n
	$H_{1\cdot0}(n-1.-n)$	2.023857	2.639706	3.175028	3.237886	3.225052	3.170212	3.088688	2.988817	2.87524	2.75144	2.61949
	$H_{0\cdot1}(n.-n+1)$	2.283141_n	2.454799_n	1.836524	2.464367	2.583580	2.591242	2.546486	2.470917	2.37459	2.26368	2.14175
	$H_{0\cdot1}(n.-n-1)$	2.283141_n	2.810509_n	3.359932_n	3.390883_n	3.360857_n	3.295300_n	3.206462_n	3.101284_n	2.98368_n	2.85672_n	2.72223_n
	$H_{2\cdot0}(n+2.-n)$	2.590848_n	2.707952_n	2.376382_n	2.175386_n	2.130812_n	2.164244_n	2.196499_n	2.203577_n	2.18377_n	2.14150_n	2.08066_n
	$H_{2\cdot0}(n.-n)$	2.754592_n	3.199457_n	2.990444_n	2.248049	3.086389	3.284047	3.355296	3.365683	3.33861	3.28624	3.21513
	$H_{2\cdot0}(n-2.-n)$	2.251669_n	3.013336_n	3.555780_n	3.787322_n	3.900910_n	3.945468_n	3.945418_n	3.914428_n	3.86046_n	3.78915_n	3.70406_n
	$H_{1\cdot1}(n+1.-n+1)$	3.062127	3.172492	2.868916	2.667577	2.657025	2.721975	2.772070	2.788666	2.77427	2.73517	2.67639
	$H_{1\cdot1}(n-1.-n+1)$	2.761928	3.197028	2.649500	3.173785_n	3.512606_n	3.644848_n	3.691986_n	3.690517_n	3.65672_n	3.60008_n	3.52617_n
	$H_{1\cdot1}(n+1.-n-1)$	2.983112	3.357877	2.854083	2.930833_n	3.328509_n	3.464872_n	3.509819_n	3.505195_n	3.46829_n	3.40891_n	3.33258_n
	$H_{1\cdot1}(n-1.-n-1)$	2.886452	3.578026	4.131226	4.308446	4.390309	4.414545	4.400460	4.359230	4.29747	4.22004	4.13004
	$H_{0\cdot2}(n.-n+2)$	2.912119_n	3.039643_n	2.771808_n	2.578026_n	2.603308_n	2.693928_n	2.756447_n	2.779086_n	2.76791_n	2.73067_n	2.67316_n
	$H_{0\cdot2}(n.-n)$	2.937465_n	3.283318_n	2.935144	3.560728	3.763936	3.846773	3.867352	3.849057	3.80356	3.73829	3.65773
	$H_{0\cdot2}(n.-n-2)$	2.912119_n	3.532772_n	4.094393_n	4.226022_n	4.279805_n	4.285230_n	4.257724_n	4.206449_n	4.13689_n	4.05324_n	3.95818_n
	$H_{0\cdot0}(n+1.-n+1)+\sigma$	2.756164_n	3.213148_n	3.209606_n	3.161233_n	3.084354_n	2.98785_n	2.87705_n	2.75548_n	2.62518_n	2.48808_n	2.34533_n
	$H_{,0}(n-1.-n-1)-\sigma$	2.404818_n	2.937464_n	3.021354_n	3.018627_n	2.969687_n	2.89202_n	2.79476_n	2.68338_n	2.56103_n	2.43031_n	2.29280_n
	$H_{\cdot0}(n+1.-n-1)+\delta$	2.756164	3.213148	3.209606	3.161233	3.084354	2.98785	2.87705	2.75548	2.62518	2.48808	2.34533
	$H_{0\,0}(n-1.-n+1)-\delta$	2.404818	2.937464	3.021354	3.018627	2.969687	2.89202	2.79476	2.68338	2.56103	2.43031	2.29280
Factor w	$H_{0\cdot0}(n.-n)$	2.10705	2.49201	2.86960	2.85902	2.80438	2.72253	2.62211	2.50826	2.38394	2.25162	2.11274
	$H_{1\cdot0}(n+1.-n)$	2.84832_n	3.11926_n	2.94344_n	2.51758_n	2.05090	2.60473	2.74368	2.77970	2.76467	2.71844	2.65095
	$H_{1\cdot0}(n-1.-n)$	2.65255_n	3.23704_n	3.62990_n	3.76534_n	3.82325_n	3.83204_n	3.80720_n	3.75803_n	3.69020_n	3.60798_n	3.51407_n
	$H_{0\cdot1}(n.-n+1)$	2.84832	3.08298	2.54339	2.72959_n	3.07532_n	3.19194_n	3.22466_n	3.21114_n	3.16786_n	3.10328_n	3.02273_n
	$H_{0\cdot1}(n.-n-1)$	2.84832	3.38959	3.79758	3.91047	3.95464	3.95433	3.92304	3.86908	3.79756	3.71242	3.61614
	$H_{2\cdot0}(n+2.-n)$	3.30153	3.47658	3.26944	3.06962	2.93550	2.89904	2.92186	2.95450	2.97290	2.97130	
	$H_{2\cdot0}(n.-n)$	3.49221	3.91454	3.87105	3.58162	3.01092_n	3.75726_n	3.97365_n	4.07117_n	4.10899_n	4.10968_n	
	$H_{2\cdot0}(n-2.-n)$	3.04285	3.70335	4.09904	4.35862	4.52456	4.62359	4.67529	4.69189	4.68170	4.65065	
	$H_{1\cdot1}(n+1.-n+1)$	3.72579_n	3.89596_n	3.68472_n	3.47854_n	3.37752_n	3.40062_n	3.46926_n	3.52757_n	3.55905_n	3.56441_n	
	$H_{1\cdot1}(n-1.-n+1)$	3.50554_n	3.91340_n	3.76928_n	2.83064	3.96157	4.23549	4.36784	4.43113	4.45108	4.44105	
	$H_{1\cdot1}(n+1.-n-1)$	3.66401_n	4.05317_n	3.93197_n	3.49986_n	[3.52156]	3.96647	4.13685	4.21423	4.24034	4.23321	
	$H_{1\cdot1}(n-1.-n-1)$	3.59196_n	4.22680_n	4.63658_n	4.86351_n	5.00543_n	5.08771_n	5.12705_n	5.13448_n	5.11710_n	5.08030_n	
	$H_{0\cdot2}(n.-n+2)$	3.54909	3.73089	3.53403	3.34318	3.29267	3.36481	3.45811	3.52608	3.56130	3.56791	
	$H_{0\cdot2}(n.-n)$	3.62947	3.99260	3.57911	3.80556_n	4.25522_n	4.44991_n	4.54840_n	4.59212_n	4.59923_n	4.57993_n	
	$H_{0\cdot2}(n.-n-2)$	3.54909	4.15411	4.57196	4.76826	4.88795	4.95411	4.98146	4.97967	4.95504	4.91236	

	$H_{0\cdot0}(n+1,-n+1)+\sigma$	3. 44221	3. 85011	3. 88349	3. 87921	3. 84673	3. 79264	3. 72171	3. 63735	3. 54179	3. 43716	3. 32184
	$H_{0\cdot0}(n-1,-n-1)-\sigma$	3. 16381	3. 62946	3. 71816	3. 74819	3. 73868	3. 70094	3. 64215	3. 56714	3. 47906	3. 38042	3. 27308
	$H_{0\cdot0}(n+1,-n-1)+\delta$	3. 44221$_n$	3. 85011$_n$	3. 88349$_n$	3. 87921$_n$	3. 84673$_n$	3. 79264$_n$	3. 72171$_n$	3. 63735$_n$	3. 54179$_n$	3. 43716$_n$	3. 32481$_n$
	$H_{0\cdot0}(n-1,-n+1)-\delta$	3. 16381$_n$	3. 62946$_n$	3. 71816$_n$	3. 74819$_n$	3. 73868$_n$	3. 70094$_n$	3. 64215$_n$	3. 56714$_n$	3. 47906$_n$	3. 38042$_n$	3. 27308$_n$
Factor u^2	$H_{0\cdot0}(n,-n)$	2. 2332$_n$	2. 6582$_n$	2. 8656$_n$	2. 9568$_n$	2. 9932$_n$	2. 9899$_n$	2. 9576$_n$	2. 9034$_n$	2. 8321$_n$	2. 7473$_n$	2. 6516$_n$
	$H_{1\cdot0}(n+1,-n)$	3. 1148	3. 4217	3. 3589	3. 2904	2. 8749	1. 6284$_n$	2. 8458$_n$	3. 0173$_n$	3. 1331$_n$	3. 1592$_n$	
	$H_{1\cdot0}(n-1,-n)$	2. 9825	3. 5079	3. 7460	3. 9255	4. 0482	4. 1224	4. 1583	4. 1644	4. 1465	4. 1102	
	$H_{0\cdot1}(n,-n+1)$	3. 1148$_n$	3. 3936$_n$	3. 2375$_n$	2. 6792$_n$	2. 9683	3. 3400	3. 4938	3. 5640	3. 5866	3. 5773	
	$H_{0\cdot1}(n,-n-1)$	3. 1148$_n$	3. 6331$_n$	3. 8807$_n$	4. 0547$_n$	4. 1709$_n$	4. 2394$_n$	4. 2707$_n$	4. 2731$_n$	4. 2524$_n$	4. 2133$_n$	
	$H_{2\cdot0}(n,-n)$			4. 3637$_n$								
	$H_{2\cdot0}(n-2,-n)$							[5. 0467$_n$]				
	$H_{1\cdot1}(n-1,-n+1)$				4. 1173							
	$H_{1\cdot1}(n+1,-n-1)$		4. 4525									
	$H_{1\cdot1}(n-1,-n-1)$						5. 3997					
	$H_{0\cdot2}(n,-n+2)$	3. 9085$_n$										
	$H_{0\cdot2}(n,-n)$			4. 3449$_n$								
	$H_{0\cdot2}(n,-n-2)$					5. 1334$_n$						
	$H_{0\cdot0}(n-1,-n-1)-\sigma$						4. 1848$_n$					
	$H_{0\cdot0}(n+1,-n-1)+\delta$		4. 2095									
	$H_{0\cdot0}(n-1,-n+1)-\delta$				4. 1683							

It is convenient to have this table in seconds of arc also

TABLE XI.

Unit = 1″

n	0	1	2	3	4	5	6	7	8
${}_{3\cdot0}(n+1.-n)$	$H=+1656$								
${}_{3\cdot0}(n-1.-n)$	$G=-1656$		$F=-1955$		$H=-5097$				
${}_{3\cdot0}(n-3.-n)$					$G=+6177$		$F=-35276$		$H=+45199$
${}_{2\cdot1}(n.-n+1)$		$H=[-9270]$							
${}_{2\cdot1}(n-2.-n+1)$		$G=+4207$		$F=-3730$		$H=+22898$			
${}_{2\cdot1}(n.-n-1)$		$F=+6733$		$H=+5323$					
${}_{2\cdot1}(n-2.-n-1)$				$G=-13730$		$F=+142854$		$H=-200024$	
${}_{1\cdot2}(n-1.-n+2)$	$F=+1634$		$H=+5239$						
${}_{1\cdot2}(n+1.-n)$	$H=+6391$								
${}_{1\cdot2}(n-1.-n)$	$G=-6391$		$F=+3445$		$H=-52183$				
${}_{1\cdot2}(n+1.-n-2)$	$F=-1634$		$H=+2712$						
${}_{1\cdot2}(n-1.-n-2)$			$G=+4409$		$F=-190467$		$H=+294332$		
${}_{0\cdot3}(n.-n+1)$		$H=-7839$							
${}_{0\cdot3}(n.-n-1)$		$F=+3479$		$H=+27512$					
${}_{0\cdot3}(n.-n-3)$		$G=+6634$		$F=+83314$		$H=-144023$			
${}_{1\cdot0}(n.-n+1)+\sigma$		$H=+11006$							
${}_{1\cdot0}(n.-n-1)-\sigma$		$F=-1732$		$H=+4022$					
${}_{1\cdot0}(n-2.-n-1)-\sigma$				$G=-3616$		$F=-7798$		$H=+9526$	
${}_{1\cdot0}(n.-n-1)+\partial$		$F=+6534$		$H=-17562$					
${}_{1\cdot0}(n.-n+1)-\partial$		$H=-6391$							
${}_{1\cdot0}(n-2.-n+1)-\partial$		$G=+6391$		$F=+4175$		$H=-10846$			
${}_{0\cdot1}(n+1.-n)+\sigma$	$H=-3012$								
${}_{0\cdot1}(n-1.-n)-\sigma$	$G=+3012$		$F=+159$		$H=+1108$				
${}_{0\cdot1}(n-1.-n-2)-\sigma$			$G=+7090$		$F=+11564$		$H=-15308$		
${}_{0\cdot1}(n+1.-n)+\partial$	$H=+3012$								
${}_{0\cdot1}(n+1.-n-2)+\partial$	$F=-1948$		$H=+21192$						
${}_{0\cdot1}(n-1.-n+2)-\partial$	$F=+1948$		$H=+5738$						
${}_{0\cdot1}(n-1.-n)-\partial$	$G=-3012$		$F=-3579$		$H=+13813$				
${}_{3\cdot0}(n+1.-n)$	$H=-11862$								
${}_{2\cdot0}(n-1.-n)$	$G=+11862$		$F=+14862$		$H=-4328$				
${}_{2\cdot0}(n-3.-n)$					$G=-16310$		$F=+189348$		$H=-304998$
${}_{2\cdot1}(n.-n+1)$		$H=[+61512]$							
${}_{2\cdot1}(n-2.-n+1)$		$G=-28940$		$F=[+1693]$		$H=-84425$			
${}_{2\cdot1}(n.-n-1)$		$F=-33547$		$H=+55061$					
${}_{2\cdot1}(n-2.-n-1)$				$G=+9080$		$F=-673242$		$H=+1218446$	
${}_{1\cdot2}(n-1.-n+2)$	$F=-7082$		$H=-35125$						
${}_{1\cdot2}(n+1.-n)$	$H=-39356$								
${}_{1\cdot2}(n-1.-n)$	$G=+39356$		$F=+15177$		$H=[+143461]$				
${}_{1\cdot2}(n+1.-n-2)$	$F=+7082$		$H=-60586$						
${}_{1\cdot2}(n-1.-n-2)$			$G=+41693$		$F=+772593$		$H=-1600036$		

Factor w	$_{0\cdot3}(n.-n+1)$		$H=+47423$							
	$_{0\cdot3}(n.-n-1)$		$F=+17883$		$H=-44330$					
	$_{0\cdot3}(n.-n-3)$		$G=-36904$		$F=-283500$		$H=+688658$			
	$_{1\cdot0}(n.-n+1)+\sigma$		$H=-60629$							
	$_{1\cdot0}(n.-n-1)-\sigma$		$F=+\ 8521$		$H=-29208$					
	$_{1\cdot0}(n-2.-n-1)-\sigma$				$G=+27235$		$F=+50227$		$H=-75678$	
	$_{1\cdot0}(n.-n-1)+\delta$		$F=-28325$		$H=+101338$					
	$_{1\cdot0}(n.-n+1)-\delta$		$H=+39356$							
	$_{1\cdot0}(n-2.-n+1)-\delta$		$G=-39356$		$F=-22400$		$H=+74714$			
	$_{0\cdot1}(n+1.-n)+\sigma$	$H=+19091$								
	$_{0\cdot1}(n-1.-n)-\sigma$	$G=-19091$		$F=-\ 2967$		$H=-\ 282$				
	$_{0\cdot1}(n-1.-n-2)-\sigma$			$G=-45771$		$F=+66719$		$H=+111481$		
	$_{0\cdot1}(n+1.-n)+\delta$	$H=-19091$								
	$_{0\cdot1}(n+1.-n-2)+\delta$	$F=+\ 9287$		$H=-111790$						
	$_{0\cdot1}(n-1.-n+2)-\delta$	$F=-\ 9287$		$H=-36971$						
	$_{0\cdot1}(n-1.-n)-\delta$	$G=+19091$		$F=+17494$		$H=-87378$				
Factor w^2	$_{1\cdot0}(n.-n+1)+\sigma$		$H=+179581$							
	$_{1\cdot0}(n.-n-1)+\delta$		$F=+64794$							
	$_{1\cdot0}(n.-n+1)-\delta$		$H=[-127808]$							
	$_{1\cdot0}(n-2.-n+1)-\delta$		$G=[+127808]$		$F=+58926$					
	$_{0\cdot1}(n+1.-n)+\sigma$	$H=-62910$								
	$_{0\cdot1}(n-1.-n)-\sigma$	$G=[+62910]$								
	$_{0\cdot1}(n+1.-n)+\delta$	$H=+62910$								
	$_{0\cdot1}(n-1.-n+2)-\delta$			$H=+123344$						
	$_{0\cdot1}(n-1.-n)-\delta$	$G=-62910$								

TABLE XII.

Unit=1″.

	n	0	1	2	3	4	5
	$F_{1\cdot0}(n+1.-n+1)+\pi'$	− 79. 1	− 191. 9	− 142. 5	− 101. 4	− 70. 4	− 48. 1
	$F_{1\cdot0}(n-1.-n+1)+\pi'$	+ 79. 1	+ 191. 9	+ 142. 5	+ 101. 4	+ 70. 4	+ 48. 1
	$F_{1\cdot0}(n+1.-n-1)-\pi'$	+ 79. 1	+ 191. 9	+ 142. 5	+ 101. 4	+ 70. 4	+ 48. 1
	$F_{1\cdot0}(n-1.-n-1)-\pi'$	− 79. 1	− 191. 9	− 142. 5	− 101. 4	− 70. 4	− 48. 1
Factor ν	$F_{1\cdot0}(n+1.-n+1)+\pi'$	+ 327	+ 705	+ 605	+ 493	+ 387	+ 295
	$F_{1\cdot0}(n-1.-n+1)+\pi'$	− 327	− 705	− 605	− 493	− 387	− 295
	$F_{1\cdot0}(n+1.-n-1)-\pi'$	− 327	− 705	− 605	− 493	− 387	− 295
	$F_{1\cdot0}(n-1.-n-1)-\pi'$	+ 327	+ 705	+ 605	+ 493	+ 387	+ 295
Factor ν^2	$F_{1\cdot0}(n-1.-n+1)+\pi'$		+1303				
	$F_{1\cdot0}(n-1.-n-1)-\pi'$				−1127		

TABLE XIII.

Unit=1″.

	0	1	2	3	4	5
$2G_{0\cdot0}(n.-n+1)+\pi'$	+ 79. 10	+ 191. 93	+ 142. 48	+ 101. 43	+ 70. 45	+ 48. 14
$2G_{0\cdot0}(n.-n-1)-\pi'$	− 79. 10	− 191. 93	− 142. 48	− 101. 43	− 70. 45	− 48. 14
$2G_{1\cdot0}(n+1.-n+1)+\pi'$	− 451. 7	− 674. 0	− 409. 2	− 232. 4	− 122. 4	− 57. 8
$2G_{1\cdot0}(n-1.-n+1)+\pi'$	− 530. 8	− 1441. 7	−1406. 6	−1246. 7	−1038. 3	− 828. 0
$2G_{1\cdot0}(n+1.-n-1)-\pi'$	+ 372. 6	+ 482. 0	+ 266. 7	+ 130. 9	+ 52. 0	+ 9. 6
$2G_{1\cdot0}(n-1.-n-1)-\pi'$	+ 609. 9	+ 1633. 6	+1549. 1	+1348. 1	+1108. 7	+ 876. 1
$2G_{0\cdot1}(n.-n+2)+\pi'$	+ 649. 5	+ 1057. 8	+ 622. 9	+ 333. 8	+ 157. 6	+ 57. 8
$2G_{0\cdot1}(n.-n)+\pi'$	+ 333. 1	+ 1057. 8	+1192. 9	+1145. 3	+1003. 0	+ 828. 0
$2G_{0\cdot1}(n.-n)-\pi'$	− 333. 1	− 290. 1	− 53. 0	+ 71. 9	+ 121. 2	+ 134. 8
$2G_{0\cdot1}(n.-n-2)-\pi'$	− 649. 5	− 1825. 5	−1762. 8	−1551. 0	−1284. 8	−1020. 6
$G_{2\cdot0}(n.-n+1)+\pi'$						
$G_{2\cdot0}(n-2.-n+1)+\pi'$		+ 2656				
$G_{1\cdot1}(n-1.-n+2)+\pi'$						
$G_{1\cdot1}(n+1.-n)+\pi'$						
$G_{1\cdot1}(n-1.-n)+\pi'$	−1506					
$G_{1\cdot1}(n+1.-n)-\pi'$						
$G_{1\cdot1}(n-1.-n)-\pi'$	+1506					
$G_{0\cdot2}(n.-n+1)+\pi'$						
$G_{0\cdot2}(n.-n+1)-\pi'$						
$G_{0\cdot0}(n-1.-n)-\delta+\pi'$	+ 266					
$G_{0\cdot0}(n+1.-n)+\delta+\pi'$						

	$G_{0\cdot0}(n-1.-n+2)-\delta+\pi'$						
	$G_{0\cdot0}(n+1.-n)+\sigma-\pi'$						
	$G_{0\cdot0}(n-1.-n)-\delta-\pi'$	+ 266					
Factor μ	$2G_{0\cdot0}(n.-n+1)+\pi'$	− 327.5	− 705.2	− 605.3	− 493.0	− 386.9	− 295.2
	$2G_{0\cdot0}(n.-n-1)-\pi'$	+ 327.5	+ 705.2	+ 605.3	+ 493.0	+ 386.9	+ 295.2
	$2G_{1\cdot0}(n+1.-n+1)+\pi'$	+2277	+ 3555	+2501	+1656	+1030	+ 594
	$2G_{1\cdot0}(n-1.-n+1)+\pi'$	+2604	+ 6376	+6739	+6586	+6059	+5318
	$2G_{1\cdot0}(n+1.-n-1)-\pi'$	−1949	− 2850	−1897	−1163	− 643	− 299
	$2G_{1\cdot0}(n-1.-n-1)-\pi'$	−2932	− 7081	−7344	−7079	−6446	−5613
	$2G_{0\cdot1}(n.-n+2)+\pi'$	−3096	− 4966	−3410	−2149	−1223	− 594
	$2G_{0\cdot1}(n.-n)+\pi'$	−1786	− 4966	−5831	−6093	−5866	−5318
	$2G_{0\cdot1}(n.-n)-\pi'$	+1786	+ 2145	+ 989	+ 177	− 325	− 587
	$2G_{0\cdot1}(n.-n-2)-\pi'$	+3096	+ 7786	+8252	+8065	+7413	+6499
	$G_{2\cdot0}(n.-n+1)+\pi'$						
	$G_{2\cdot0}(n-2.-n+1)+\pi'$		−14804				
	$G_{1\cdot1}(n-1.-n+2)+\pi'$						
	$G_{1\cdot1}(n+1.-n)+\pi'$						
	$G_{1\cdot1}(n-1.-n)+\pi'$	+9545					
	$G_{1\cdot1}(n+1.-n)-\pi'$						
	$G_{1\cdot1}(n-1.-n)-\pi'$	−9545					
	$G_{0\cdot2}(n.-n+1)+\pi'$						
	$G_{0\cdot2}(n.-n+1)-\pi'$						
	$G_{0\cdot0}(n-1.-n)-\sigma+\pi'$	−1640					
	$G_{0\cdot0}(n+1.-n)+\delta+\pi'$						
	$G_{0\cdot0}(n-1.-n+2)-\delta+\pi'$						
	$G_{0\cdot0}(n+1.-n)+\sigma-\pi'$						
	$G_{0\cdot0}(n-1.-n)-\delta-\pi'$	−1640					
Factor μ²	$2G_{1\cdot0}(n-1.-n-1)-\pi'$		+12292				
	$2G_{0\cdot1}(n.-n-2)-\pi'$	−7474					

TABLE XIV.

Unit=1″.

	n	0	1	2	3	4	5
	$2H_{0\cdot0}(n.-n+1)+\pi'$	− 79.10	− 191.93	− 142.48	− 101.43	− 70.45	− 48.14
	$2H_{0\cdot0}(n.-n-1)-\pi'$	+ 79.10	+ 191.93	+ 142.48	+ 101.43	+ 70.45	+ 48.14
	$2H_{1\cdot0}(n+1.-n+1)+\pi'$	+ 609.9	+ 1057.8	+ 694.2	+ 435.2	+ 263.3	+ 154.0
	$2H_{1\cdot0}(n-1.-n+1)+\pi'$	+ 372.6	+ 1057.8	+ 1121.6	+ 1043.8	+ 897.4	+ 731.7
	$2H_{1\cdot0}(n+1.-n-1)-\pi'$	− 530.8	− 865.9	− 551.7	− 333.8	− 192.9	− 105.9
	$2H_{1\cdot0}(n-1.-n-1)-\pi'$	− 451.7	− 1249.8	− 1264.1	− 1145.2	− 967.8	− 779.8
	$2H_{0\cdot1}(n.-n+2)+\pi'$	− 649.4	− 1057.8	− 623.0	− 338.2	− 157.7	− 57.8
	$2H_{0\cdot1}(n.-n)+\pi'$	− 333.0	− 1057.8	− 1192.9	− 1145.2	− 1003.0	− 828.0
	$2H_{0\cdot1}(n.-n)-\pi'$	+ 333.0	+ 290.2	+ 53.0	− 71.9	− 124.2	− 134.8
	$2H_{0\cdot1}(n.-n-2)-\pi'$	+ 649.4	+ 1825.6	+ 1762.8	+ 1551.0	+ 1284.8	+ 1020.6
	$H_{2\cdot0}(n.-n+1)+\pi'$		− 3292				
	$H_{2\cdot0}(n-2.-n+1)+\pi'$						
	$H_{1\cdot1}(n-1.-n+2)+\pi'$			+ 2871			
	$H_{1\cdot1}(n+1.-n)+\pi'$	+1506					
	$H_{1\cdot1}(n-1.-n)+\pi'$						
	$H_{1\cdot1}(n+1.-n)-\pi'$	−1506					
	$H_{1\cdot1}(n-1.-n)-\pi'$						
	$H_{0\cdot2}(n.-n+1)+\pi'$		− 3292				
	$H_{0\cdot2}(n.-n+1)-\pi'$		+ 828				
	$H_{0\cdot0}(n-1.-n)-\sigma+\pi'$						
	$H_{0\cdot0}(n+1.-n)+\delta+\pi'$	+ 266					
	$H_{0\cdot0}(n-1.-n+2)-\delta+\pi'$			+ 437			
	$H_{0\cdot0}(n+1.-n)+\sigma-\pi'$	+ 266					
	$H_{0\cdot0}(n-1.-n)-\delta-\pi'$						
Factor w	$2H_{0\cdot0}(n.-n+1)+\pi'$	+ 327.5	+ 705.2	+ 605.3	+ 493.0	+ 386.9	+ 295.2
	$2H_{0\cdot0}(n.-n-1)-\pi'$	− 327.5	− 705.2	− 605.3	− 493.0	− 386.9	− 295.2
	$2H_{1\cdot0}(n+1.-n+1)+\pi'$	−2932	− 4966	− 3713	− 2642	− 1803	− 1185
	$2H_{1\cdot0}(n-1.-n+1)+\pi'$	−1949	− 4966	− 5529	− 5600	− 5285	− 4728
	$2H_{1\cdot0}(n+1.-n-1)-\pi'$	+2604	+ 4261	+ 3108	+ 2149	+ 1416	+ 889
	$2H_{1\cdot0}(n-1.-n-1)-\pi'$	+2277	+ 5671	+ 6134	+ 6093	+ 5672	+ 5023
	$2H_{0\cdot1}(n.-n+2)+\pi'$	+3096	+ 4966	+ 3410	+ 2149	+ 1223	+ 594
	$2H_{0\cdot1}(n.-n)+\pi'$	+1786	+ 4966	+ 5831	+ 6093	+ 5866	+ 5318
	$2H_{0\cdot1}(n.-n)-\pi'$	−1786	− 2145	− 989	− 177	+ 325	+ 587
	$2H_{0\cdot1}(n.-n-2)-\pi'$	−3096	− 7786	− 8252	− 8065	− 7413	− 6499
	$H_{2\cdot0}(n.-n+1)+\pi'$		+20030				
	$H_{2\cdot0}(n-2.-n+1)+\pi'$						
	$H_{1\cdot1}(n-1.-n+2)+\pi'$			−18486			
	$H_{1\cdot1}(n+1.-n)+\pi'$	−9545					

	$H_{1\cdot1}(n-1.-n)+\pi'$							
	$H_{1\cdot1}(n+1.-n)-\pi'$	+9545						
	$H_{1\cdot1}(n-1.-n)-\pi'$							
	$H_{0\cdot2}(n.-n+1)+\pi'$		+20030					
	$H_{0\cdot2}(n.-n+1)-\pi'$		− 5931					
	$H_{0\cdot0}(n-1.-n)-\sigma+\pi'$							
	$H_{0\cdot0}(n+1.-n)+\delta+\pi'$	−1640						
	$H_{0\cdot0}(n-1.-n+2)-\delta+\pi'$			− 2927				
	$H_{0\cdot0}(n+1.-n)+\sigma-\pi'$	−1640						
	$H_{0\cdot0}(n-1.-n)-\delta-\pi'$							
Factor u^2	$2H_{0\cdot0}(n.-n+1)+\pi'$		− 1302					
	$2H_{0\cdot0}(n.-n-1)-\pi'$				+ 1128			
	$2H_{1\cdot0}(n+1.-n+1)+\pi'$							
	$2H_{1\cdot0}(n-1.-n+1)+\pi'$				+14730			
	$2H_{1\cdot0}(n+1.-n-1)-\pi'$		−10988					
	$2H_{1\cdot0}(n-1.-n-1)-\pi'$							−15302
	$2H_{0\cdot1}(n.-n+2)+\pi'$	−7474						
	$2H_{0\cdot1}(n.-n)+\pi'$			−14432				
	$2H_{0\cdot1}(n.-n)-\pi'$			+ 4532				
	$2H_{0\cdot1}(n.-n-2)-\pi'$					+20360		

TABLE XV.

Unit=1″

	n	0	1	2	3	4	5	6	7
	$S_{0\cdot0}(n.-n)$	——	+ 15.71	+ 90.50	+ 72.04	+ 53.29	+ 37.93	+ 26.37	+ 18.04
	$S_{1\cdot0}(n+1.-n)$	——	− 19.7	+ 134.2	+ 140.7	+ 129.7	+ 110.8	+ 89.9	
	$S_{1\cdot0}(n-1.-n)$	——	− 66.8	− 408.6	− 507.7	− 509.8	− 458.2	− 384.7	− 308.4
	$S_{0\cdot1}(n.-n+1)$	——	+ 43.2	− 43.7	− 104.6	− 129.7	− 129.7	− 116.3	
	$S_{0\cdot1}(n.-n-1)$	——	+ 106.1	+ 680.3	+ 759.8	+ 723.0	+ 628.9	+ 516.5	
	$S_{2\cdot0}(n+2.-n)$								
	$S_{2\cdot0}(n.-n)$			− 777					
	$S_{2\cdot0}(n-2.-n)$			+ 838				+ 2678	
	$S_{1\cdot1}(n+1.-n+1)$								
	$S_{1\cdot1}(n-1.-n+1)$				+ 560				
	$S_{1\cdot1}(n+1.-n-1)$		− 116						
	$S_{1\cdot1}(n-1.-n-1)$		− 540				− 7982		
	$S_{0\cdot2}(n.-n+2)$	——							
	$S_{0\cdot2}(n.-n)$			− 499					
	$S_{0\cdot2}(n.-n-2)$	——				+ 5930			
	$S_{0\cdot0}(n+1.-n+1+\sigma$		——				+ 192.6		
	$S_{0\cdot0}(n-1.-n-1)-\sigma$		− 383.9						
	$S_{0\cdot0}(n+1.-n-1)+\delta$								
	$S_{0\cdot0}(n-1.-n+1)-\delta$				− 202.9				
Factor w'	$S_{0\cdot0}(n.-n)$	——	− 49.8	− 212.2	− 218.4	− 197.8	− 166.4	− 133.4	− 103.4
	$S_{1\cdot0}(n+1.-n)$	——	+ 87	− 216	− 343	− 417	− 439	− 422	
	$S_{1\cdot0}(n-1.-n)$	——	+ 236	+1057	+1622	+ 1956	+ 2057	+ 1980	+1791
	$S_{0\cdot1}(n.-n+1)$	——	− 161	+ 4	+ 234	+ 417	+ 522	+ 555	
	$S_{0\cdot1}(n.-n-1)$	——	− 360	−1693	−2387	− 2747	− 2806	− 2647	
	$S_{2\cdot0}(n+2.-n)$								
	$S_{2\cdot0}(n.-n)$			+1194					
	$S_{2\cdot0}(n-2.-n)$			−2573				−14096	
	$S_{1\cdot1}(n+1.-n+1)$								
	$S_{1\cdot1}(n-1.-n+1)$				− 908				
	$S_{1\cdot1}(n+1.-n-1)$		+ 651						
	$S_{1\cdot1}(n-1.-n-1)$		+2092				+36591		
	$S_{0\cdot2}(n.-n+2)$	——							
	$S_{0\cdot2}(n.-n)$			+ 150					
	$S_{0\cdot2}(n.-n-2)$	——				−23115			

	$S_{0\cdot0}(n+1.-n+1)+\sigma$ $S_{0\cdot0}(n-1.-n-1)-\sigma$ $S_{0\cdot0}(n+1.-n-1)+\delta$ $S_{0\cdot0}(n-1.-n+1)-\delta$		$\overline{}$ +1410		+ 986		− 1181		
Factor v^2	$S_{0\cdot0}(n.-n)$	——	+ 62	+ 175	+ 250	+ 289	+ 297	+ 282	
	$S_{1\cdot0}(n+1.-n)$	——	− 176	− 63	+ 155	+ 405	+ 617	+ 760	
	$S_{1\cdot0}(n-1.-n)$	——	− 362	−1117	−2092	− 3070	− 3844	− 4314	
	$S_{0\cdot1}(n.-n+1)$	——	+ 269	+ 239	− 30	− 404	− 765	− 1042	
	$S_{0\cdot1}(n.-n-1)$	——	+ 517	+1643	+2966	+ 4227	+ 5183	+ 5724	

INTEGRATION OF THE DIFFERENTIAL EQUATION FOR W.

With the exception of Tables LVI and LVII all the following tables are concerned with the integration of functions whose coefficients can be derived, more or less directly, from the preceding tables. The terms of first order in the mass, before and after integration, are of the type

$$\Sigma C_{p\cdot q}(n+r.-n+s)\eta^p\eta'^q j^{2t}\begin{Bmatrix}\sin\\ \cos\end{Bmatrix}\begin{cases}A\\ A+\varepsilon-\psi\\ A-\varepsilon+\psi\end{cases}$$

where $$C_{p\cdot q}=C_{0\cdot p\cdot q}+C_{1\cdot p\cdot q}\cdot w+C_{2\cdot p\cdot q}\cdot w^2+\cdots (\text{see } Z\ 25)$$

and $$A=[n+r-\tfrac{1}{2}(n-s)]\varepsilon+(n-s)\theta+i\ \Pi+i'\ \Pi'$$

In the argument A the factor n is always a positive integer; the factors r, s, i, and i' are positive and negative integers. Evidently, the factor of ε is $\pm\frac{k}{2}$ where k is any positive integer, and the arguments in a series are $\Sigma_n\Sigma_r\Sigma_s A$. Within the extent of Bohlin's tables all of the coefficients can be written in symbolic form from B 188, XVII, XVIII. In the notation for the coefficients the particular values of r and s are given, and there remains to be found only the positive value of n, if there is one, for each multiple of $\frac{\varepsilon}{2}$.

The following tables present, in skeleton form, any series of the given type. There are properly two tables, one for perturbations in the plane of the orbit, and the other for perturbations perpendicular to the same. The headings Δ and Σ are defined by

$$\Delta=\Pi-\Pi'$$
$$\Sigma=\Pi+\Pi'$$

Considering first the tables referring to the plane of the orbit, omitting for the moment the arguments bearing the subscripts $\pm\delta$ or $\pm\sigma$, the argument A for any term is read from a main heading $\pm\frac{k\varepsilon}{2}$ and the first two columns under this heading. The tabulated numbers are the respective factors of θ, Δ, and Σ. The degree of the factors in the eccentricities is indicated in the subscripts $p\cdot q$ in the symbol for the coefficient. Further, when $j^{2t}=1$

$$i\ \Pi+i'\Pi'=n(\Pi-\Pi')=n\Delta.$$

Hence the coefficient of Δ is also the number n in the proper table of the numerical values of the coefficients. For instance, in the function T_2 (Z 41, eq. 82) we have for one term

$$F_{1\cdot 0}(n-1.-n)\eta\sin(\varepsilon+4\theta+4\Delta)$$

where F, taken from Table VIII, is numerically

$$F_{1\cdot 0}(n-1.-n)_{n=4}=-1514''+5780''w-8976''w^2.$$

Adding $\varepsilon-\psi$ to the argument and taking the coefficients from Table IX, we have also in the function T,

$$G_{1\cdot 0}(n-1.-n)_{n=4}\eta\sin(2\varepsilon-\psi+4\theta+4\Delta)$$

where $$G_{1\cdot 0}(n-1.-n)_{n=4}=+452''-1475''w+1451''w^2.$$

In this manner the series is built up.

The coefficients having subscripts $\pm\delta$ and $\pm\sigma$ belong to terms depending upon the mutual inclination of the orbit planes. They differ from the preceding type of terms in three ways. In the first place the subscript signifies the addition of $\pm\Delta$ and $\pm\Sigma$ to the argument, respectively. Evidently, if $\pm\Delta$ is added to the argument, the factor of Δ is not n but $n\pm1$, from which we determine n. Lastly, these terms contain the factor j^2, i. e., within the extent of our tables the exponent t is not greater than unity.

For the tables referring to functions which concern the perturbations in the third coordinate the same explanations hold, with the exception that the additional subscript $\pm\pi'$ signifies the addition of $\pm\Pi'$ to the argument.

These tables, in connection with the proper tables of numerical coefficients, enable the computer to write a complete series by inspection or segregate any term of given degree and given argument.

TABLE XVa.

$$\Sigma C_{p\cdot q}(n+r.-n+s)\eta^p\eta'^q j^{2t} \begin{matrix}\sin\\ \cos\end{matrix}\begin{cases}A\\ A+\epsilon-\psi\\ A-\epsilon+\psi\end{cases}$$

In the Plane of the Orbit.

				$+\epsilon$			$-\epsilon$			$+2\epsilon$			-2ϵ		
$C_{p\cdot q}$	θ	Δ	Σ	θ	Δ	Σ	θ	Δ	Σ	θ	Δ	Σ	θ	Δ	Σ
$_{0.0}(n.-n)$	0	0		2	2					4	4				
$_{1.0}(n+1.-n)$				0	0					2	2				
$_{1.0}(n-1.-n)$	2	2		4	4		0	0		6	6				
$_{0.1}(n.-n+1)$				0	1					2	3				
$_{0.1}(n.-n-1)$	2	1		4	3					6	5				
$_{2.0}(n+2.-n)$										0	0				
$_{2.0}(n.-n)$	0	0		2	2					4	4				
$_{2.0}(n-2.-n)$	4	4		6	6		2	2		8	8		0	0	
$_{1.1}(n+1.-n+1)$										0	1				
$_{1.1}(n-1.-n+1)$	0	1		2	3					4	5				
$_{1.1}(n+1.-n-1)$				2	1					4	3				
$_{1.1}(n-1.-n-1)$	4	3		6	5		2	1		8	7				
$_{0.2}(n.-n+2)$				−2	0					0	2				
$_{0.2}(n.-n)$	0	0		2	2					4	4				
$_{0.2}(n.-n-2)$	4	2		6	4		2	0		8	6				
$_{0.0}(n+1.-n+1)+\sigma$										0	1	1			
$_{0.0}(n-1.-n-1)-\sigma$	4	3	−1	6	5	−1	2	1	−1	8	7	−1			
$_{0.0}(n+1.-n-1)+\partial$				2	2					4	4				
$_{0.0}(n-1.-n+1)-\partial$	0	0		2	2					4	4				
$_{3.0}(n+1.-n)$				0	0					2	2				
$_{3.0}(n-1.-n)$	2	2		4	4		0	0		6	6				
$_{3.0}(n-3.-n)$	6	6		8	8		4	4		10	10		2	2	
$_{2.1}(n.-n+1)$				0	1					2	3				
$_{2.1}(n-2.-n+1)$	2	3		4	5		0	1		6	7				
$_{2.1}(n.-n-1)$	2	1		4	3					6	5				
$_{2.1}(n-2.-n-1)$	6	5		8	7		4	3		10	9		2	1	
$_{1.2}(n-1.-n+2)$	−2	0		0	2					2	4				
$_{1.2}(n+1.-n)$				0	0					2	2				
$_{1.2}(n-1.-n)$	2	2		4	4		0	0		6	6				
$_{1.2}(n+1.-n-2)$	2	0		4	2					6	4				
$_{1.2}(n-1.-n-2)$	6	4		8	6		4	2		10	8		2	0	
$_{0.3}(n.-n+1)$				0	1					2	3				
$_{0.3}(n.-n-1)$	2	1		4	3					6	5				
$_{0.3}(n.-n-3)$	6	3		8	5		4	1		10	7				
$_{1.0}(n.-n+1)+\sigma$				0	1	1				2	3	1			
$_{1.0}(n.-n-1)-\sigma$	2	1	−1	4	3	−1				6	5	−1			
$_{1.0}(n-2.-n-1)-\sigma$	6	5	−1	8	7	−1	4	3	−1	10	9	−1	2	1	−1
$_{1.0}(n.-n-1)+\partial$	2	2		4	4					6	6				
$_{1.0}(n.-n+1)-\partial$				0	0					2	2				
$_{1.0}(n-2.-n+1)-\partial$	2	2		4	4		0	0		6	6				
$_{0.1}(n+1.-n)+\sigma$				0	0	1				2	2	1			
$_{0.1}(n-1.-n)-\sigma$	2	2	−1	4	4	−1	0	0	−1	6	6	−1			
$_{0.1}(n-1.-n-2)-\sigma$	6	4	−1	8	6	−1	4	2	−1	10	8	−1	2	0	−1
$_{0.1}(n+1.-n)+\partial$				0	1					2	3				
$_{0.1}(n+1.-n-2)+\partial$	2	1		4	3					6	5				
$_{0.1}(n-1.-n+2)-\partial$	−2	−1		0	1					2	3				
$_{0.1}(n-1.-n)-\partial$	2	1		4	3		0	−1		6	5				

	$+3\epsilon$			$+\frac{1}{2}\epsilon$			$-\frac{1}{2}\epsilon$			$+\frac{3}{2}\epsilon$			$-\frac{3}{2}\epsilon$			$+\frac{5}{2}\epsilon$			$-\frac{5}{2}\epsilon$			$+\frac{7}{2}\epsilon$		
$C_{p\cdot q}$	θ	Δ	Σ	θ	Δ	Σ	θ	Δ	Σ	θ	Δ	Σ	θ	Δ	Σ	θ	Δ	Σ	θ	Δ	Σ	θ	Δ	Σ
$_{0.0}(n.-n)$	6	6		1	1					3	3					5	5					7	7	
$_{1.0}(n+1.-n)$	4	4								1	1					3	3					5	5	
$_{1.0}(n-1.-n)$	8	8		3	3		1	1		5	5					7	7					9	9	
$_{0.1}(n.-n+1)$	4	5		−1	0					1	2					3	4					5	6	
$_{0.1}(n.-n-1)$	8	7		3	2		1	0		5	4					7	6					9	8	
$_{2.0}(n+2.-n)$																1	1							
$_{2.0}(n.-n)$				1	1					3	3					5	5							
$_{2.0}(n-2.-n)$				5	5		3	3		7	7		1	1		9	9							
$_{1.1}(n+1.-n+1)$										−1	0					1	2							
$_{1.1}(n-1.-n+1)$				1	2		−1	0		3	4					5	6							
$_{1.1}(n+1.-n-1)$				1	0					3	2					5	4							
$_{1.1}(n-1.-n-1)$				5	4		3	2		7	6		1	0		9	8							
$_{0.2}(n.-n+2)$										−1	1					1	3							
$_{0.2}(n.-n)$				1	1					3	3					5	5							
$_{0.2}(n.-n-2)$				5	3		3	1		7	5					9	7							
$_{0.0}(n+1.-n+1)+\sigma$										−1	0	1				1	2	1						
$_{0.0}(n-1.-n-1)-\sigma$				5	4	−1	3	2	−1	7	6	−1	1	0	−1	9	8	−1						
$_{0.0}(n+1.-n-1)+\partial$				1	1					3	3					5	5							
$_{0.0}(n-1.-n+1)-\partial$				1	1		−1	−1		3	3					5	5							

TABLE XVb.

$$\Sigma C_{p \cdot q}(n+r.-n+s)\eta^p\eta'^q j^{2t} \begin{smallmatrix}\sin\\ \cos\end{smallmatrix} \begin{cases} A \\ A+\varepsilon-\psi \\ A-\varepsilon+\psi \end{cases}$$

Perpendicular to the Plane of the Orbit.

				$+\iota$			$-\iota$			$+2\iota$			-2ι			$+\frac{1}{2}\iota$			$-\frac{1}{2}\iota$			$+\frac{3}{2}\iota$			$-\frac{3}{2}\iota$		
$C_{p\,q}$	θ	Δ	Σ	θ	Δ	Σ	θ	Δ	Σ	θ	Δ	Σ	θ	Δ	Σ	θ	Δ	Σ	θ	Δ	Σ	θ	Δ	Σ	θ	Δ	Σ
$_{0\cdot0}(n.-n+1)+\pi'$				0	1					2	3					−1	0					1	2				
$_{0\cdot0}(n.-n-1)-\pi'$	2	1		4	3					6	5					3	2		1	0		5	4				
$_{1\cdot0}(n+1.-n+1)+\pi'$										0	1											−1	0				
$_{1\cdot0}(n-1.-n+1)+\pi'$	0	1		2	3					4	5					1	2		−1	0		3	4				
$_{1\cdot0}(n+1.-n-1)-\pi'$				2	1					4	3					1	0					3	2				
$_{1\cdot0}(n-1.-n-1)-\pi'$	4	3		6	5		2	1		8	7					5	4		3	2		7	6		1	0	
$_{0\cdot1}(n.-n+2)+\pi'$				−2	0					0	2											−1	1				
$_{0\cdot1}(n.-n)+\pi'$	0	0		2	2					4	4					1	1					3	3				
$_{0\cdot1}(n.-n)-\pi'$	0	0		2	2					4	4					1	1					3	3				
$_{0\cdot1}(n.-n-2)-\pi'$	4	2		6	4		2	0		8	6					5	3		3	1		7	5				
$_{2\cdot0}(n+2.-n+1)+\pi'$																											
$_{2\cdot0}(n.-n+1)+\pi'$				0	1					2	3					−1	0					1	2				
$_{2\cdot0}(n-2.-n+1)+\pi'$	2	3		4	5		0	1		6	7					3	4		1	2		5	6		−1	0	
$_{2\cdot0}(n+2.-n-1)-\pi'$										2	1											1	0				
$_{2\cdot0}(n.-n-1)-\pi'$	2	1		4	3					6	5					3	2		1	0		5	4				
$_{2\cdot0}(n-2.-n-1)-\pi'$	6	5		8	7		4	3		10	9		2	1		7	6		5	4		9	8		3	2	
$_{1\cdot1}(n+1.-n+2)+\pi'$										−2	0																
$_{1\cdot1}(n-1.-n+2)+\pi'$	−2	0		0	2					2	4					−1	1					1	3				
$_{1\cdot1}(n+1.-n)+\pi'$				0	0					2	2											1	1				
$_{1\cdot1}(n-1.-n)+\pi'$	2	2		4	4		0	0		6	6					3	3		1	1		5	5				
$_{1\cdot1}(n+1.-n)-\pi'$				0	0					2	2											1	1				
$_{1\cdot1}(n-1.-n)-\pi'$	2	2		4	4		0	0		6	6					3	3		1	1		5	5				
$_{1\cdot1}(n+1.-n-2)-\pi'$	2	0		4	2					6	4					3	1					5	3				
$_{1\cdot1}(n-1.-n-2)-\pi'$	6	4		8	6		4	2		10	8		2	0		7	5		5	3		9	7		3	1	
$_{0\cdot2}(n.-n+3)+\pi'$										−2	1											−3	0				
$_{0\cdot2}(n.-n+1)+\pi'$				0	1					2	3					−1	0					1	2				
$_{0\cdot2}(n.-n-1)+\pi'$	2	1		4	3					6	5					3	2		1	0		5	4				
$_{0\cdot2}(n.-n+1)-\pi'$				0	1					2	3					−1	0					1	2				
$_{0\cdot2}(n.-n-1)-\pi'$	2	1		4	3					6	5					3	2		1	0		5	4				
$_{0\cdot2}(n.-n-3)-\pi'$	6	3		8	5		4	1		10	7					7	4		5	2		9	6		3	0	
$_{0\cdot0}(n+1.-n+2)+\sigma+\pi'$										−2	0	1															
$_{0\cdot0}(n-1.-n)-\sigma+\pi'$	2	2	−1	4	4	−1	0	0	−1	6	6	−1				3	3	−1	1	1	−1	5	5	−1			
$_{0\cdot0}(n+1.-n)+\partial+\pi'$				0	1					2	3											1	2				
$_{0\cdot0}(n-1.-n+2)-\partial+\pi'$	−2	−1		0	1					2	3					−1	0					1	2				
$_{0\cdot0}(n+1.-n)+\sigma-\pi'$				0	0	1				2	2	1										1	1	1			
$_{0\cdot0}(n-1.-n-2)-\sigma-\pi'$	6	4	−1	8	6	−1	4	2	−1	10	8	−1	2	0	−1	7	5	−1	5	3	−1	9	7	−1	3	1	−1
$_{0\cdot0}(n+1.-n-2)+\partial-\pi'$	2	1		4	3					6	5					3	2					5	4				
$_{0\cdot0}(n-1.-n)-\partial-\pi'$	2	1		4	3		0	−1		6	5					3	2		1	0		5	4				

Our problem is now the integration of the partial differential equations Z 7, eq. (33), Z 8, eqs. (37) and (39), and Z 9, eq. (47[1]).

In the trigonometric series to be integrated the argument is a function of θ, ε, ψ, J, Σ. The last two are constants. According to the principles of Hansen, ψ occurs outside the operation. Numerically, however, it is equal to ε. The argument θ contains ε implicitly. See Z 9, eq. (43). Hence we must, in general, write

$$F(\varepsilon, \theta)$$

and

$$\frac{dF}{d\varepsilon}=\frac{\partial F}{\partial \varepsilon}+\frac{\partial F}{\partial \theta}\cdot\frac{d\theta}{d\varepsilon}$$

In order to set up the partial differential equations from the total derivative, the following notation is introduced:

$$F(\varepsilon, \theta)=[F(\varepsilon, \theta)]+F(\varepsilon, \theta)-[F(\varepsilon, \theta)]$$

where $[F(\varepsilon, \theta)]$ signifies that part of the function which is independent of ε. Again, since ε has the period of the planet, there can be no secular terms in ε (with the exception of the function θ), i. e.,

$$\left[\frac{\partial F}{\partial \varepsilon}\right]=0$$

On the other hand, the argument θ varies much more slowly, and there may be secular terms in θ. Hence

$$\left[\frac{\partial F}{\partial \theta}\right]\neq 0$$

and θ may occur outside the sign of integration.

Owing to the presence of the required function in the differential equation, the integrations must be performed rank by rank where rank is defined as follows:

In the course of the developments there arise negative powers of w. Since w is a small quantity, these factors increase the numerical value of the terms, or, in other words, they lower the order. Therefore, it is better to define order in terms of both the disturbing mass m' and w. For this purpose v. Zeipel makes the assumption that both w and $\sqrt{m'}$ are quantities of the first order. Order so defined is called "rank," and the word "order" is reserved as usual for the powers of m'. The factors $\frac{m'^{\alpha}}{w^{\beta}}$ are arranged according to rank in Z 53.

Any function is then written in the form

$$F(\varepsilon, \theta)=F_1(\varepsilon, \theta)+F_2(\varepsilon,\theta)+F_3(\varepsilon, \theta)+\cdot\cdot\cdot\cdot\cdot\cdot$$

where the subscript denotes the term of *lowest* rank, for F_i (ε, θ) contains terms of more than one rank since each coefficient is itself a Taylor's series in w. In assigning rank it is to be noted that the coefficients in all the preceding tables contain the factor m' implicitly. The implicit mass factor is indicated at the foot of each table which follows.

On the basis of the foregoing principles, the differential equation for W,

$$\frac{dW}{d\varepsilon}=\frac{\partial W}{\partial \varepsilon}+\frac{\partial W}{\partial \theta}\cdot\frac{d\theta}{d\varepsilon}=T$$

expressed in Z 52, eq. (91), is broken up into four equations, Z 53, eqs. (95_1 — 95_4), according to rank, and before integration they are again subdivided according to parts which contain ε and parts which are independent of ε. The total derivative is then in the form of eight equivalent equations, and the integration can be performed in the following order:

$$W_1;\ \ W_2-[W_2];\ [W_2];\ \ W_3-[W_3];\ \text{etc.}$$

It is possible to avoid the computation of T_3, as v. Zeipel did, by the introduction of some auxiliary functions, but we found it preferable to tabulate them.

Employing Table XVa, and by inspection of Tables VIII, IX, X, XI, T_2 is written directly. (T has no terms of first rank.)

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TABLE XVc.

T_2 Unit = 1″

	Sin	w^0	w	w^1
	$\epsilon-\psi$	+ 43.1	− 128.0	+ 171
	$-\epsilon+\psi$	− 43.1	+ 128.0	− 171
	$\epsilon+2\theta+2\Delta$	+ 271.5	− 636.6	+ 526
	$2\epsilon-\psi+2\theta+2\Delta$	− 67.1	+ 108.2	+ 32
	$\psi+2\theta+2\Delta$	− 294.9	+ 740.6	− 734
	$2\epsilon+4\theta+4\Delta$	+ 159.9	− 593.3	+ 869
	$3\epsilon-\psi+4\theta+4\Delta$	− 45.7	+ 122.1	− 174
	$\epsilon+\psi+4\theta+4\Delta$	− 167.4	+ 637.4	− 984
	$2\epsilon+\psi+6\theta+6\Delta$	− 81.7	+ 418.9	− 907
η	$2\theta+2\Delta$	−1180	+2962	− 2935
	$\epsilon-\psi+2\theta+2\Delta$	+ 273	− 179	− 1170
	$-\epsilon+\psi+2\theta+2\Delta$	+1496	−4265	+ 5572
η	ϵ	− 173	+ 512	− 684
	$2\epsilon-\psi$	− 211	+ 899	− 1921
	ψ	+ 384	−1410	+ 2605
η	$\epsilon+4\theta+4\Delta$	−1514	+5780	− 8976
	$2\epsilon-\psi+4\theta+4\Delta$	+ 452	−1475	+ 1451
	$\psi+4\theta+4\Delta$	+1679	−6656	+11172
η	$2\epsilon+2\theta+2\Delta$	− 6	+ 408	− 1307
	$3\epsilon-\psi+2\theta+2\Delta$	− 83	+ 262	− 564
	$\epsilon+\psi+2\theta+2\Delta$	+ 136	− 878	+ 2285
η	$2\epsilon+6\theta+6\Delta$	−1149	+5902	−12820
	$3\epsilon-\psi+6\theta+6\Delta$	+ 360	−1734	+ 3301
	$\epsilon+\psi+6\theta+6\Delta$	+1227	−6415	+14400
η	$2\epsilon+\psi+4\theta+4\Delta$	− 102	+ 112	
η	$2\epsilon+\psi+8\theta+8\Delta$	+ 750	−4900	
η'	$2\theta+\Delta$	+ 318	−1081	+ 1552
	$\epsilon-\psi+2\theta+\Delta$	+ 222	−1012	+ 2227
	$-\epsilon+\psi+2\theta+\Delta$	− 646	+2452	− 4296
η'	$\epsilon+\Delta$	+ 130	− 484	+ 808
	$2\epsilon-\psi+\Delta$	+ 112	− 565	+ 1393
	$\psi+\Delta$	− 285	+1211	− 2475
η'	$\epsilon+4\theta+3\Delta$	+2279	−7160	+ 8896
	$2\epsilon-\psi+4\theta+3\Delta$	− 580	+1410	− 520
	$\psi+4\theta+3\Delta$	−2460	+8138	−11342
η'	$2\epsilon+2\theta+3\Delta$	− 314	+ 702	− 90
	$3\epsilon-\psi+2\theta+3\Delta$	+ 127	− 399	+ 598
	$\epsilon+\psi+2\theta+3\Delta$	+ 291	− 537	− 478
η'	$2\epsilon+6\theta+5\Delta$	+1887	−8417	+15550
	$3\epsilon-\psi+6\theta+5\Delta$	− 542	+2221	− 3377
	$\epsilon+\psi+6\theta+5\Delta$	−1974	+9002	−17350
η'	$2\epsilon+\psi+4\theta+5\Delta$	+ 390	−1556	
η'	$2\epsilon+\psi+8\theta+7\Delta$	−1263	+7397	
η^2	$\epsilon-\psi$	+ 568	− 3106	
	$-\epsilon+\psi$	− 568	+ 3106	
η^2	$4\theta+4\Delta$	+6716	− 26627	+ 44700
	$\epsilon-\psi+4\theta+4\Delta$	−2114	+ 6488	
	$-\epsilon+\psi+4\theta+4\Delta$	−7960	+ 33462	
η^2	$\epsilon+2\theta+2\Delta$	+ 128	− 3166	
	$2\epsilon-\psi+2\theta+2\Delta$	+ 535	− 2505	
	$\psi+2\theta+2\Delta$	− 978	+ 7431	− 23105
			m'	

TABLE XVc—Continued.

T_2

Unit=1″

	Sin	w^0	w	w^2
η^2	$\epsilon+6\theta+6\Delta$	+7969	−41736	
	$2\epsilon-\psi+6\theta+6\Delta$	−2624	+12577	
	$\psi+6\theta+6\Delta$	−8819	+47347	−111337
η^2	$-\epsilon+2\theta+2\Delta$	+2245	−6168	
	$-\psi+2\theta+2\Delta$	−396	−1494	+9351
	$-2\epsilon+\psi+2\theta+2\Delta$	−3596	+12561	
η^2	2ϵ	+423	−1797	
	$3\epsilon-\psi$	+357	−2207	
	$\epsilon+\psi$	−780	+4005	
η^2	$2\epsilon+4\theta+4\Delta$	−1783	+3946	
	$3\epsilon-\psi+4\theta+4\Delta$	+924	−3327	
	$\epsilon+\psi+4\theta+4\Delta$	+1220	−1026	
η^2	$2\epsilon+8\theta+8\Delta$	+6749	−44127	
	$3\epsilon-\psi+8\theta+8\Delta$	−2247	+14052	
	$\epsilon+\psi+8\theta+8\Delta$	−7252	+48051	
$\eta\eta'$	Δ	−285	+1210	−2475
	$\epsilon-\psi+\Delta$	−1004	+5771	
	$-\epsilon+\psi+\Delta$	+1574	−8192	
$\eta\eta'$	$4\theta+3\Delta$	−17218	+56961	−79400
	$\epsilon-\psi+4\theta+3\Delta$	+4253	−8340	
	$-\epsilon+\psi+4\theta+3\Delta$	+20345	−73031	
$\eta\eta'$	$\epsilon+2\theta+\Delta$	−1429	+6138	
	$2\epsilon-\psi+2\theta+\Delta$	−523	+3792	
	$\psi+2\theta+\Delta$	+2280	−11302	+28347
$\eta\eta'$	$\epsilon+2\theta+3\Delta$	+1725	−3054	
	$2\epsilon-\psi+2\theta+3\Delta$	−1003	+3753	
	$\psi+2\theta+3\Delta$	−1492	+677	+13097
$\eta\eta'$	$\epsilon+6\theta+5\Delta$	−23773	+108605	
	$2\epsilon-\psi+6\theta+5\Delta$	+7038	−28427	
	$\psi+6\theta+5\Delta$	+25974	−122380	+251019
$\eta\eta$	$-\epsilon+2\theta+\Delta$	−965	+3533	
	$-\psi+2\theta+\Delta$	−2068	+10582	+39011
	$-2\epsilon+\psi+2\theta+\Delta$	+3785	−16928	
$\eta\eta'$	$2\epsilon+\Delta$	−820	+3797	
	$3\epsilon-\psi+\Delta$	−470	+3185	
	$\epsilon+\psi+\Delta$	+1488	−7870	
$\eta\eta'$	$2\epsilon+4\theta+3\Delta$	+1815	−1190	
	$3\epsilon-\psi+4\theta+3\Delta$	−1181	+3807	
	$\epsilon+\psi+4\theta+3\Delta$	−853	−3161	
$\eta\eta'$	$2\epsilon+4\theta+5\Delta$	+4294	−17092	
	$3\epsilon-\psi+4\theta+5\Delta$	−1571	+6629	
	$\epsilon+\psi+4\theta+5\Delta$	−4414	+17198	
$\eta\eta'$	$2\epsilon+8\theta+7\Delta$	−21544	+126397	
	$3\epsilon-\psi+8\theta+7\Delta$	+6700	−37167	
	$\epsilon+\psi+8\theta+7\Delta$	+22868	−136294	
η'^2	$\epsilon-\psi$	+866	−4261	
	$-\epsilon+\psi$	−866	+4261	
η'^2	$4\theta+2\Delta$	+10682	−28347	+32120
	$\epsilon-\psi+4\theta+2\Delta$	−1815	+474	
	$-\epsilon+\psi+4\theta+2\Delta$	−12428	+37322	
η'^2	$\epsilon+2\theta+2\Delta$	−1498	+450	
	$2\epsilon-\psi+2\theta+2\Delta$	+1136	−4394	
	$\psi+2\theta+2\Delta$	+861	+3794	−22127
		m'		

XVc—Continued.

T_2

Unit = 1″

	Sin	w^0	w	w^2
η'^2	$\epsilon+\quad 6\theta+4⅃$	+17790	− 69344	
	$2\epsilon-\psi+6\theta+4⅃$	− 4675	+ 15200	
	$\psi+6\theta+4⅃$	−19046	+ 77260	−135954
η'^2	$-\psi+2\theta$	+ 1634	− 7081	+ 16199
	$-2\epsilon+\psi+2\theta$	− 1634	+ 7081	
η'^2	$2\epsilon+\quad 2⅃$	+ 328	− 1710	
	$3\epsilon-\psi+\quad 2⅃$	+ 154	− 1141	
	$\epsilon+\psi+\quad 2⅃$	− 591	+ 3420	
η'^2	$2\epsilon+\quad 4\theta+4⅃$	− 5879	+ 19019	
	$3\epsilon-\psi+4\theta+4⅃$	+ 2032	− 7361	
	$\epsilon+\psi+4\theta+4⅃$	+ 5807	− 17998	
η'^2	$2\epsilon+\quad 8\theta+6⅃$	+17340	− 69064	
	$3\epsilon-\psi+8\theta+6⅃$	− 5018	+ 24266	
	$\epsilon+\psi+8\theta+6⅃$	−18102	+ 95320	
j^2	$\epsilon-\psi$	− 866	+ 4260	
	$-\ \epsilon+\psi$	+ 866	− 4260	
j^2	$4\theta+3⅃-\Sigma$	+ 609	− 2958	+ 6763
	$\epsilon-\psi+4\theta+3⅃-\Sigma$	+ 232	− 1656	
	$-\ \epsilon+\psi+4\theta+3⅃-\Sigma$	− 1044	+ 5600	
j^2	$\epsilon+\quad 2\theta+2⅃$	− 1760	+ 7189	
	$2\epsilon-\psi+2\theta+2⅃$	− 331	+ 3096	
	$\psi+2\theta+2⅃$	+ 2677	− 12681	+ 30930
j^2	$\epsilon+\quad 6\theta+5⅃-\Sigma$	+ 578	− 3543	
	$2\epsilon-\psi+6\theta+5⅃-\Sigma$	+ 10	− 299	
	$\psi+6\theta+5⅃-\Sigma$	− 780	+ 5023	− 15302
j^2	$-\psi+2\theta+⅃-\Sigma$	+ 866	− 4260	+ 10988
	$-2\epsilon+\psi+2\theta+⅃-\Sigma$	− 866	+ 4260	
j^2	$2\epsilon+\quad ⅃+\Sigma$	+ 1152	− 4231	
	$3\epsilon-\psi+\quad ⅃+\Sigma$	+ 98	− 1440	
	$\epsilon+\psi+\quad ⅃+\Sigma$	− 1634	+ 7081	
j^2	$2\epsilon+\quad 4\theta+4⅃$	− 1795	+ 9459	
	$3\epsilon-\psi+4\theta+4⅃$	+ 164	− 17	
	$\epsilon+\psi+4\theta+4⅃$	+ 2229	− 12595	
j^2	$2\epsilon+\quad 8\theta+7⅃-\Sigma$	+ 392	− 2914	
	$3\epsilon-\psi+8\theta+7⅃-\Sigma$	− 40	+ 194	
	$\epsilon+\psi+8\theta+7⅃-\Sigma$	− 482	+ 3691	
	$\frac{1}{2}\epsilon+\quad \theta+\ ⅃$	+ 47.1	− 149.3	+ 186
	$\frac{3}{2}\epsilon-\psi+\ \theta+\ ⅃$	+ 27.5	− 111.4	+ 207
	$-\frac{1}{2}\epsilon+\psi+\ \theta+\ ⅃$	− 90.4	+ 310.5	− 455
	$\frac{3}{2}\epsilon+\quad 3\theta+3⅃$	+ 216.1	− 655.2	+ 749
	$\frac{5}{2}\epsilon-\psi+3\theta+3⅃$	− 58.9	+ 150.7	− 93
	$\frac{1}{2}\epsilon+\psi+3\theta+3⅃$	− 229.3	+ 722.8	− 905
	$\frac{5}{2}\epsilon+\quad 5\theta+5⅃$	+ 113.8	− 499.2	+ 892
	$\frac{7}{2}\epsilon-\psi+5\theta+5⅃$	− 33.5	+ 137.7	− 213
	$\frac{3}{2}\epsilon+\psi+5\theta+5⅃$	− 118.2	+ 527.9	− 977
	$\frac{7}{2}\epsilon+\quad 7\theta+7⅃$	+ 54.1	− 310.2	+ 757
	$\frac{9}{2}\epsilon-\psi+7\theta+7⅃$	− 16.5	+ 91.3	− 209
	$\frac{5}{2}\epsilon+\psi+7\theta+7⅃$	− 55.7	+ 322.3	− 801
	$\frac{9}{2}\epsilon+\quad 9\theta+9⅃$	+ 24.5	− 173.5	+ 537
	$\frac{11}{2}\epsilon-\psi+9\theta+9⅃$	− 7.6	+ 52.7	− 157
	$\frac{7}{2}\epsilon+\psi+9\theta+9⅃$	− 25.1	+ 178.5	− 559
		$v.'$		

XVc—Continued.

T_2 Unit = 1″

	sin	w^0	w	u^2
η	$\frac{1}{2}\varepsilon+\quad 3\theta+3J$	−1497	+ 4732	− 5963
	$\frac{3}{2}\varepsilon-\psi+3\theta+3J$	+ 419	− 966	+ 131
	$-\frac{1}{2}\varepsilon+\psi+3\theta+3J$	+1729	− 5826	+ 8424
η	$-\frac{1}{2}\varepsilon+\quad \theta+J$	− 114	+ 385	− 548
	$\frac{1}{2}\varepsilon-\psi+\theta+J$	− 224	+ 1006	− 2186
	$-\frac{3}{2}\varepsilon+\psi+\theta+J$	+ 436	− 1726	+ 3220
η	$\frac{3}{2}\varepsilon+\quad \theta+J$	− 208	+ 781	− 1315
	$\frac{5}{2}\varepsilon-\psi+\theta+J$	− 55	+ 349	− 1026
	$\frac{1}{2}\varepsilon+\psi+\theta+J$	+ 314	− 1316	+ 2641
η	$\frac{3}{2}\varepsilon+\quad 5\theta+5J$	−1366	+ 6114	−11363
	$\frac{5}{2}\varepsilon-\psi+5\theta+5J$	+ 420	− 1711	+ 2548
	$\frac{1}{2}\varepsilon+\psi+5\theta+5J$	+1480	− 6793	+13254
η	$\frac{5}{2}\varepsilon+\quad 3\theta+3J$	+ 108	− 20	− 847
	$\frac{7}{2}\varepsilon-\psi+3\theta+3J$	− 85	+ 256	− 395
	$\frac{3}{2}\varepsilon+\psi+3\theta+3J$	− 19	− 329	+ 1586
η	$\frac{5}{2}\varepsilon+\quad 7\theta+7J$	− 922	+ 5348	−13320
	$\frac{7}{2}\varepsilon-\psi+7\theta+7J$	+ 292	− 1618	+ 3692
	$\frac{3}{2}\varepsilon+\psi+7\theta+7J$	+ 975	− 5728	+14602
η	$\frac{7}{2}\varepsilon+\quad 5\theta+5J$	+ 172	− 594	+ 491
	$\frac{9}{2}\varepsilon-\psi+5\theta+5J$	− 74	+ 298	− 470
	$\frac{5}{2}\varepsilon+\psi+5\theta+5J$	− 133	+ 402	− 42
η	$\frac{7}{2}\varepsilon\quad +9\theta+9J$	− 541	+ 3856	−12092
	$\frac{9}{2}\varepsilon-\psi+9\theta+9J$	+ 174	− 1205	+ 3608
	$\frac{5}{2}\varepsilon+\psi+9\theta+9J$	+ 564	− 4055	+12887
η'	$\frac{1}{2}\varepsilon\quad +3\theta+2J$	+2041	− 5080	+ 4928
	$\frac{3}{2}\varepsilon-\psi+3\theta+2J$	− 431	+ 499	− 1026
	$-\frac{1}{2}\varepsilon+\psi+3\theta+2J$	−2290	+ 6274	− 7597
η'	$\frac{1}{2}\varepsilon-\psi+\theta$	+ 384	− 1410	+ 2605
	$-\frac{3}{2}\varepsilon+\psi+\theta$	− 384	+ 1410	− 2605
η'	$\frac{3}{2}\varepsilon\quad +\theta+2J$	− 131	+ 12	+ 717
	$\frac{5}{2}\varepsilon-\psi+\theta+2J$	+ 106	− 366	+ 772
	$\frac{1}{2}\varepsilon+\psi+\theta+2J$	+ 69	+ 350	− 1728
η'	$\frac{3}{2}\varepsilon\quad +5\theta+4J$	+2169	− 8241	+12680
	$\frac{5}{2}\varepsilon-\psi+5\theta+4J$	− 596	+ 1980	− 2086
	$\frac{1}{2}\varepsilon+\psi+5\theta+4J$	−2295	+ 9008	−14823
η'	$\frac{5}{2}\varepsilon\quad +3\theta+4J$	− 389	+ 1251	− 1212
	$\frac{7}{2}\varepsilon-\psi+3\theta+4J$	+ 135	− 479	+ 687
	$\frac{3}{2}\varepsilon+\psi+3\theta+4J$	+ 383	− 1189	+ 930
η'	$\frac{5}{2}\varepsilon\quad +7\theta+6J$	+1550	− 7940	+17170
	$\frac{7}{2}\varepsilon-\psi+7\theta+6J$	− 457	+ 2211	− 4245
	$\frac{3}{2}\varepsilon+\psi+7\theta+6J$	−1609	+ 8376	−18650
η'	$\frac{7}{2}\varepsilon\quad +5\theta+6J$	− 349	+ 1665	− 3127
	$\frac{9}{2}\varepsilon-\psi+5\theta+6J$	+ 113	− 543	+ 1052
	$\frac{5}{2}\varepsilon+\psi+5\theta+6J$	+ 352	− 1678	+ 3117
η'	$\frac{7}{2}\varepsilon\quad +9\theta+8J$	+ 937	− 6044	+16950
	$\frac{9}{2}\varepsilon-\psi+9\theta+8J$	− 286	+ 1784	− 4724
	$\frac{5}{2}\varepsilon+\psi+9\theta+8J$	− 963	+ 6274	−17880
η^2	$\frac{1}{2}\varepsilon\quad +\theta+J$	+ 757	− 3272	
	$\frac{3}{2}\varepsilon-\psi+\theta+J$	+ 514	− 3644	
	$-\frac{1}{2}\varepsilon+\psi+\theta+J$	−1583	+ 8214	
η^2	$\frac{1}{2}\varepsilon\quad +5\theta+5J$	+7767	−35692	
	$\frac{3}{2}\varepsilon-\psi+5\theta+5J$	−2522	+10085	
	$-\frac{1}{2}\varepsilon+\psi+5\theta+5J$	−8820	+42033	
			w'	

TABLE XVc—Continued.

T_2

Unit=1″

	Sin	w^0	w	w^2
η^2	$-\frac{1}{2}\varepsilon\quad+3\theta+3\Delta$	+ 4758	− 15945	
	$\frac{1}{2}\varepsilon-\psi+3\theta+3\Delta$	− 1369	+ 2307	
	$-\frac{3}{2}\varepsilon+\psi+3\theta+3\Delta$	− 6128	+ 22836	
η^2	$\frac{3}{2}\varepsilon\quad+3\theta+3\Delta$	− 882	− 280	
	$\frac{5}{2}\varepsilon-\psi+3\theta+3\Delta$	+ 732	− 2580	
	$\frac{1}{2}\varepsilon+\psi+3\theta+3\Delta$	+ 177	+ 3816	
η^2	$\frac{3}{2}\varepsilon\quad+7\theta+7\Delta$	+ 7549	− 44427	
	$\frac{5}{2}\varepsilon-\psi+7\theta+7\Delta$	− 2504	+ 13864	
	$\frac{1}{2}\varepsilon+\psi+7\theta+7\Delta$	− 8212	+ 49192	
η^2	$-\frac{3}{2}\varepsilon\quad+\theta+\Delta$	− 32	+ 220	
	$-\frac{1}{2}\varepsilon-\psi+\theta+\Delta$	+ 784	− 4194	
	$-\frac{5}{2}\varepsilon+\psi+\theta+\Delta$	− 1031	+ 5051	
η^2	$\frac{5}{2}\varepsilon\quad+9\theta+9\Delta$	+ 5780	− 41583	
	$\frac{7}{2}\varepsilon-\psi+9\theta+9\Delta$	− 1929	+ 13412	
	$\frac{3}{2}\varepsilon+\psi+9\theta+9\Delta$	− 6154	+ 44735	
$\eta\,\eta'$	$\frac{1}{2}\varepsilon\quad+\theta$	− 768	+ 2821	
	$\frac{3}{2}\varepsilon-\psi+\theta$	− 1156	+ 6406	
	$-\frac{1}{2}\varepsilon+\psi+\theta$	+ 1924	− 9227	
$\eta\,\eta'$	$\frac{1}{2}\varepsilon\quad+\theta+2\Delta$	+ 209	+ 1404	
	$\frac{3}{2}\varepsilon-\psi+\theta+2\Delta$	− 771	+ 3774	
	$-\frac{1}{2}\varepsilon+\psi+\theta+2\Delta$	+ 446	− 5879	
$\eta\,\eta'$	$\frac{1}{2}\varepsilon\quad+5\theta+4\Delta$	−21869	+ 85960	
	$\frac{3}{2}\varepsilon-\psi+5\theta+4\Delta$	+ 6125	− 19358	
	$-\frac{1}{2}\varepsilon+\psi+5\theta+4\Delta$	+24564	−101260	
$\eta\,\eta'$	$-\frac{1}{2}\varepsilon\quad+3\theta+2\Delta$	−10182	+ 27638	
	$\frac{1}{2}\varepsilon-\psi+3\theta+2\Delta$	+ 1576	+ 2276	
	$-\frac{3}{2}\varepsilon+\psi+3\theta+2\Delta$	+13528	− 43309	
$\eta\,\eta'$	$\frac{3}{2}\varepsilon\quad+3\theta+2\Delta$	+ 384	+ 3626	
	$\frac{5}{2}\varepsilon-\psi+3\theta+2\Delta$	− 939	+ 3382	
	$\frac{1}{2}\varepsilon+\psi+3\theta+2\Delta$	+ 715	− 8550	
$\eta\,\eta'$	$\frac{3}{2}\varepsilon\quad+3\theta+4\Delta$	+ 3250	− 10017	
	$\frac{5}{2}\varepsilon-\psi+3\theta+4\Delta$	− 1333	+ 4996	
	$\frac{1}{2}\varepsilon+\psi+3\theta+4\Delta$	− 3256	+ 9153	
$\eta\,\eta'$	$\frac{3}{2}\varepsilon\quad+7\theta+6\Delta$	−23414	+122108	
	$\frac{5}{2}\varepsilon-\psi+7\theta+6\Delta$	+ 7146	− 34410	
	$\frac{1}{2}\varepsilon+\psi+7\theta+6\Delta$	+25145	−133985	
$\eta\,\eta'$	$-\frac{3}{2}\varepsilon\quad+\theta$	+ 768	− 2821	
	$-\frac{1}{2}\varepsilon-\psi+\theta$	− 2308	+ 10637	
	$-\frac{5}{2}\varepsilon+\psi+\theta$	+ 1540	− 7816	
$\eta\,\eta'$	$\frac{5}{2}\varepsilon\quad+9\theta+8\Delta$	−18847	+122928	
	$\frac{7}{2}\varepsilon-\psi+9\theta+8\Delta$	+ 5935	− 37138	
	$\frac{3}{2}\varepsilon+\psi+9\theta+8\Delta$	+19837	−130949	
η'^2	$\frac{1}{2}\varepsilon\quad+\theta+\Delta$	+ 761	− 3333	
	$\frac{3}{2}\varepsilon-\psi+\theta+\Delta$	+ 906	− 5387	
	$-\frac{1}{2}\varepsilon+\psi+\theta+\Delta$	− 1920	+ 9831	
η'^2	$\frac{1}{2}\varepsilon\quad+5\theta+3\Delta$	+15303	− 49954	
	$\frac{3}{2}\varepsilon-\psi+5\theta+3\Delta$	− 3577	+ 7957	
	$-\frac{1}{2}\varepsilon+\psi+5\theta+3\Delta$	−16828	+ 58649	
η'^2	$-\frac{1}{2}\varepsilon\quad+3\theta+\Delta$	+ 1582	− 5765	
	$\frac{1}{2}\varepsilon-\psi+3\theta+\Delta$	+ 1300	− 6572	
	$-\frac{3}{2}\varepsilon+\psi+3\theta+\Delta$	− 3410	+ 14260	
η'^2	$\frac{3}{2}\varepsilon\quad-\theta+\Delta$	+ 451	− 1890	
	$\frac{5}{2}\varepsilon-\psi-\theta+\Delta$	+ 494	− 2861	
	$\frac{1}{2}\varepsilon+\psi-\theta+\Delta$	− 1096	+ 5381	
		m'		

TABLE XVc—Continued.

T_2 Unit=1″

	Sin	w^0	w	w^2
η'^2	$\frac{3}{2}\varepsilon+3\theta+3\Delta$	− 3918	+ 8760	
	$\frac{5}{2}\varepsilon-\psi+3\theta+3\Delta$	+ 1588	− 5289	
	$\frac{1}{2}\varepsilon+\psi+3\theta+3\Delta$	+ 3637	− 6391	
η'^2	$\frac{3}{2}\varepsilon+7\theta+5\Delta$	+ 18292	− 83098	
	$\frac{5}{2}\varepsilon-\psi+7\theta+5\Delta$	− 5104	+ 20825	
	$\frac{1}{2}\varepsilon+\psi+7\theta+5\Delta$	− 19286	+ 89973	
j^2	$\frac{1}{2}\varepsilon+\theta+\Delta$	− 902	+ 3781	
	$\frac{3}{2}\varepsilon-\psi+\theta+\Delta$	− 988	+ 5721	
	$-\frac{1}{2}\varepsilon+\psi+\theta+\Delta$	+ 2191	− 10762	
j^2	$\frac{1}{2}\varepsilon+5\theta+4\Delta-\Sigma$	+ 634	− 3482	
	$\frac{3}{2}\varepsilon-\psi+5\theta+4\Delta-\Sigma$	+ 87	− 836	
	$-\frac{1}{2}\varepsilon+\psi+5\theta+4\Delta-\Sigma$	− 933	+ 5479	
j^2	$-\frac{1}{2}\varepsilon+3\theta+2\Delta-\Sigma$	+ 428	− 1816	
	$\frac{1}{2}\varepsilon-\psi+3\theta+2\Delta-\Sigma$	+ 480	− 2805	
	$-\frac{3}{2}\varepsilon+\psi+3\theta+2\Delta-\Sigma$	− 1050	+ 5226	
j^2	$\frac{3}{2}\varepsilon+3\theta+3\Delta$	− 1916	+ 8929	
	$\frac{5}{2}\varepsilon-\psi+3\theta+3\Delta$	+ 2	+ 1220	
	$\frac{1}{2}\varepsilon+\psi+3\theta+3\Delta$	+ 2553	− 13126	
j^2	$\frac{3}{2}\varepsilon+7\theta+6\Delta-\Sigma$	+ 488	− 3307	
	$\frac{5}{2}\varepsilon-\psi+7\theta+6\Delta-\Sigma$	− 27	+ 23	
	$\frac{1}{2}\varepsilon+\psi+7\theta+6\Delta-\Sigma$	− 623	+ 4387	
j^2	$-\frac{3}{2}\varepsilon+\theta-\Sigma$	− 475	+ 1965	
	$-\frac{1}{2}\varepsilon-\psi+\theta-\Sigma$	+ 1141	− 5536	
	$-\frac{5}{2}\varepsilon+\psi+\theta-\Sigma$	− 508	+ 2916	
j^2	$\frac{5}{2}\varepsilon+\theta+2\Delta+\Sigma$	+ 1282	− 5447	
	$\frac{7}{2}\varepsilon-\psi+\theta+2\Delta+\Sigma$	− 90	− 384	
	$\frac{3}{2}\varepsilon+\psi+\theta+2\Delta+\Sigma$	− 1620	+ 7647	
j^2	$\frac{5}{2}\varepsilon+5\theta+5\Delta$	− 1544	+ 9111	
	$\frac{7}{2}\varepsilon-\psi+5\theta+5\Delta$	+ 222	− 735	
	$\frac{3}{2}\varepsilon+\psi+5\theta+5\Delta$	+ 1838	− 11413	
j^2	$\frac{5}{2}\varepsilon+9\theta+8\Delta-\Sigma$	+ 304	− 2460	
	$\frac{7}{2}\varepsilon-\psi+9\theta+8\Delta-\Sigma$	− 42	+ 266	
	$\frac{3}{2}\varepsilon+\psi+9\theta+8\Delta-\Sigma$	− 364	+ 3013	
η^3	$2\theta+2\Delta$	− 1955	+ 14862	
	$6\theta+6\Delta$	− 35276	+ 189348	
	ψ	+ 3312	− 23724	
	$\psi+4\theta+4\Delta$	− 5097	− 4328	
	$-\psi+4\theta+4\Delta$	+ 6177	− 16310	
	$\psi+8\theta+8\Delta$	+ 45199	− 304998	
$\eta^2\eta'$	$2\theta+\Delta$	+ 6733	− 33547	
	$2\theta+3\Delta$	− 3730	+ 1693	
	$6\theta+5\Delta$	+142854	− 673242	
	$\psi+\Delta$	− 9270	+ 61512	
	$-\psi+\Delta$	+ 4207	− 28940	
	$\psi+4\theta+3\Delta$	+ 5323	+ 55061	
	$-\psi+4\theta+3\Delta$	− 13730	+ 9080	
	$\psi+4\theta+5\Delta$	+ 22898	− 84425	
	$\psi+8\theta+7\Delta$	−200024	+1218446	
$\eta\eta'^2$	2θ	− 3268	+ 14164	
	$2\theta+2\Delta$	+ 3445	+ 15177	
	$6\theta+4\Delta$	−190467	+ 772593	
	ψ	+ 12782	− 78712	
	$\psi+2\Delta$	+ 5239	− 35125	
	$\psi+4\theta+2\Delta$	+ 2712	− 60586	
	$-\psi+4\theta+2\Delta$	+ 4409	+ 41693	
	$\psi+4\theta+4\Delta$	− 52183	+ 143461	
	$\psi+8\theta+6\Delta$	+294332	−1600036	
			m'	

TABLE XVc—Continued.

T_2

Unit = 1″

	Sin	w^0	w	w^2
η'^3	$2\theta+\Delta$	+ 3479	− 17883	
	$6\theta+3\Delta$	+ 83314	− 283500	
	$\psi+\Delta$	− 7839	+ 47423	
	$-\psi+4\theta+\Delta$	+ 6634	− 36904	
	$\psi+4\theta+3\Delta$	+ 27512	− 44330	
	$\psi+8\theta+5\Delta$	− 144023	+ 688658	
$j^2\eta$	$2\theta+2\Delta$	+ 10709	− 50725	
	$2\theta+\Delta-\Sigma$	− 1732	+ 8521	
	$6\theta+5\Delta-\Sigma$	− 7799	+ 50227	
	ψ	− 12782	+ 78712	
	$\psi+\Delta+\Sigma$	+ 11006	− 60629	
	$\psi+4\theta+3\Delta-\Sigma$	+ 4022	− 29208	
	$-\psi+4\theta+3\Delta-\Sigma$	− 3616	+ 27235	
	$\psi+4\theta+4\Delta$	− 28408	+ 176052	
	$\psi+8\theta+7\Delta-\Sigma$	+ 9526	− 75678	
$j^2\ \eta'$	$2\theta+\Delta$	− 7475	+ 36068	
	$2\theta+2\Delta-\Sigma$	+ 159	− 2967	
	$6\theta+4\Delta-\Sigma$	+ 11564	− 66719	
	$\psi+\Delta$	+ 11762	− 75153	
	$\psi+\Sigma$	− 6024	+ 38182	
	$-\psi+4\theta+2\Delta-\Sigma$	+ 7090	− 45771	
	$\psi+4\theta+3\Delta$	+ 35006	− 199168	
	$\psi+4\theta+4\Delta-\Sigma$	+ 1108	− 281	
	$\psi+8\theta+6\Delta-\Sigma$	− 15308	+ 111481	
			m'	

An inspection of the preceding table, which is typical of all the trigonometric series under consideration, shows readily that any function of this type is of the form

$$\Sigma k' \sin K' + \Sigma k \sin (K \pm \psi) = \Sigma k' \sin K' + \Sigma k \sin K \cos \psi \pm \Sigma k \cos K \sin \psi$$

or

$$\Sigma k' \cos K' + \Sigma k \cos (K \pm \psi) = \Sigma k' \cos K' + \Sigma k \cos K \cos \psi \mp \Sigma k \sin K \sin \psi$$

or, more briefly,

$$a + b \cos \psi + c \sin \psi$$

where a, b, c are trigonometric series and can be written by inspection from the tabulated function.

Hence, in v. Zeipel's notation (Z 54, eq. 96),

$$T_i = X_i + Y_i \cos \psi + Z_i \sin \psi$$

and the integral may be written

$$W_i = x_i + y_i \cos \psi + z_i \sin \psi$$

The functions T and W are to be used in this form in solving equations (95).

Considering only first order in the mass in T

$$T_2 = X_2 + Y_2 \cos \psi + Z_2 \sin \psi$$

where

$$X_2 = \Sigma k' \sin K';\quad Y_2 = \Sigma k \sin K;\quad Z_2 = \pm \Sigma k \cos K$$

or, X_2 is the part of T_2 which is independent of ψ, Y_2 is a trigonometric sine series having the same numerical coefficients as the part of T_2 which contains ψ in the argument, but in which ψ is omitted from the argument, and Z_2 is the corresponding cosine series.

Considering the first two of the eqs. (95), the first one states that W_1 is not a function of ε alone, or,

$$W_1 - [W_1] = 0;\quad W_1 = [W_1].$$

Making use of this fact in the second, W_1 can be obtained from (95_2). (See Z 54.) Introducing the auxiliary functions ψ_1 and u_1, defined by (99) and (101), the differential equation for W_1 is replaced by the equivalent differential equations, (100) and (102), for ψ_1 and u_1.

The series

$$[X_2] - \eta[Y_2]$$

and

$$[Y_2] \cos \psi + [Z_2] \sin \psi$$

can be written by inspection from T_2, or, better, the integration itself can be performed in part at the same time.

The function ϕ_1 is given by Z 59, eq. (103), or,

$$\phi_1 - w = \frac{2}{w}\int^{\theta} ([X_2] - \eta[Y_2])d\theta - \frac{2}{w^3}\left\{\int^{\theta}([X_2] - \eta[Y_2])d\theta\right\}^2 + \frac{4}{w^5}\left\{\int^{\theta}([X_2] - \eta[Y_2])d\theta\right\}^3 + \dots$$

From the table of T_2, page 82, it is not difficult to write immediately

$$\int^{\theta}([X_2] - \eta[Y_2])d\theta$$

The terms of higher order must be obtained by the usual method for the mechanical multiplication of series. A logarithmic multiplication is the most direct.

In each term in the expression for ϕ_1 the terms of lowest rank must be of the first rank. Recalling the tabulation of factors in Z 53, w, $\frac{m'}{w}$, $\frac{m'^2}{w^3}$, $\frac{m'^3}{w^5}$, etc., are all of first rank. But the coefficient for a given argument consists of three terms in ascending powers of w. Hence $\phi_1 - w$, within the limits of the given tabulation for T_2, is of rank 1, 2, 3 for each order in the mass. Table XVI, giving $\phi_1 - w$, is tabulated with double headings. The three subheadings indicate the expansion of the coefficients in a Taylor's series and the main headings give the factors in the development of the radical in Z 59, eq. (103).

Having found ϕ_1, its reciprocal, ϕ_1^{-1}, inclusive of first order in the mass, is given by

$$\phi_1^{-1} = \frac{1}{w} - \frac{2}{w^3}\int^{\theta} ([X_2] - \eta[Y_2])d\theta$$

The second term is the negative of the first three columns of Table XVI multiplied by w^{-2}. The product of $2\phi_1^{-1}$ and that part of T_2 which contains ψ gives $\frac{du_1}{d\theta}$, and integration with respect to θ gives u_1, tabulated in XVIII. The function u_1 is of first and higher rank because the factor ϕ_1^{-1} is of rank minus one and T_2 is of second rank.

From Table XVIII y_1 can be read by inspection, and ηy_1 added to Table XVI gives x_1, tabulated in Table XVII. The function W_1 is the sum of Tables XVII and XVIII.

In the integration those terms whose arguments are independent of θ are of the nature of constants. In accordance with the condition that there may be secular terms in θ, the integral contains such terms as the following:

$$\theta \cdot k \sin (\psi + \Delta).$$

As the constant of integration

$$\theta_0 \cdot k \sin (\psi + \Delta)$$

is added. Hence the integral contains terms such as

$$(\theta - \theta_0)\ k \sin (\psi + \Delta,$$

where θ_0 is the value of θ for the time $t=0$.

In passing, it should be noted that, in order that the expansion of Z 59, eq. (103), shall represent the function, we must have

$$\left|\frac{4}{w^2}\int^{\theta} ([X_2] - \eta[Y_2])\, d\,\theta\right| < 1$$

and this condition should be tested for a given planet before applying this method of determining the perturbations.

To the computer the extent of auxiliary tables, the arrangement of series in logarithms or natural numbers, in seconds of arc or radians, inclusive or exclusive of numerical factors, and foresight in combining operations—all these are of the greatest importance. But considerations of this kind would carry the reader into complicated details which are best left to the computer's own judgment.

On the other hand, general considerations about the extent of the published tables are of importance in the discussion of the accuracy of the final tables. Yet, for a given limit of accuracy, it is so difficult to determine, for each table, the highest powers of m', w, η, η', and j^2 that little or nothing is said about it in connection with individual tables, but the discussion is reserved until later.

Table XVI.

$\phi_1 - w = x_1 - \eta y_1 = [(1 - e \cos \epsilon) \overline{W}_1]$

Unit = 4th decimal of a radian.

	Cos	w^{-1}			w^{-2}			w^{-4}	
		w^0	w	w^2	w^0	w	w^2	w^0	w
η^2					−0.0460	+0.231	−0.52		
η'^2					−0.0060	+0.040	−0.127		
$\eta\ \eta'$	Δ				+0.0331	−0.195	+0.53		
η	$2\theta+2\Delta$	+ 42.889	− 107.72	+ 106.7					
η'	$2\theta+\ \Delta$	− 15.427	+ 52.39	− 75.2					
η^2	$4\theta+4\Delta$	− 122.10	+ 484.1	− 813	−0.0460	+0.231	−0.52		
$\eta\ \eta'$	$4\theta+3\Delta$	+ 357.75	− 1183.5	+1650	+0.0331	−0.195	+0.53		
η'^2	$4\theta+2\Delta$	− 258.93	+ 687.2	− 779	−0.0060	+0.040	−0.127		
j^2	$4\theta+3\Delta-\Sigma$	− 14.75	+ 71.7	− 164					
η^3	$2\theta+2\Delta$	+ 28.2	− 433		+0.262	−1.70		+0.0003	−0.0022
η^3	$6\theta+6\Delta$	+ 428	− 2295		+0.262	−1.70		+0.0001	−0.0008
$\eta^2\eta'$	$2\theta+\ \Delta$	− 316.1	+ 1592		−0.767	+4.46		−0.00021	+0.0018
$\eta^2\eta'$	$2\theta+3\Delta$	+ 108.5	− 49		−0.094	+0.69		−0.00011	+0.0009
$\eta^2\eta'$	$6\theta+5\Delta$	−1889	+ 8902		−0.86	+5.2		−0.0001	+0.001
$\eta\ \eta'^2$	2θ	+ 237.6	− 1030		+0.555	−2.87		+0.00004	−0.0002
$\eta\ \eta'^2$	$2\theta+2\Delta$	− 125.3	− 552		+0.276	−1.85		+0.00008	−0.0009
$\eta\ \eta'^2$	$6\theta+4\Delta$	+2770	−11237		+0.83	−4.7			
η'^3	$2\theta+\ \Delta$	− 168.7	+ 867		−0.200	+1.21			
η'^3	$6\theta+3\Delta$	−1346	+ 4581		−0.200	+1.21			
$j^2\eta$	$2\theta+2\Delta$	− 389.4	+ 1846	−4498					
$j^2\eta$	$2\theta+\ \Delta-\Sigma$	+ 126.0	− 620		+0.032	−0.23			
$j^2\eta$	$6\theta+5\Delta-\Sigma$	+ 113	− 731		+0.032	−0.23			
$j^2\ \eta'$	$2\theta+\ \Delta$	+ 362.4	− 1749						
$j^2\ \eta'$	$2\theta+2\Delta-\Sigma$	− 7.7	+ 144		−0.011	+0.09			
$j^2\ \eta'$	$6\theta+4\Delta-\Sigma$	− 187	+ 1078		−0.011	+0.1			
		m'			m'^2			m'^3	

Table XVII.

x_1

Unit = 1″.

	Cos	w^{-1}			w^{-3}		
		w^0	w	w^2	w^0	w	w^2
η	$2\theta+2\Delta$	+ 1179.6	− 2963	+ 2935			
η'	$2\theta+\ \Delta$	− 318.2	+ 1081	− 1552			
η^2					− 0.95	+ 4.8	
η^2	$4\theta+4\Delta$	− 3358	+ 13313	− 22356	− 1.27	+ 6.4	
$\eta\ \eta'$	Δ				+ 0.68	− 4.0	
$\eta\ \eta'$	$4\theta+3\Delta$	+ 8609	− 28481	+ 39702	+ 0.79	− 4.7	
η'^2					− 0.12	+ 0.8	
η'^2	$4\theta+2\Delta$	− 5341	+ 14175	− 16063	− 0.12	+ 0.8	
j^2	$4\theta+3\Delta-\Sigma$	− 304	+ 1479	− 3383			
η^3	$2\theta+2\Delta$	+ 1955	− 14861		+ 7.2	− 46.6	
η^3	$6\theta+6\Delta$	+11758	− 63112		+ 7.2	− 46.6	
$\eta^2\eta'$	$2\theta+\ \Delta$	− 6732	+ 33547		−15.2	+ 88.0	
$\eta^2\eta'$	$2\theta+3\Delta$	+ 3730	− 1691		− 3.8	+ 27.9	
$\eta^2\eta'$	$6\theta+5\Delta$	−47616	+224423		−21.7	+130.0	
$\eta\ \eta'^2$	2θ	+ 3267	− 14165		+ 7.4	− 37.8	
$\eta\ \eta'^2$	$2\theta+2\Delta$	− 3446	− 15176		+ 7.8	− 52.5	
$\eta\ \eta'^2$	$6\theta+4\Delta$	+63489	−257533		+19.0	−108.1	
$j^2\eta$	$2\theta+\ \Delta-\Sigma$	+ 1733	− 8522		+ 0.4	− 3.1	
$j^2\eta$	$6\theta+5\Delta-\Sigma$	+ 2599	− 16744		+ 0.7	− 5.2	
$j^2\eta$	$2\theta+2\Delta$	−10709	+ 50748	−123705			
η'^3	$2\theta+\ \Delta$	− 3479	+ 17880		− 4.1	+ 24.9	
η'^3	$6\theta+3\Delta$	−27772	+ 94500		− 4.1	+ 24.9	
$j^2\ \eta'$	$2\theta+2\Delta-\Sigma$	− 159	+ 2966		− 0.2	+ 1.9	
$j^2\ \eta'$	$6\theta+4\Delta-\Sigma$	− 3855	+ 22240		− 0.2	+ 1.9	
$j^2\ \eta'$	$2\theta+\ \Delta$	+ 7475	− 36070				
	$(\theta-\theta_0)$ sin						
$\eta\ \eta'$	Δ	− 570	+ 2421	− 4950	− 0.45	+ 2.7	−7.2
		m'			m'^2		

TABLE XVIII.

$u_1 = y_1 \cos \psi + z_1 \sin \psi$

Unit = 1″.

	Cos	w^{-1}			w^{-2}		
		w^0	w	w^2	w^0	w	w^2
	$\psi+2\theta+2\Gamma$	+ 294.89	− 740.6	+ 734			
η	$\psi+4\theta+4\Gamma$	− 839.5	+ 3328	− 5586	− 0.316	+ 1.59	− 3.6
η'	$\psi+4\theta+3\Gamma$	+ 1229.8	− 4069	+ 5671	+ 0.114	− 0.67	+ 1.8
η^2	$-\psi+2\theta+2\Gamma$	+ 396	+ 1494	− 9351	− 2.62	+ 16.8	
η^2	$\psi+2\theta+2\Gamma$	+ 978	− 7431	+ 23105	+ 4.42	− 28.4	
η^2	$\psi+6\theta+6\Gamma$	+ 2940	− 15782	[+ 37112]	+ 1.80	− 11.7	
$\eta\ \eta'$	$-\psi+2\theta+\ \Gamma$	+ 2068	− 10582	− 39010	+ 6.18	− 36.9	
$\eta\ \eta'$	$\psi+2\theta+3\Gamma$	+ 1492	− 677	− 13098	− 1.91	+ 13.6	
$\eta\ \eta'$	$\psi+2\theta+\ \Gamma$	− 2280	+ 11302	− 28348	− 5.57	+ 32.8	
$\eta\ \eta'$	$\psi+6\theta+5\Gamma$	− 8658	+ 40793	− 83730	− 3.95	+ 23.6	
η'^2	$-\psi+2\theta$	− 1634	+ 7081	− 16199	− 4.04	+ 21.4	
η'^2	$\psi+2\theta+2\Gamma$	− 861	− 3794	+ 22127	+ 2.12	− 14.4	
η'^2	$\psi+6\theta+4\Gamma$	+ 6349	− 25753	+ 45318	+ 1.90	− 10.8	
j^2	$-\psi+2\theta+\ \Gamma-\Sigma$	− 866	+ 4260	− 10988	− 0.22	+ 1.6	
j^2	$\psi+6\theta+5\Gamma-\Sigma$	+ 260	− 1674	+ 5101	+ 0.07	− 0.5	
j^2	$\psi+2\theta+2\Gamma$	− 2677	+ 12681	− 30930			
η^3	$\psi+4\theta+4\Gamma$	+ 2549	+ 2164		[−11.9]	[+ 89]	
η^3	$-\psi+4\theta+4\Gamma$	− 3089	+ 8155		[+ 3.9]	[− 25]	
η^3	$\psi+8\theta+8\Gamma$	−11300	+ 76250		− 8.9	[+ 70]	
$\eta^2\eta'$	$\psi+4\theta+5\Gamma$	−11449	+ 42212		+ 1.9	− 23	
$\eta^2\eta'$	$\psi+4\theta+3\Gamma$	− 2661	− 27530		[+36.4]	[−241]	
$\eta^2\eta'$	$-\psi+4\theta+3\Gamma$	+ 6865	− 4540		−20.3	[+118]	
$\eta^2\eta'$	$\psi+8\theta+7\Gamma$	+50005	−304611		[+33.8]	[−248]	
$\eta\ \eta'^2$	$\psi+4\theta+4\Gamma$	+26091	− 71730		−10.1	+ 83	
$\eta\ \eta'^2$	$\psi+4\theta+2\Gamma$	− 1356	+ 30293		[−25.5]	[+153]	
$\eta\ \eta'^2$	$-\psi+4\theta+2\Gamma$	− 2204	− 20846		[+28.0]	[−153]	
$\eta\ \eta'^2$	$\psi+8\theta+6\Gamma$	−73583	+400009		[−41.9]	[+284]	
η'^3	$\psi+4\theta+3\Gamma$	−13756	+ 22165		+10.1	− 65	
η'^3	$-\psi+4\theta+\ \Gamma$	− 3317	+ 18452		−12.4	+ 64	
η'^3	$\psi+8\theta+5\Gamma$	+36006	−172164		+16.6	−104	
$j^2\eta$	$\psi+4\theta+3\Gamma-\Sigma$	− 2011	+ 14604		− 1.9	+ 14	
$j^2\eta$	$-\psi+4\theta+3\Gamma-\Sigma$	+ 1808	− 13617		[+ 1.9]	[− 14]	
$j^2\eta$	$\psi+8\theta+7\Gamma-\Sigma$	− 2381	+ 18919		− 1.1	+ 10	
$j^2\eta$	$\psi+4\theta+4\Gamma$	+14204	− 88026		+ 5.7	[− 42]	
$j^2\ \eta'$	$\psi+4\theta+4\Gamma-\Sigma$	− 554	+ 140		+ 0.5	− 4	
$j^2\ \eta'$	$-\psi+4\theta+2\Gamma-\Sigma$	− 3545	+ 22886		− 1.8	+ 14	
$j^2\ \eta'$	$\psi+8\theta+6\Gamma-\Sigma$	+ 3827	− 27870		+ 1.3	− 11	
$j^2\ \eta'$	$\psi+4\theta+3\Gamma$	−17503	+ 99584		[− 3.7]	[+ 28]	
	$(\theta-\theta_0)$ sin						
η	ψ	+ 767.72	− 2820.9	+ 5210	+ 1.265	− 6.35	+14.3
η'	$\psi\quad+\Gamma$	− 569.95	+ 2421.1	− 4950	− 0.455	+ 2.69	− 7.2
η^3	ψ	+ 6624	− 47448		+23.8	[−221.9]	
$\eta^2\eta'$	$\psi\quad+\Gamma$	[−18540]	[+123024]		−73.4	+572.4	
$\eta^2\eta'$	$-\psi\quad+\Gamma$	+ 8414	− 57880		+36.0	−282.2	
$\eta\ \eta'^2$	$\psi\quad+2\Gamma$	+10478	− 70250		+55.2	−374.8	
$\eta\ \eta'^2$	ψ	+25564	−157424		+87.3	−652.8	
η'^3	$\psi\quad+\Gamma$	−15678	+ 94846		−69.9	+438.6	
$j^2\eta$	$\psi\quad+\Gamma+\Sigma$	+22012	−121258	+359162	+ 9.9	− 77.0	
$j^2\eta$	ψ	−25564	+157424	[−511232]	−23.1	+165.0	
$j^2\ \eta'$	$\psi\quad+\Sigma$	−12048	+ 76364	−251640	− 5.2	+ 45.8	
$j^2\ \eta'$	$\psi\quad+\Gamma$	+23524	−150306	+498328	+14.8	−112.0	
		m'			m'^2		

After the determination of W_1, the function $W_2-[W_2]$ is obtained from the solution of Z 53, eq. (95_2). The integral may be written as in Z 63, eqs. (105), (106), or, quite as simply, as follows:

$$W_2-[W_2]=W_2'+\int(T_2-[T_2])d\varepsilon$$

$$W_2'=-\int\left\{\frac{1}{2}(1-e\cos\varepsilon)(w+W_1)-\frac{1}{2}[(1-e\cos\varepsilon)(w+W_1)]\right\}\frac{dW_1}{d\theta}\,d\varepsilon$$

The function W_2' is given in Table XIX.

Anticipating some later developments, for which we shall need

$$[(1-e\cos\varepsilon)W]$$

the function

$$[(1-e\cos\varepsilon)W_2']$$

is tabulated in Table XX.

The determination of $[W_2]$ may be accomplished according to Z 65, eq. (108) – Z 67, eq. (116), or in the manner outlined below, which we regard as preferable.

Repeating Z 65, eq. (107),

$$\phi_1\frac{d[W_2]}{d\theta}+[x_2-\eta y_2+w^2]\frac{dW_1}{d\theta}=\left\{w\phi_1+\frac{3}{4}\left[\left(W_1-\frac{1}{3}\,\Xi_1\right)\left(W_1+\frac{1}{9}\,\Xi_1\right)(1-e\cos\varepsilon)\right]\right\}\frac{dW_1}{d\theta}$$

$$-\left[(1-e\cos\varepsilon)\left\{(w+W_1)\frac{\partial}{\partial\theta}\int(T_2-[T_2])d\varepsilon+\frac{dW_1}{d\theta}\int\overline{(T_2-[T_2])d\varepsilon}\right\}\right]+2[T_3]$$

in which all the known parts are contained on the right-hand side, the development of equivalent equations proceeds in a manner analogous to that for W_1.

Writing

$$T_3=X_3+Y_3\cos\psi+Z_3\sin\psi$$

and introducing

$$\phi_2=[x_2]-\eta[y_2]+w^2$$

and equating parts independent of ψ, coefficients of $\cos\psi$ and coefficients of $\sin\psi$, the three equivalent equations are:

$$\phi_1\frac{d[x_2]}{d\theta}+\phi_2\frac{dx_1}{d\theta}=w\phi_1\frac{dx_1}{d\theta}+\frac{3}{4}\left[(1-e\cos\varepsilon)\left(W_1-\frac{1}{3}\,\Xi_1\right)\left(W_1+\frac{1}{9}\,\Xi_1\right)\right]\frac{dx_1}{d\theta}$$

$$-\left[(1-e\cos\varepsilon)(w+W_1)\frac{\partial}{\partial\theta}\int(X_2-[X_2])d\varepsilon\right]-\left[(1-e\cos\varepsilon)\int\overline{(T_2-[T_2])d\varepsilon}\right]\frac{dx_1}{d\theta}+2[X_3]$$

$$\phi_1\frac{d[y_2]}{d\theta}+\phi_2\frac{dy_1}{d\theta}=w\phi_1\frac{dy_1}{d\theta}+\frac{3}{4}\left[(1-e\cos\varepsilon)\left(W_1-\frac{1}{3}\,\Xi_1\right)\left(W_1+\frac{1}{9}\,\Xi_1\right)\right]\frac{dy_1}{d\theta}$$

$$-\left[(1-e\cos\varepsilon)(w+W_1)\frac{\partial}{\partial\theta}\int(Y_2-[Y_2])d\varepsilon\right]-\left[(1-e\cos\varepsilon)\int\overline{(T_2-[T_2])d\varepsilon}\right]\frac{dy_1}{d\theta}+2[Y_3]$$

$$\phi_1\frac{d[z_2]}{d\theta}+\phi_2\frac{dz_1}{d\theta}=w\phi_1\frac{dz_1}{d\theta}+\frac{3}{4}\left[(1-e\cos\varepsilon)\left(W_1-\frac{1}{3}\,\Xi_1\right)\left(W_1+\frac{1}{9}\,\Xi_1\right)\right]\frac{dz_1}{d\theta}$$

$$-\left[(1-e\cos\varepsilon)(w+W_1)\frac{\partial}{\partial\theta}\int(Z_2-[Z_2])d\varepsilon\right]-\left[(1-e\cos\varepsilon)\int\overline{(T_2-[T_2])d\varepsilon}\right]\frac{dz_1}{d\theta}+2[Z_3].$$

Multiplying the second of these by η and subtracting from the first:

$$\frac{d}{d\theta}\left\{\phi_1\left(\phi_2-\frac{w}{2}\phi_1\right)\right\}=\frac{3}{4}\left[(1-e\cos\varepsilon)\left(W_1-\frac{1}{3}\,\Xi_1\right)\left(W_1+\frac{1}{9}\,\Xi_1\right)\right]\frac{d\phi_1}{d\theta}$$

$$-\left[(1-e\cos\varepsilon)(w+W_1)\frac{\partial}{\partial\theta}\int\{X_2-\eta Y_2-[X_2-\eta Y_2]\}d\varepsilon\right]$$

$$-\left[(1-e\cos\varepsilon)\int\overline{(T_2-[T_2])d\varepsilon}\right]\frac{d\phi_1}{d\theta}+2[X_3-\eta Y_3].$$

Multiplying the second by $\cos\psi$, the third by $\sin\psi$ and adding:

$$\phi_1\frac{d}{d\theta}(u_2-wu_1)=-\phi_2+\frac{3}{4}\left[(1-e\cos\varepsilon)\left(W_1-\frac{1}{3}\Xi_1\right)\left(W_1+\frac{1}{9}\Xi_1\right)\right]\frac{du_1}{d\theta}+2([Y_3]\cos\psi+[Z_3]\sin\psi)$$

$$-\left[(1-e\cos\varepsilon)(w+W_1)\frac{\partial}{\partial\theta}\int\{Y_2\cos\psi+Z_2\sin\psi-[Y_2\cos\psi+Z_2\sin\psi]\}d\varepsilon\right]$$

$$-\left[(1-e\cos\varepsilon)\int\overline{(T_2-[T_2])d\varepsilon}\right]\frac{du_1}{d\theta}$$

in which

$$u_2=[y_2]\cos\psi+[z_2]\sin\psi$$

and $[X_3]$, $[Y_3]$, $[Z_3]$ are read by inspection from T_3, which is to be determined as follows:

If Z 50, eqs. (89), (90), are written in the form

$$\frac{1}{\cos^2\varphi}\,\frac{4\rho r^{-2}}{a^3\cos^2\varphi}[1-\cos(f-\omega)]=4\{1-2\eta\cos\varepsilon-\cos(\varepsilon-\psi)+\eta\cos(2\varepsilon-\psi)+\eta\cos\psi+\ldots\ldots\}$$

$$\frac{1}{\cos^2\varphi}\left\{\frac{r^{-2}}{a^2}+\frac{2\rho r^{-2}}{a^3\cos^2\varphi}[1-\cos(f-\omega)]\right\}=3+14\eta^2-8\eta\cos\varepsilon+2\eta^2\cos2\varepsilon-2\cos(\varepsilon-\psi)$$
$$-8\eta^2\cos(\varepsilon-\psi)+2\eta\cos(2\varepsilon-\psi)+2\eta\cos\psi+\ldots\ldots$$

then $T_{\overline{W}}$ and T'_ν, given by Z 49, eqs. (84), (85), in connection with Z 50, eq. (87), are given by

$$T_{\overline{W}}=-T_2-4\{1-2\eta\cos\varepsilon-\cos(\varepsilon-\psi)+\eta\cos(2\varepsilon-\psi)+\eta\cos\psi+\ldots\}$$
$$\Sigma S_{p.q}(n+r.-n+s)\eta^p\eta'^q j^{2t}\sin A$$

$$T'_\nu=\{3+14\eta^2-8\eta\cos\varepsilon+2\eta^2\cos2\varepsilon-2\cos(\varepsilon-\psi)-8\eta^2\cos(\varepsilon-\psi)+2\eta\cos(2\varepsilon-\psi)+2\eta\cos\psi$$
$$+\ldots\ldots\}\Sigma S_{p.q}(n+r.-n+s)\eta^p\eta'^q j^{2t}\sin A-\frac{3}{2}(1-w)\frac{\partial T_2}{\partial w}$$

and T_3 (Table XVIIIa) is computed by Z 53, eq. (94), in which

$$\Xi=x+2\eta y,\qquad \Xi_i=x_i+2\eta y_i$$

The function

$$\phi_2-u^2=[x_2]-\eta[y_2]$$

is tabulated in Table XXI; the function

$$u_2=[y_2]\cos\psi+[z_2]\sin\psi$$

is tabulated in Table XXII.

From the latter $[y_2]$ can be read by inspection, and $\eta[y_2]$ added to the former gives $[x_2]$. Finally, (Table XXIIa),

$$[W_2]=[x_2]+[y_2]\cos\psi+[z_2]\sin\psi$$

TABLE XVIIIa.

T_3. Unit=1″

	Sin	w^{-3}			w^{-1}		
		w^0	w	w^2	w^0	w	w^2
	$-\varepsilon+\psi$				+0. 339	− 2. 01	
	$\varepsilon\quad+2\theta+2\Delta$				−0. 375	+ 2. 403	
	$2\varepsilon-\psi+2\theta+2\Delta$				−0. 137	+ 0. 847	
	$\psi+2\theta+2\Delta$				+0. 498	− 3. 223	+ 7. 72
	$2\varepsilon\quad+4\theta+4\Delta$				−0. 438	+ 2. 234	
	$\varepsilon+\psi+4\theta+4\Delta$				+0. 429	− 2. 338	
	$2\varepsilon+\psi+6\theta+6\Delta$				+0. 361	− 2. 372	
η	$2\theta+2\Delta$	−0. 00047	+0. 0036	−0. 0123	+2. 199	−14. 58	+ 33. 34
η	$\varepsilon-\psi+2\theta+2\Delta$				+0. 286	− 3. 85	
η	$-\varepsilon+\psi+2\theta+2\Delta$				−3. 294	+23. 82	
η	ε				−2. 811	+12. 20	
η	ψ	−0. 00038	+0. 0035	−0. 0136	−0. 688	+ 1. 67	− 16. 51
η	$\varepsilon\quad+4\theta+4\Delta$				+0. 432	− 2. 58	
η	$\psi+4\theta+4\Delta$	−0. 00015	+0. 0013	−0. 0048	−4. 536	+35. 80	− 95. 79
η	$\varepsilon+\psi+2\theta+2\Delta$				+1. 017	− 6. 333	
η	$\varepsilon+\psi+6\theta+6\Delta$				−3. 219	+22. 43	
η'	$2\theta+\ \Delta$	+0. 00017	−0. 0014	+0. 0055	−2. 520	+14. 78	− 31. 95
η'	$\varepsilon-\psi+2\theta+\ \Delta$				−1. 253	+10. 20	
η'	$-\varepsilon+\psi+2\theta+\ \Delta$				+4. 372	−28. 56	
η'	$\varepsilon\quad+\ \Delta$				−0. 404	+ 4. 06	
η'	$\psi\quad+\ \Delta$	+0. 00014	−0. 0014	+0. 0060	+1. 188	−11. 30	+ 34. 07
η'	$\varepsilon\quad+4\theta+3\Delta$				−0. 224	+ 1. 53	
η'	$\psi+4\theta+3\Delta$	+0. 00005	−0. 0005	+0. 0021	+6. 480	−47. 37	+120. 37
η'	$\varepsilon+\psi+2\theta+3\Delta$				+0. 214	− 1. 66	
η'	$\varepsilon+\psi+6\theta+5\Delta$				+5. 977	−36. 82	
	$(\theta-\theta_0)$ cos						
η	$2\theta+2\Delta$	−0. 00188	+0. 0143	−0. 0489	−1. 141	+ 7. 14	− 20. 54
η	$\varepsilon-\psi+2\theta+2\Delta$				+0. 235	− 1. 12	
η	$-\varepsilon+\psi+2\theta+2\Delta$				+1. 12	− 7. 39	
η	ψ	−0. 00059	+0. 0051	−0. 0189	−0. 357	+ 2. 62	− 8. 20
η	$\varepsilon\quad+4\theta+4\Delta$				−0. 975	+ 7. 39	
η	$\psi+4\theta+4\Delta$	+0. 00155	−0. 0141	+0. 0540	+0. 939	− 7. 27	+ 23. 43
η	$\varepsilon+\psi+2\theta+2\Delta$				−1. 12	+ 7. 39	
η'	$2\theta+\ \Delta$	+0. 00068	−0. 0058	+0. 0222	+0. 847	− 5. 79	+ 18. 12
η'	$\varepsilon-\psi+2\theta+\ \Delta$				−0. 17	+ 0. 93	
η'	$-\varepsilon+\psi+2\theta+\ \Delta$				−0. 828	+ 5. 96	
η'	$\psi\quad+\ \Delta$	+0. 00021	−0. 0020	+0. 0085	+0. 265	− 2. 10	+ 7. 15
η'	$\varepsilon\quad+4\theta+3\Delta$				+0. 724	− 5. 90	
η'	$\psi+4\theta+3\Delta$	−0. 00056	+0. 0056	−0. 0239	−0. 697	+ 5. 80	− 20. 35
η'	$\varepsilon+\psi+2\theta+3\Delta$				+0. 828	− 5. 96	
		m'^3			m'^2		

TABLE XIX.

W_2'

Unit=1″.

Cos		w^0			w^{-2}			w^{-4}	
		w^0	w	w^2	w^0	w	w^2	w^0	w
	$-\psi+\epsilon$				+0. 2108	− 1. 059	+2. 379		
	$\psi+\epsilon+4\theta+4\Delta$				−0. 2108	+ 1. 059	−2. 379		
η	ϵ				+0. 843	− 4. 236			
η	$\epsilon+4\theta+4\Delta$				−0. 843	+ 4. 236			
η	$-\psi+\epsilon-2\theta-2\Delta$	− 294. 9	+ 740. 6	−733. 9	−1. 200	+ 7. 772			
η	$-\psi+\epsilon+2\theta+2\Delta$				−0. 875	+ 5. 583			
η	$\psi+\epsilon+2\theta+2\Delta$	+ 294. 9	− 740. 6	+733. 9	+0. 274	− 1. 697			
η	$\psi+\epsilon+6\theta+6\Delta$				+1. 800	−11. 658			
η	$-\psi+2\epsilon$				−0. 105	+ 0. 529			
η	$\psi+2\epsilon+4\theta+4\Delta$				+0. 105	− 0. 529			
η'	$\epsilon\quad+\Delta$				−0. 227	+ 1. 344			
η'	$\epsilon+4\theta+3\Delta$				+0. 227	− 1. 344			
η'	$-\psi+\epsilon-2\theta-\Delta$				+1. 758	−10. 233			
η'	$-\psi+\epsilon+2\theta+\Delta$				+1. 083	− 6. 493			
η'	$\psi+\epsilon+2\theta+3\Delta$				−0. 204	+ 1. 377			
η'	$\psi+\epsilon+6\theta+5\Delta$				−2. 637	+15. 350			
η^2	$-\psi+\epsilon$	+ 384	−1410						
η^2	$-\psi+\epsilon-4\theta-4\Delta$	+1679	−6656						
η^2	$\epsilon+2\theta+2\Delta$	+1180	−2963						
η^2	$-\epsilon+2\theta+2\Delta$	−1180	+2963						
η^2	$\psi+\epsilon$	− 384	+1410						
η^2	$\psi+\epsilon+4\theta+4\Delta$	−1679	+6656						
$\eta\,\eta'$	$-\psi+\epsilon\quad-\Delta$	− 285	+1210						
$\eta\,\eta'$	$-\psi+\epsilon-4\theta-3\Delta$	−2460	+8138						
$\eta\,\eta'$	$\epsilon+2\theta+\Delta$	− 318	+1081						
$\eta\,\eta'$	$\epsilon-2\theta-\Delta$	+ 318	−1081						
$\eta\,\eta'$	$\psi+\epsilon\quad+\Delta$	+ 285	−1210						
$\eta\,\eta'$	$\psi+\epsilon+4\theta+3\Delta$	+2460	−8138						
	$(\theta-\theta_0)$ sin								
η	$-\psi+\epsilon-2\theta-2\Delta$				+0. 549	− 3. 40		+0. 00090	−0. 0068
η	$\psi+\epsilon+2\theta+2\Delta$				−0. 549	+ 3. 40		−0. 00090	+0. 0068
η'	$-\psi+\epsilon-2\theta-\Delta$				−0. 407	+ 2. 75		−0. 00032	+0. 0027
η'	$\psi+\epsilon+2\theta+3\Delta$				+0. 407	− 2. 75		+0. 00032	−0. 0027
		m'			m'^2			m'^3	

Table XX.

$[(1-e\cos\varepsilon)\,W'_2]$ Unit=4th decimal of a radian

Cos		w^0			w^{-2}			w^{-4}	
		w^0	w	w^2	w^0	w	w^2	w^0	w
					+0.01022	−0.0513	+0.115		
η^2		+ 18.6	− 68		+0.187	−1.76		+0.00043	−0.0039
η'^2					+0.296	−2.46		+0.00020	−0.0020
j^2					−0.186	+1.34	−4.8		
$\eta\eta'$	Δ	− 13.8	+ 59		−0.529	+4.34		−0.00075	+0.0065
η	$2\theta+2\Delta$	− 14.29	+ 35.9	− 36	−0.1006	+0.647	−1.91	−0.000055	+0.00041
η'	$2\theta+\Delta$				+0.1377	−0.811	+2.19	+0.000020	−0.00017
η^2	$4\theta+4\Delta$	+ 81.4	− 323		+0.477	−3.64			
$\eta\eta'$	$4\theta+3\Delta$	− 119.2	+ 395		−1.295	+9.36			
η'^2	$4\theta+2\Delta$				+0.921	−6.09			
j^2	$4\theta+3\Delta-\Sigma$				+0.036	−0.32			
	$(\theta-\theta_0)$ sin								
η	$2\theta+2\Delta$				−0.0266	+0.165	−0.49	−0.000044	+0.00033
η'	$2\theta+\Delta$				+0.0198	−0.134	+0.43	+0.000016	−0.00013
η^2	$4\theta+4\Delta$				+0.151	−1.16		+0.00031	−0.0027
$\eta\eta'$	$4\theta+3\Delta$				−0.334	+2.47		−0.00052	+0.0045
η'^2	$4\theta+2\Delta$				+0.165	−1.24		+0.00015	−0.0014
		m'			m'^2			m'^3	

TABLE XXI.

$\phi_2 - w^2 = [x_2] - \eta[y_2] = [(1 - e \cos \varepsilon)\ [\overline{W_2}]]$.

Unit = 4th decimal of a radian.

	Cos	w^{-4}			w^{-2}			w^0		
		w^0	w	w^2		w	w^2	w^0	w	w^2
η^2		+0. 000378	−0. 00343		[+0. 0052]	[− 0. 022]	+ 0. 03			
η'^2		+0. 000204	−0. 00188		[−0. 0030]	[+ 0. 020]	− 0. 06			
$\eta\eta'$	Δ	−0. 000702	+0. 00609		[+0. 0063]	[− 0. 039]	+ 0. 12			
η	$2\theta+2\Delta$	———	———	+0. 00002	−0. 1765	+ 1. 155	− 3. 001	− 13. 162	[+30. 862]	− 25. 5
η'	$2\theta+\Delta$	———	———	−0. 00001	+0. 2640	− 1. 547	+ 3. 747			
η^2	$4\theta+4\Delta$	[+0. 000108]	[−0. 00104]		[+0. 5874]	[− 4. 946]	+ 8. 72	+ 65. 300	−249. 32	−431
$\eta\eta'$	$4\theta+3\Delta$	[−0. 000034]	[+0. 00047]		[−1. 9416]	[+14. 295]	−25. 3	−110. 51	+ 347. 1	+388
η'^2	$4\theta+2\Delta$	[−0. 000262]	[+0. 00172]		[+1. 4798]	[− 9. 83]	+19. 8			
j^2	$4\theta+3\Delta-\Sigma$	[−0. 000026]	[+0. 00026]		+0. 0164	− 0. 222	+ 0. 52			
	$(\theta-\theta_0)$ sin									
η	$2\theta+2\Delta$	−0. 000132	+0. 00099	−0. 00338	[−0. 0798]	+ 0. 4937	− 1. 421			
η'	$2\theta+\Delta$	+0. 000047	−0. 00040	+0. 00154	+0. 0592	− 0. 4005	+ 1. 252			
η^2	$4\theta+4\Delta$	+0. 000690	−0. 00616		+0. 3408	− 2. 604	+ 8. 04			
$\eta\eta'$	$4\theta+3\Delta$	−0. 001166	+0. 01017		[−0. 7523]	+ 5. 564	−17. 283			
$\eta\eta'$	Δ	[+0. 000002]	[−0. 00001]		[0. 0000]	[− 0. 39]	− 5. 97			
η'^2	$4\theta+2\Delta$	+0. 000330	−0. 00308		+0. 3707	− 2. 801	+ 9. 024			
	$(\theta-\theta_0)^2$ cos									
η^2		[+0. 000171]	[−0. 00149]		+0. 0520	− 0. 382	+ 1. 055			
$\eta\eta'$	Δ	[−0. 000188]	[+0. 00177]		−0. 0771	+ 0. 610	− 1. 798			
η'^2		[+0. 000046]	[−0. 00046]		+0. 0286	− 0. 243	+ 0. 765			
		m'^3			m'^2			m'		

TABLE XXII

$u_2=[y_2]\cos\psi+[z_2]\sin\psi$

Unit=1″.

	Cos	w^{-4}			w^{-3}			w^0		
		w^0	w	w^2	w^0	w	w^2	w	w	w^2
	$\psi+2\theta+2\Delta$	+0.00022	−0.00170	+0.0059	−0.6139	+ 4.059	−10.30			
η	$\psi+4\theta+4\Delta$	+0.00132	−0.0123	+0.046	+2.790	[−23.39]	+70.40	+167.4	−637	+985
η'	$\psi+4\theta+3\Delta$	−0.00092	+0.0083	−0.032	[−4.559]	[+33.80]				
	$(\theta-\theta_0)\sin$									
η	ψ	−0.00671	[+0.0620]	−0.240	+1.015	[+ 8.43]				
η	$\psi+4\theta+4\Delta$	+0.00158	−0.0141	+0.056	+0.781	− 5.96	+20.02			
η'	$\psi\quad+\ \Delta$	+0.00668	[−0.0577]	+0.220	+3.249	[−30.12]				
η'	$\psi+4\theta+3\Delta$	−0.00068	+0.0066	−0.029	−0.580	+ 4.76				
	$(\theta-\theta_0)^2\cos$									
η	ψ	−0.00118	+0.01023	−0.040	−0.3572	+ 2.624				
η'	$\psi\quad+\ \Delta$	+0.00064	−0.0061	+0.026	+0.2652	− 2.100				
		m'^3			m'^2			m'		

TABLE XXII*a*.

$[W_2]$. Unit=1″

	Cos	w^{-1}			w^0	
		w^0	w	w^2	w^0	w
	$\psi+2\theta+2\Delta$	− 0.614	+ 4.059	−10.3		
η	$2\theta+2\Delta$	− 4.255	+27.89		− 271.5	+ 636.6
η	$\psi+4\theta+4\Delta$	+ 2.791	−23.39		+ 167.4	− 637.4
η'	$2\theta+\Delta$	+ 5.444	−31.91			
η'	$\psi+4\theta+3\Delta$	− 4.558	+33.80			
η^2		+ 0.11				
η^2	$4\theta+4\Delta$	+14.90			+1514	−5780
η^2	$\psi+2\theta+2\Delta$				+1360	−3387
η^2	$\psi+6\theta+6\Delta$				−1227	+6415
η^2	$-\psi+2\theta+2\Delta$				− 273	+ 179
$\eta\eta'$	Δ	+ 0.13				
$\eta\eta'$	$4\theta+3\Delta$	−44.62			−2279	+7160
$\eta\eta'$	$\psi+2\theta+\Delta$				− 646	+2452
$\eta\eta'$	$\psi+2\theta+3\Delta$				− 291	+ 536
$\eta\eta'$	$\psi+6\theta+5\Delta$				+1974	−9002
$\eta\eta'$	$-\psi+2\theta+\Delta$				− 222	+1012
η'^2		− 0.06				
η'^2	$4\theta+2\Delta$	+30.53				
j^2	$4\theta+3\Delta-\Sigma$	+ 0.34				
	$(\theta-\theta_0)$ sin					
η	$2\theta+2\Delta$	− 1.64	+10.18			
η	ψ	+ 1.014	+ 8.43			
η	$\psi+4\theta+4\Delta$	+ 0.782	− 5.96			
η'	$2\theta+\Delta$	+ 1.22	− 8.26			
η'	$\psi+\Delta$	+ 3.249	−30.12			
η'	$\psi+4\theta+3\Delta$	− 0.579	+ 4.74			
η^2	$4\theta+4\Delta$	+ 7.81				
$\eta\eta'$	Δ	+ 3.25				
$\eta\eta'$	$4\theta+3\Delta$	−16.10				
η'^2	$4\theta+2\Delta$	+ 7.64				
	$(\theta-\theta_0)^2$ cos					
η	ψ	− 0.356	+ 2.62			
η'	$\psi+\Delta$	+ 0.266	− 2.10			
		m'^2			m'	

In the construction of Tables XXI and XXII it is necessary to compute

$$\int (T_2-[T_2])d\varepsilon$$

as one factor of a product, but the more complete tabulation is best arranged as follows. This function gives all of the terms of the first order in the mass in $W_2-[W_2]$. Let

$$W_2''=\int (T_2-[T_2])d\varepsilon$$

and denote first order terms in $W_3-[W_3]$ and $W_4-[W_4]$ by W_3'' and W_4'', respectively. Then because of the similarity in the equations for these functions of successive ranks, the sum

$$W_2''+W_3''+W_4''$$

can be computed by Z 70, eqs. (117), (118), (119). The coefficients $\tilde{F}$, $\tilde{G}$, $\tilde{H}$ are tabulated in Tables XXIII, XXIV, XXV. The mass factor m' is, of course, implicitly contained in the tables.

Eliminating the distinction between ψ and ε, the function is

$$\overline{W}_2''+\overline{W}_3''+\overline{W}_4''$$

in which the coefficients $\overline{A}_{p,q}$, determined by Z 71, eq. (121), are tabulated in Table XXVI.

The coefficients $A_{p,q}$ in the function

$$(1-e\cos\varepsilon)\ (\overline{W}_2''+\overline{W}_3''+\overline{W}_4'')$$

are computed by Z 71, eq. (123) and are tabulated in Table XXVII.

By means of Table XXVII we readily compute

$$[(1-e\cos\varepsilon)\ (\overline{W}_2''+\overline{W}_3''+\overline{W}_4'')]$$

tabulated in Table XXVIII.

Proceeding now to the determination of

$$[(1-e\cos\varepsilon)\ \overline{W}_3]$$

(from which we shall subtract $[(1-e\cos)\ \overline{W}_3'']$, already included in Table XXVIII), we have by Z 53, eq. (95)

$$\begin{aligned}\frac{\partial}{\partial\varepsilon}(W_3-[W_3])=(T_3-[T_3])\ (1-e\cos\varepsilon)(w+\overline{W}_1)\frac{\partial \overline{W}_2}{\partial\theta}-\frac{1}{2}\frac{d\,\overline{W}_1}{d\theta}\Big\{&(1-e\cos\varepsilon)(\overline{W}_3-w\overline{W}_1)\\ -\frac{3}{4}(1-e\cos\varepsilon)(\overline{W}_1-\frac{1}{3}\Xi_1)\ (\overline{W}_1+\frac{1}{9}\Xi_1)-&\Big[(1-e\cos\varepsilon)(w+\overline{W}_1)\frac{\partial \overline{W}_2}{\partial\theta}\Big]\\ -[(1-e\cos\varepsilon)(\overline{W}_3-w\overline{W}_1)]-\frac{3}{4}[(1-e\cos\varepsilon)(&\overline{W}_1-\frac{1}{3}\Xi_1)\ (\overline{W}_1+\frac{1}{9}\Xi_1)]\Big\}\end{aligned}$$

in which all quantities are known. The integration gives $W_3-[W_3]$.

Having computed $W_3-[W_3]$, $[W_3]$ can be obtained from Z 53, eq. (95).

$$\begin{aligned}[(1-e\cos\varepsilon)(w+\overline{W}_1)]\frac{\partial[W_3]}{\partial\theta}+[(1-e\cos\varepsilon)[\overline{W}_3]]\frac{d\,\overline{W}_1}{d\theta}=2[T_4]-\Big[(1-e\cos\varepsilon)(\overline{W}_3-w\overline{W}_1)\frac{\partial \overline{W}_2}{\partial\theta}\Big]\\ +[1-e\cos\varepsilon)w\,\overline{W}_2]\frac{d\,\overline{W}_1}{d\theta}-[(1-e\cos\varepsilon)(w+\overline{W}_1)\frac{\partial}{\partial\theta}(\overline{W}_3-[\overline{W}_3])]-[(1-e\cos\varepsilon)(\overline{W}_3-[\overline{W}_3])]\frac{d\,\overline{W}_1}{d\theta}\end{aligned}$$

The function $[T_4]$, computed from Z 53, eq. (94), is tabulated in Table XXVIII*a*.

In a manner similar to the development of equations for W_1 and $[W_2]$, the right-hand side of this equation, when computed, can be segregated into portions independent of ψ, terms multiplied by $\cos\psi$, and terms multiplied by $\sin\psi$. It is of the form

$$A+B\cos\psi+C\sin\psi$$

where A, B, C are too complicated to be written analytically, but can be written by inspection after the computation has been performed.

The equation can then be written in the three following equivalent equations:

$$\begin{aligned}\phi_1\frac{d[x_3]}{d\theta}+(\phi_3-w^3)\frac{dx_1}{d\theta}&=A\\ \phi_1\frac{d[y_3]}{d\theta}+(\phi_3-w^3)\frac{dy_1}{d\theta}&=B\\ \phi\frac{d[z_3}{d\theta}+(\phi_3-w^3)\frac{dz_1}{d\theta}&=C\end{aligned}$$

in which we define

$$\phi_3-w^3=[x_3]-\eta[y_3]\,.$$

From the first two equations we compute

$$\phi_3 - w^3 = \phi_1^{-1} \int (A - \eta B) d\theta$$

Let

$$u_3 = [y_3] \cos \psi + [z_3] \sin \psi.$$

Then from the second and the third equations

$$u_3 = \int \phi_1^{-1} \left\{ B \cos \psi + C \sin \psi - (\phi_3 - w^3) \frac{du_1}{d\theta} \right\} d\theta$$

By inspection of u_3 the function $[y_3]$ can be written, and $\eta[y_3]$ added to $[x_3] - \eta[y_3]$ gives $[x_3]$. Finally,

$$[W_3] = [x_3] + [y_3] \cos \psi + [z_3] \sin \psi$$

and

$$[(1 - e \cos \varepsilon)\, \overline{W}_3]$$

is readily computed from $\overline{W}_3$, which is tabulated in Table XXVIII*b*.

But this function contains $[(1 - e \cos \varepsilon)\, \overline{W}_3'']$, already included in Table XXVIII. By *Z* 69

$$[(1 - e \cos \varepsilon)\, \overline{W}_3''] = -\frac{w}{2} \left[(1 - e \cos \varepsilon) \int \overline{\left\{ (1 - e \cos \varepsilon) \frac{\partial W_2}{\partial \theta} - [(1 - e \cos \varepsilon) \frac{\partial W_2}{\partial \theta}] \right\} d\varepsilon} \right]$$

Subtracting Table XXVIII*c* from $[(1 - e \cos \varepsilon)\, \overline{W}_3]$ we have

$$[(1 - e \cos \varepsilon)\, (\overline{W}_3 - \overline{W}_3'')]$$

which is tabulated in Table XXIX.

TABLE XXIII.[1]

Unit=1″

	n	0	1	2	3	4	5	6	7	8	9	10
	$\tilde{F}_{0\cdot0}(n.-n)$		−94.26	−271.49	−144.07	−79.94	−45.52	−26.37	−15.46	−9.15	−5.45	−3.26
	$\tilde{F}_{1\cdot0}(n+1.-n)$	+86.3	+138.9	+3.1	−43.0	−53.6	−49.1	−39.8	−30.2	−22.0	−15.7	−10.9
	$\tilde{F}_{1\cdot0}(n-1.-n)$	+86.3	−227.8		+2993.4	+1514.3	+910.6	+574.4	+368.9	+238.4	+154.6	+100.3
	$\tilde{F}_{0\cdot1}(n.-n+1)$	——	−129.7	+87.4	+157.0	+155.6	+129.7	+99.7	+73.1	+52.0	+36.2	+24.9
	$\tilde{F}_{0\cdot1}(n.-n-1)$	——		−4081.7	−2279.4	−1445.9	−943.4	−619.8	−407.6	−267.6	−175.4	−114.6
	$\tilde{F}_{2\cdot0}(n+2.-n)$	−106	−130	−72	−54	−50	−47	−43	−38	−32	−26	−20
	$\tilde{F}_{2\cdot0}(n.-n)$		−1513	−128	+588	+891	+944	+865	+733	+590	+458	+347
	$\tilde{F}_{2\cdot0}(n-2.-n)$	−106	−22	+2245	+9516		−15534	−7969	−5033	−3375	−2312	−1595
	$\tilde{F}_{1\cdot1}(n+1.-n+1)$	+256	+410	+209	+153	+152	+158	+154	+140	+121	+100	+79
	$\tilde{F}_{1\cdot1}(n-1.-n+1)$	+768		−418	−1725	−2167	−2147	−1904	−1581	−1255	−966	−725
	$\tilde{F}_{1\cdot1}(n+1.-n-1)$	+768	+1429	−256	−908	−1171	−1184	−1064	−893	−714	−553	−417
	$\tilde{F}_{1\cdot1}(n-1.-n-1)$	+256	−965	−20365		+43739	+23772	+15609	+10772	+7539	+5290	+3704
	$\tilde{F}_{0\cdot2}(n.-n+2)$	——	−301	−164	−108	−118	−135	−140	−132	−116	−97	−78
	$\tilde{F}_{0\cdot2}(n.-n)$		−1521	+1498	+2612	+2940	+2796	+2428	+1991	+1568	+1199	+897
	$\tilde{F}_{0\cdot2}(n.-n-2)$	——	+3165		−30607	−17790	−12195	−8670	−6206	−4428	−3142	−2216
	$\tilde{F}_{0\cdot0}(n+1.-n+1)+\sigma$	−158	−576	−513	−406	−302	−217	−152	−105	−71	−48	−32
	$\tilde{F}_{0\cdot0}(n-1.-n-1)-\sigma$	−158	——	+855		−1268	−578	−325	−196	−122	−77	−49
	$\tilde{F}_{0\cdot0}(n+1.-n-1)+\delta$	+475	+1152	+855	+609	+423	+289	+195	+131	+87	+57	+38
	$\tilde{F}_{0\cdot0}(n-1.-n+1)-\delta$	+475		+855	+609	+423	+289	+195	+131	+87	+57	+38
Factor w	$\tilde{F}_{0\cdot0}(n.-n)$		+392.9	+908.1	+580.8	+376.6	+245.2	+159.8	+104.1	+67.7	+44.0	+28.5
	$\tilde{F}_{1\cdot0}(n+1.-n)$	−256	−598	−341	−53	+106	+172	+184	+168	+141	+113	+87
	$\tilde{F}_{1\cdot0}(n-1.-n)$	−256	+637		−18877	−8969	−5669	−3852	−2677	−1871	−1307	−911
	$\tilde{F}_{0\cdot1}(n.-n+1)$	——	+483	−37	−429	−594	−609	−547	−456	−363	−279	−209
	$\tilde{F}_{0\cdot1}(n.-n-1)$	——		+22406	+11719	+7904	+5624	+4044	+2903	+2071	+1467	+1032
	$\tilde{F}_{2\cdot0}(n+2.-n)$	+449	+747	+455	+293	+231	+218	+218	+214	+201	+182	
	$\tilde{F}_{2\cdot0}(n.-n)$		[+7969]	(+1846)	+2549	−1403	−3325	−4111	−4200	−3889	−3388	
	$\tilde{F}_{2\cdot0}(n-2.-n)$	+449	+216	(−4830)	[−12320]		+153610	+67367	+42222	+29290	+21073	
	$\tilde{F}_{1\cdot1}(n+1.-n+1)$	−855	−1898	−1183	−759	−663	−712	−778	−803	−778	−714	
	$\tilde{F}_{1\cdot1}(n-1.-n+1)$	−3589		−2303	[+4936]	+9000	+10823	+11073	+10350	+9108	+7680	
	$\tilde{F}_{1\cdot1}(n+1.-n-1)$	−3589	(−7249)	−6243	−777	+2574	+4411	+5139	+5139	+4711	+4092	
	$\tilde{F}_{1\cdot1}(n-1.-n-1)$	−855	(+2886)	+6428		−397850	−182750	−119275	−85560	−63223	−47093	
	$\tilde{F}_{0\cdot2}(n.-n+2)$	——	+1160	+855	+571	+558	+661	+759	+802	+786	+719	
	$\tilde{F}_{0\cdot2}(n.-n)$		+8188	−1947	−8452	−12449	−14128	−14095	−12992	−11336	−9504	
	$\tilde{F}_{0\cdot2}(n.-n-2)$	——	−2036		+252940	+122720	+83854	+62371	+47468	+36194	+27464	

	$\tilde{F}_{0\cdot0}(n+1.-n+1)+\sigma$	+ 602	+2116	+ 2281	+ 2107	+ 1788	+ 1437	+ 1113	+ 840	+ 620	+ 451	
	$\tilde{F}_{0\cdot0}(n-1.-n-1)-\sigma$	+ 602	———	− 1067		+ 13304	+ 5276	+ 2964	+ 1849	+ 1203	+ 809	
	$\tilde{F}_{0\cdot0}(n+1.-n-1)+\delta$	−2440	−5383	− 4487	− 3566	− 2744	− 2060	− 1518	− 1102	− 790	− 560	
	$\tilde{F}_{0\cdot0}(n-1.-n+1)-\delta$	−2440		− 4487	− 3566	− 2744	− 2060	− 1518	− 1102	− 790	− 560	
Factor n^2	$\tilde{F}_{0\cdot0}(n.-n)$		− 765	− 1434	− 1080	− 811	− 602	− 442	− 320	− 230	− 163	− 115
	$\tilde{F}_{1\cdot0}(n+1.-n)$	+ 342	+1207	+ 1261	+ 719	+ 239	− 88	− 276	− 359	− 373	− 349	
	$\tilde{F}_{1\cdot0}(n-1.-n)$	+ 342	− 852		+ 70290	+ 27660	+ 17430	+ 12430	+ 9221	+ 6920	+ 5193	
	$\tilde{F}_{0\cdot1}(n.-n+1)$	———	− 808	− 466	+ 260	+ 841	+ 1171	+ 1291	+ 1248	+ 1126	+ 966	
	$\tilde{F}_{0\cdot1}(n.-n-1)$	———		−77090	− 32330	− 21630	− 16210	− 12530	− 9728	− 7506	− 5740	

TABLE XXIV.

Unit=1''

$\tilde{G}_{0\cdot0}(n.-n)$	− 43.141	− 18.35	+ 33.55	+ 23.56	+ 15.25	+ 9.58	+ 5.94	+ 3.67	+ 2.25	+ 1.39	+ 0.85
$\tilde{G}_{1\cdot0}(n+1.-n)$	+ 52.8	+ 22.0	+ 27.6	+ 24.2	+ 20.5	+ 16.5	+ 12.6	+ 9.3	+ 6.7	+ 4.8	+ 3.3
$\tilde{G}_{1\cdot0}(n-1.-n)$		+ 448.2	− 273.0	− 279.4	− 225.9	− 168.1	− 119.9	− 83.5	− 57.1	− 38.6	− 25.9
$\tilde{G}_{0\cdot1}(n.-n+1)$	− 128.0	− 56.0	− 42.5	− 42.4	− 38.7	− 32.2	− 25.2	− 18.8	− 13.7	− 9.7	− 6.8
$\tilde{G}_{0\cdot1}(n.-n-1)$	− 383.9	− 222.1	+ 287.1	+ 289.7	+ 238.6	+ 180.6	+ 130.7	+ 92.0	+ 63.5	+ 43.2	+ 29.2
$\tilde{G}_{2\cdot0}(n+2.-n)$	− 60	− 32	[− 1][1]	+ 10	+ 15	+ 16	+ 15	+ 13	+ 12	+ 9	+ 7
$\tilde{G}_{2\cdot0}(n.-n)$	− 568	− 343	− 267	− 293	− 308	− 294	− 261	− 219	− 176	− 137	−104
$\tilde{G}_{2\cdot0}(n-2.-n)$	+ 390	+ 1567		+ 2738	+ 2114	+ 1682	+ 1312	+ 1002	+ 749	+ 551	+400
$\tilde{G}_{1\cdot1}(n+1.-n+1)$	+ 308	+ 157	+ 33	− 15	− 37	− 46	− 47	− 43	− 37	− 31	− 24
$\tilde{G}_{1\cdot1}(n-1.-n+1)$	+1924	+ 1004	+ 514	+ 501	+ 533	+ 524	+ 474	+ 404	+ 329	+ 258	+198
$\tilde{G}_{1\cdot1}(n+1.-n-1)$	+ 385	+ 261	+ 376	+ 394	+ 393	+ 364	+ 317	+ 263	+ 210	+ 163	+123
$\tilde{G}_{1\cdot1}(n-1.-n-1)$	−2308		−3152	− 4253	− 4080	− 3519	− 2858	− 2233	− 1696	− 1261	−922
$\tilde{G}_{0\cdot2}(n.-n+2)$	− 408	− 198	− 51	− 6	+ 18	+ 30	+ 35	+ 34	+ 31	+ 26	+ 22
$\tilde{G}_{0\cdot2}(n.-n)$	− 866	− 604	− 568	− 635	− 677	− 655	− 586	− 495	− 400	− 313	−239
$\tilde{G}_{0\cdot2}(n.-n-2)$		− 2600	+1815	+ 2385	+ 2337	+ 2042	+ 1673	+ 1314	+ 1002	+ 747	+547
$\tilde{G}_{0\cdot0}(n+1.-n+1)+\sigma$	− 102	− 33	+ 26	+ 43	+ 43	+ 37	+ 29	+ 21	+ 15	+ 11	+ 7
$\tilde{G}_{0\cdot0}(n-1.-n-1)-\sigma$	+1141		− 961	− 232	− 58	− 5	+ 11	+ 13	+ 12	+ 9	+ 7
$\tilde{G}_{0\cdot0}(n+1.-n-1)+\delta$	+ 169	+ 49	− 36	− 58	− 56	− 46	− 35	− 25	− 18	− 13	− 9
$\tilde{G}_{0\cdot0}(n-1.-n+1)-\delta$	+1141	+ 866	+ 320	+ 116	+ 35	+ 3	− 8	− 10	− 9	− 7	− 6

[1] The terms inclosed by ( ) contain quantities which are functions of $W_1'+[W_2]$. See Z 69.

TABLE XXIV—Continued.

Unit=1″

	n	0	1	2	3	4	5	6	7	8	9	10
Factor w	$\tilde{G}_{0\cdot0}(n.-n)$	+ 127.95	+ 80.4	− 70.9	− 74.4	− 61.4	− 46.2	− 33.1	− 23.1	− 15.8	− 10.7	− 7.2
	$\tilde{G}_{1\cdot0}(n+1.-n)$	− 225	− 148	− 85	− 73	− 73	− 70	− 63	− 54	− 44	− 34	− 26
	$\tilde{G}_{1\cdot0}(n-1.-n)$		− 2478	+ 485.5	+ 947	+ 979	+ 862	+ 704	+ 549	+ 416	+ 308	+224
	$\tilde{G}_{0\cdot1}(n.-n+1)$	+ 427	+ 283	+ 155	+ 147	+ 153	+ 149	+ 135	+ 114	+ 93	+ 73	+ 56
	$\tilde{G}_{0\cdot1}(n.-n-1)$	+1794	+ 1234	− 620	− 995	− 1031	− 921	− 762	− 602	− 460	− 343	−251
	$\tilde{G}_{2\cdot0}(n+2.-n)$	+ 368	+ 270	+ 85	+ 5	− 37	− 59	− 69	− 71	− 68	− 61	
	$\tilde{G}_{2\cdot0}(n.-n)$	+3106	+ 2700	+1264	+ 1055	+ 1180	+ 1307	+ 1346	+ 1290	+ 1167	+ 1009	
	$\tilde{G}_{2\cdot0}(n-2.-n)$	−2002	− 7269		−13668	−11168	− 9806	− 8437	− 7065	− 5759	− 4590	
	$\tilde{G}_{1\cdot1}(n+1.-n+1)$	−1476	− 1062	− 360	− 65	+ 87	+ 176	+ 224	+ 241	+ 237	+ 218	
	$\tilde{G}_{1\cdot1}(n-1.-n+1)$	−7175	− 5771	[−2701]	− 2148	− 2341	− 2580	− 2650	− 2538	− 2296	− 1985	
	$\tilde{G}_{1\cdot1}(n+1.-n-1)$	−2392	− 2138	−1406	− 1339	− 1497	− 1624	− 1641	− 1553	− 1394	− 1201	
	$\tilde{G}_{1\cdot1}(n-1.-n-1)$	+8713		+5765	+17425	+20109	[+19763]	+17949	+15489	+12872	+10396	
	$\tilde{G}_{0\cdot2}(n.-n+2)$	+1566	+ 1105	+ 380	+ 107	− 33	− 121	− 174	− 200	− 205	− 194	
	$\tilde{G}_{0\cdot2}(n.-n)$	+4261	+ 3793	+2481	+ 2497	+ 2905	+ 3209	+ 3269	+ 3106	+ 2792	+ 2403	
	$\tilde{G}_{0\cdot2}(n.-n-2)$		+20946	−4103	− 9279	−11106	−11195	−10319	− 8994	− 7525	− 6108	
	$\tilde{G}_{0\cdot0}(n+1.-n+1)+\sigma$	+ 563	+ 480	+ 106	− 90	− 173	− 191	− 178	− 151	− 121	− 94	
	$\tilde{G}_{0\cdot0}(n-1.-n-1)-\sigma$	−4396		+8493	+ 2121	+ 654	+ 157	− 24	− 83	− 91	− 82	
	$\tilde{G}_{0\cdot0}(n+1.-n-1)+\delta$	−1029	− 744	− 132	+ 144	+ 243	+ 255	+ 228	+ 189	+ 148	+ 113	
	$\tilde{G}_{0\cdot0}(n-1.-n+1)-\delta$	−4396	− 4260	−1977	− 886	− 355	− 102	+ 12	+ 56	+ 66	+ 62	
Factor w^2	$\tilde{G}_{0\cdot0}(n.-n)$	− 171	− 165	+ 20	+ 82	+ 99	+ 94	+ 80	+ 64	+ 50	+ 37	+ 28
	$\tilde{G}_{1\cdot0}(n+1.-n)$	+ 480	+ 456	+ 193	+ 112	+ 105	+ 118	+ 128	+ 129	+ 121	+ 108	
	$\tilde{G}_{1\cdot0}(n-1.-n)$		+ 6930	+ 613	− 1108	− 1766	− 1928	− 1837	− 1627	− 1375	− 1120	
	$\tilde{G}_{0\cdot1}(n.-n+1)$	− 726	− 696	− 340	− 248	− 262	− 293	− 309	− 302	− 278	− 243	
	$\tilde{G}_{0\cdot1}(n.-n-1)$	−4399	− 3461	[− 64]	+ 1255	+ 1866	+ 2047	+ 1975	+ 1771	+ 1510	+ 1242	

TABLE XXV.[1]

Unit=1″

	n	0	1	2	3	4	5	6	7	8	9	10
	$\tilde{H}_{0\cdot0}(n.-n)$	−43.141	−180.72		+ 458.50	+ 167.42	+ 78.80	+ 40.85	+ 22.26	+ 12.5	+ 7.2	+ 4.2
	$\tilde{H}_{1\cdot0}(n+1.-n)$		− 628.9	− 135.8	+ 13.0	+ 50.9	+ 53.0	+ 44.4	+ 34.0	+ 24.8	+ 17.5	+ 12.2
	$\tilde{H}_{1\cdot0}(n-1.-n)$	+ 52.8	+ 290.8	+ 1496.3	+ 3458.7		− 2959.7	− 1226.6	− 649.7	− 375.2	− 225.7	− 138.8
	$\tilde{H}_{0\cdot1}(n.-n+1)$	− 383.9		− 137.3	− 291.3	− 255.6	− 195.1	− 140.8	− 98.6	− 67.7	− 45.9	− 30.8
	$\tilde{H}_{0\cdot1}(n.-n-1)$	− 128.7	− 646.4	− 4581.0		+ 4590.8	+ 1973.8	+ 1072.4	+ 631.3	+ 385.3	+ 239.7	+ 150.7

	$\tilde{H}_{2\cdot0}(n+2.-n)$	+ 390	+ 340	+ 119	+ 60	+ 45	+ 42	+ 39	+ 36	+ 31	+ 25	+ 20
	$\underset{\sim}{H}_{2\cdot0}(n.-n)$	− 568	− 3166		− 354	− 1220	− 1282	− 1133	− 928	− 727	− 552	− 410
	$H_{2\cdot0}(n-2.-n)$	− 60	− 412	− 1798	− 4085	− 7960	− 17640		+ 16423	+ 7252	+ 4103	+2530
	$\tilde{H}_{1\cdot1}(n+1.-n+1)$	−2308	[− 1488]	− 493	− 233	− 182	− 176	− 169	− 154	− 132	− 109	− 86
	$\underset{\sim}{H}_{1\cdot1}(n-1.-n+1)$	+ 385	+ 1574	+ 892		+ 6511	+ 4414	+ 3280	+ 2452	+ 1815	+ 1327	+ 960
	$\underset{\sim}{H}_{1\cdot1}(n+1.-n-1)$	+1924		− 1429	+ 853	+ 1420	+ 1458	+ 1294	+ 1067	+ 840	+ 641	+ 478
	$H_{1\cdot1}(n-1.-n-1)$	+ 308	+ 1892	+ 9019	+20344	+ 49129		− 50291	− 22868	− 13225	− 8299	−5396
	$\tilde{H}_{0\cdot2}(n.-n+2)$		+ 2191	+ 591	+ 252	+ 201	+ 198	+ 190	+ 172	+ 147	+ 120	+ 94
	$\underset{\sim}{H}_{0\cdot2}(n.-n)$	− 866	− 3840		− 7274	− 5807	− 4685	− 3684	− 2826	− 2121	− 1564	−1137
	$H_{0\cdot2}(n.-n-2)$	− 408	− 2273	−12428	−33655		+ 38571	+ 18102	+ 10724	+ 6853	+ 4522	+3027
	$\tilde{H}_{0\cdot0}(n+1.-n+1)+\sigma$	+1141	+ 1634	+ 1080	+ 725	+ 486	+ 324	+ 215	+ 142	+ 94	+ 62	+ 40
	$\underset{\sim}{H}_{0\cdot0}(n-1.-n-1)-\sigma$	− 102	− 433	− 700	− 1044	− 1865		+ 1247	+ 482	+ 243	+ 135	+ 78
	$\underset{\sim}{H}_{0\cdot0}(n+1.-n-1)+\delta$	+1141		− 3241	− 1450	− 810	− 486	− 301	− 190	− 121	− 77	− 49
	$H_{0\cdot0}(n-1.-n+1)-\delta$	+ 169	+ 866	+ 2101		− 1865	− 780	− 416	− 241	− 146	− 90	− 56
Factor w	$\tilde{H}_{0\cdot0}(n.-n)$	+ 127. 95	+ 440. 2		− 2821. 1	− 972. 2	− 483. 2	− 270. 7	− 160. 1	− 97. 4	− 60. 2	− 37. 6
	$\tilde{H}_{1\cdot0}(n+1.-n)$		+ 3080	(+ 719)	+ 665	+ 60	− 135	− 188	− 184	− 158	− 127	− 98
	$H_{1\cdot0}(n-1.-n)$	− 225	− 993	(− 3063. 4)	− 2651		+ 28777	+ 10217	+ 5383	+ 3225	+ 2041	+1327
	$\tilde{H}_{0\cdot1}(n.-n+1)$	+1794		− 562	+ 828	+ 1048	+ 973	+ 812	+ 611	+ 488	+ 363	+ 265
	$H_{0\cdot1}(n.-n-1)$	+ 427	+ 1806. 0	− 1194		− 40971	− 14923	− 8086	− 4961	− 3203	− 2119	−1417
	$\tilde{H}_{2\cdot0}(n+2.-n)$	−2002	− 2321	(− 910)	− 498	− 283	− 218	− 205	− 201	− 193	− 176	
	$\underset{\sim}{H}_{2\cdot0}(n.-n)$	+3106	+13600		+ 3845	(+ 7260. 3)	+ 1105	+ 4632	+ 5150	+ 4786	+ 4120	
	$H_{2\cdot0}(n-2.-n)$	+ 368	+ 1880	(+ 4781)	+ 7680	(+ 20565)	+ 10663		[−217895]	− 78562	−42808	
	$\tilde{H}_{1\cdot1}(n+1.-n+1)$	+8713	+ 7870	+ 3343	+ 1476	+ 910	+ 826	+ 862	+ 884	+ 855	+ 784	
	$\underset{\sim}{H}_{1\cdot1}(n-1.-n+1)$	−2392	− 8192	−10728		− 38605	− 26417	− 21252	− 17318	− 13937	−11034	
	$\underset{\sim}{H}_{1\cdot1}(n+1.-n-1)$	−7175		+ 7645	(− 3464)	[+ 3068]	− 3855	− 5792	− 6040	− 5553	− 4779	
	$H_{1\cdot1}(n-1.-n-1)$	−1476	− 7195	−15273	(−37263)	+ 20179		+627529	+230294	+128129	+81500	
	$\tilde{H}_{0\cdot2}(n.-n+2)$		− 8572	− 3420	− 1553	− 1081	− 1045	− 1084	− 1082	− 1020	− 915	
	$\underset{\sim}{H}_{0\cdot2}(n.-n)$	+4261	+15822		+34602	+ 29611	+ 26593	+ 23202	+ 19594	+ 16074	+12871	
	$H_{0\cdot2}(n.-n-2)$	+1566	+ 7232	+12466	−50979		−449940	−168230	− 95788	− 62214	−42638	
	$\tilde{H}_{0\cdot0}(n+1.-n+1)+\sigma$	−4396	− 7081	− 5458	− 4148	− 3102	− 2284	− 1659	− 1191	− 847	− 596	
	$\underset{\sim}{H}_{0\cdot0}(n-1.-n-1)-\sigma$	+ 563	+ 1914	+ 2784	+ 3512	+ 1632		− 17501	− 5620	− 2737	− 1537	
	$\underset{\sim}{H}_{0\cdot0}(n+1.-n-1)+\delta$	−4396		+25016	+10471	+ 6033	+ 3831	+ 2529	+ 1699	+ 1150	+ 780	
	$H_{0\cdot0}(n-1.-n+1)-\delta$	−1029	− 4261	− 8351		+ 16553	+ 6582	+ 3617	+ 2207	+ 1409	+ 920	
Factor w^2	$\tilde{H}_{0\cdot0}(n.-n)$	− 171	− 470		+10273	+ 2929	+ 1457	+ 860	+ 544	+ 356	+ 237	+ 159
	$\tilde{H}_{1\cdot0}(n+1.-n)$		− 7922	(− 2262)	− 4544	− 1407	− 331	+ 151	+ 342	+ 400	+ 387	
	$H_{1\cdot0}(n-1.-n)$	+ 480	+ 1669	(+ 3250)	+17358		[−172780]	− 45846	− 22667	− 13652	− 8939	
	$\tilde{H}_{0\cdot1}(n.-n+1)$	−4400		+ 4017	− 350	− 1668	− 2067	− 2059	− 1862	− 1591	− 1307	
	$H_{0\cdot1}(n.-n-1)$	− 726	− 2491	−18776		+234480	+ 62130	+ 31304	+ 19302	+ 12918	+ 8978	

[1] The terms enclosed by ( ) contain quantities which are functions of $W_2'+[W_2]$. See *Z* 69.

TABLE XXVI.[1]

Unit = 1″

	n	0	1	2	3	4	5	6	7	8	9	10
	$\bar{A}_{0\cdot0}(n.-n)$	− 86.282	− 293.33	− 237.94	+ 337.99	+ 102.73	+ 42.86	+ 20.42	+ 10.47	+ 5.50	+ 3.14	+ 1.79
	$\bar{A}_{1\cdot0}(n+1.-n)$	+ 139.1	− 468.0	− 105.1	− 5.8	+ 17.8	+ 20.4	+ 17.2	+ 13.1	+ 9.5	+ 6.6	+ 4.6
	$\bar{A}_{1\cdot0}(n-1.-n)$	+ 139.1	+ 511.1	+ 1223.30	+ 6172.8	+ 1288.5	− 2217.1	− 772.0	− 364.3	− 193.9	− 109.7	− 64.4
	$\bar{A}_{0\cdot1}(n.-n+1)$	− 511.8	− 185.7	− 92.3	− 176.8	− 138.6	− 97.5	− 66.3	− 44.3	− 29.4	− 19.4	− 12.7
	$\bar{A}_{0\cdot1}(n.-n-1)$	− 511.8	− 868.54	− 8375.6	− 1989.6	+ 3383.4	+ 1211.0	+ 583.3	+ 315.7	+ 181.2	+ 107.5	+ 65.3
	$\bar{A}_{2\cdot0}(n+2.-n)$	+ 225	+ 178	[+ 46]	+ 16	+ 10	+ 10	+ 11	+ 11	+ 11	+ 8	+ 7
	$\bar{A}_{2\cdot0}(n.-n)$	− 1136.6	− 5022	− 396	− 59	− 637	− 633	− 529	− 414	− 313	− 231	− 167
	$\bar{A}_{2\cdot0}(n-2.-n)$	+ 225	+ 1133	+ 447	+ 8169	− 5845.9	− 31492	− 6657	+ 12392	+ 4626	+ 2342	+1335
	$\bar{A}_{1\cdot1}(n+1.-n+1)$	− 1744	[− 921]	− 252	− 95	− 66	− 64	− 62	− 56	− 48	− 40	− 31
	$\bar{A}_{1\cdot1}(n-1.-n+1)$	+ 3077	+ 2578.2	+ 988	− 1224	+ 4878	+ 2791	+ 1851	+ 1275	+ 889	+ 619	+ 433
	$\bar{A}_{1\cdot1}(n+1.-n-1)$	+ 3077	+ 1690	− 1309	+ 339	+ 642	+ 639	+ 547	+ 437	+ 336	+ 251	+ 184
	$\bar{A}_{1\cdot1}(n-1.-n-1)$	− 1744	+ 928	−14498	+ 16091.9	+ 88785	+ 20253	− 37540	− 14329	− 7382	− 4270	−2614
	$\bar{A}_{0\cdot2}(n.-n+2)$	− 408	+ 1693	+ 376	+ 139	+ 100	+ 93	+ 85	+ 74	+ 62	+ 49	+ 38
	$\bar{A}_{0\cdot2}(n.-n)$	− 1731.8	− 5965	+ 930	− 5297	− 3545	− 2544	− 1842	− 1330	− 953	− 678	− 479
	$\bar{A}_{0\cdot2}(n.-n-2)$	− 408	− 1709	−10613.2	− 61877	− 15453	+ 28418	+ 11105	+ 5832	+ 3427	+ 2126	+1358
	$\bar{A}_{0\cdot0}(n+1.-n+1)+\sigma$	+ 881	+ 1025	+ 593	+ 363	+ 227	+ 144	+ 92	+ 58	+ 38	+ 25	+ 15
	$\bar{A}_{0\cdot0}(n-1.-n-1)-\sigma$	+ 881	− 433	− 806	− 1276.2	− 3191	− 582	+ 932	+ 299	+ 133	+ 67	+ 36
	$\bar{A}_{0\cdot0}(n+1.-n-1)+\delta$	+ 1785	+ 1201	− 2422	− 899	− 442	− 243	− 141	− 84	− 52	− 33	− 20
	$\bar{A}_{0\cdot0}(n-1.-n+1)-\delta$	+ 1785	+ 1731.8	+ 3276	+ 725	− 1408	− 488	− 228	− 120	− 68	− 40	− 24
Factor w	$\bar{A}_{0\cdot0}(n.-n)$	+ 255.90	+ 913.5	+ 837.2	− 2314.7	− 657.0	− 284.2	− 144.0	− 79.1	− 45.5	− 26.9	− 16.3
	$\bar{A}_{1\cdot0}(n+1.-n)$	− 481	+ 2334	(+ 293)	+ 539	+ 93	− 33	− 67	− 70	− 61	− 48	− 37
	$\bar{A}_{1\cdot0}(n-1.-n)$	− 481	− 2834	(− 2577.9)	− 20581	− 7990	+ 23970	+ 7069	+ 3255	+ 1770	+ 1042	+ 640
	$\bar{A}_{0\cdot1}(n.-n+1)$	+ 2222	+ 766	− 444	+ 546	+ 607	+ 513	+ 400	+ 299	+ 218	+ 157	+ 112
	$\bar{A}_{0\cdot1}(n.-n-1)$	+ 2222	+ 3040.0	+20592	+ 10724	− 34098	− 10220	− 4804	− 2660	− 1592	− 995	− 636
	$\bar{A}_{2\cdot0}(n+2.-n)$	− 1185	− 1304	(− 370)	− 200	− 89	− 59	− 56	− 58	− 59	− 55	
	$\bar{A}_{2\cdot0}(n.-n)$	+ 6212	[+24269]	(+ 3109)	+ 7449	(+ 7038)	− 913	+ 1867	+ 2240	+ 2064	+ 1741	
	$\bar{A}_{2\cdot0}(n-2.-n)$	− 1185	− 5173	(− 49)	[− 18308]	(+ 9397)	+154467	+ 58930	[−182738]	−55031	−26325	
	$\bar{A}_{1\cdot1}(n+1.-n+1)$	+ 6382	+ 4910	+ 1800	+ 652	+ 334	+ 290	+ 308	+ 322	+ 314	+ 288	
	$\bar{A}_{1\cdot1}(n-1.-n+1)$	−13156	−13963	−15732	[+ 2788]	− 31946	− 18174	− 12829	− 9506	− 7125	− 5339	
	$\bar{A}_{1\cdot1}(n+1.-n-1)$	−13156	(− 9387)	− 4	(− 5580)	[+ 4145]	− 1068	− 2294	− 2454	− 2236	− 1888	
	$\bar{A}_{1\cdot1}(n-1.-n-1)$	+ 6382	(− 4309)	− 3080	(− 19838)	−357562	−162987	+526203	+160223	+77778	+44803	
	$\bar{A}_{0\cdot2}(n.-n+2)$	+ 1566	− 6307	− 2185	− 875	− 556	− 505	− 499	− 480	− 439	− 390	
	$\bar{A}_{0\cdot2}(n.-n)$	+ 8522	+27803	+ 534	+ 28647	+ 20067	+ 15674	+ 12376	+ 9708	+ 7530	+ 5770	
	$\bar{A}_{0\cdot2}(n.-n-2)$	+ 1566	+26142	+ 8363	+192682	+111614	−377281	−116178	− 57314	−33545	−21282	

	$\bar{A}_{0\cdot0}(n+1.-n+1)+\sigma$	− 3231	− 4485	− 3071	− 2131	− 1487	− 1038	− 724	− 502	− 348	− 239	
	$\bar{A}_{0\cdot0}(n-1.-n-1)-\sigma$	− 3231	+ 1914	+10207	+ 5633	+ 15590	+ 5433	− 14561	− 3854	− 1625	− 810	
	$\bar{A}_{0\cdot0}(n+1.-n-1)+\delta$	− 7865	− 6127	+20397	+ 7050	+ 3532	+ 2026	+ 1239	+ 786	+ 508	+ 333	
	$\bar{A}_{0\cdot0}(n-1.-n+1)-\delta$	− 7865	− 8522	−14815	− 4453	+ 13454	+ 4420	+ 2111	+ 1161	+ 685	+ 422	
Factor w^2	$\bar{A}_{0\cdot0}(n.-n)$	− 342	− 1400	− 1414	+ 9275	+ 2217	+ 949	+ 498	+ 288	+ 176	+ 111	+ 72
	$\bar{A}_{1\cdot0}(n+1.-n)$	+ 822	− 6260	(− 808)	− 3713	− 1063	− 301	+ 3	+ 112	+ 148	+ 146	
	$\bar{A}_{1\cdot0}(n-1.-n)$	+ 822	+ 7747	(+ 3863)	+ 86540	+ 25894	[−157278]	− 35253	− 15073	− 8107	− 4866	
	$\bar{A}_{0\cdot1}(n.-n+1)$	− 5126	− 1504	+ 3211	− 338	− 1089	− 1189	− 1077	− 916	− 743	− 584	
	$\bar{A}_{0\cdot1}(n.-n-1)$	− 5126	− 5952	−95930	− 31075	+214716	+ 47967	+ 20749	+ 11345	+ 6922	+ 4480	

[1] The terms enclosed by ( ) contain quantities which are functions of $W_2'+[W_2]$. See Z 69.

Table XXVII.[1]

Unit = 1″

$A_{0\cdot0}(n.-n)$	− 86.282	− 293.33	− 237.94	+ 337.99	+ 102.73	+ 42.86	+ 20.42	+ 10.47	+ 5.50	+ 3.14	+ 1.79
$A_{1\cdot0}(n+1.-n)$	+ 225.4	− 174.7	+ 132.9	− 343.8	− 84.9	− 22.4	− 3.2	+ 2.6	+ 4.0	+ 3.5	+ 2.8
$A_{1\cdot0}(n-1.-n)$	+ 225.4	+ 804.4	+ 1461.21	+ 5834.8	+ 1185.8	− 2260.0	− 792.4	− 374.8	− 199.4	− 112.8	− 66.2
$A_{0\cdot1}(n.-n+1)$	− 511.8	− 185.7	− 92.3	− 176.8	− 138.6	− 97.5	− 66.3	− 44.3	− 29.4	− 19.4	− 12.7
$A_{0\cdot1}(n.-n-1)$	− 511.8	− 868.54	− 8375.6	− 1989.6	+ 3383.4	+ 1211.0	+ 583.3	+ 315.7	+ 181.2	+ 107.5	+ 65.3
$A_{2\cdot0}(n+2.-n)$	+ 86	+ 646	+ 151	+ 22	− 8	− 10	− 6	− 2	+ 1	+ 1	+ 2
$A_{2\cdot0}(n.-n)$	− 1415	− 5065	− 1514	− 6226	− 1943	+ 1564	+ 226	− 63	− 129	− 128	− 107
$A_{2\cdot0}(n-2.-n)$	+ 86	+ 622	− 776	+ 1996	− 7134.4	− 29275	− 5885	+ 12756	+ 4820	+ 2452	+1399
$A_{1\cdot1}(n+1.-n+1)$	− 1232	[− 735]	− 160	+ 82	+ 73	+ 34	+ 4	− 12	− 19	− 21	− 18
$A_{1\cdot1}(n-1.-n+1)$	+ 3589	+ 2764.2	+ 1080	− 1047	+ 5017	+ 2889	+ 1917	+ 1319	+ 918	+ 638	+ 446
$A_{1\cdot1}(n+1.-n-1)$	+ 3589	+ 2559	+ 7067	+ 2329	− 2741	− 572	− 36	+ 121	+ 155	+ 144	+ 119
$A_{1\cdot1}(n-1.-n-1)$	− 1232	+ 1797	− 6122	+ 18082	+ 85402	+ 19042	− 38123	− 14645	− 7563	− 4377	−2679
$A_{0\cdot2}(n.-n+2)$	− 408	+ 1693	+ 376	+ 139	+ 100	+ 93	+ 85	+ 74	+ 62	+ 49	+ 38
$A_{0\cdot2}(n.-n)$	− 1731.8	− 5965	+ 930	− 5297	− 3545	− 2544	− 1842	− 1330	− 953	− 678	− 479
$A_{0\cdot2}(n.-n-2)$	− 408	− 1709	−10613.2	− 61877	− 15453	+ 28418	+ 11105	+ 5832	+ 3427	+ 2126	+1358
$A_{0\cdot0}(n+1.-n+1)+\sigma$	+ 881	+ 1025	+ 593	+ 363	+ 227	+ 144	+ 92	+ 58	+ 38	+ 25	+ 15
$A_{0\cdot0}(n-1.-n-1)-\sigma$	+ 881	− 433	− 806	− 1276.2	− 3191	− 582	+ 932	+ 299	+ 133	+ 67	+ 36
$A_{0\cdot0}(n+1.-n-1)+\delta$	+ 1785	+ 1201	− 2422	− 899	− 442	− 243	− 141	− 84	− 52	− 33	− 20
$A_{0\cdot0}(n-1.-n+1)-\delta$	+ 1785	+ 1731.8	+ 3276	+ 725	− 1408	− 488	− 228	− 120	− 68	− 40	− 24

TABLE XXVII[1]—Continued.

Unit = 1″

	n	0	1	2	3	4	5	6	7	8	9	10
Factor w	$A_{0\cdot0}(n.-n)$	+ 255.90	+ 913.5	+ 837.2	− 2314.7	− 657.0	− 284.2	− 144.0	− 79.1	− 45.5	− 26.9	− 16.3
	$A_{1\cdot0}(n+1.-n)$	− 737	+ 1421	(− 544)	+ 2854	+ 750	+ 251	+ 77	+ 9	− 15	− 21	− 21
	$A_{1\cdot0}(n-1.-n)$	− 737	− 3748	(− 3415.1)	− 18266	− 7333	+ 24254	+ 7213	+ 3334	+ 1815	+ 1069	+ 656
	$A_{0\cdot1}(n.-n+1)$	+ 2222	+ 766	− 444	+ 546	+ 607	+ 513	+ 400	+ 299	+ 212	+ 157	+ 112
	$A_{0\cdot1}(n.-n-1)$	+ 2222	+ 3040.0	+20592	+ 10724	− 34098	− 10220	− 4804	− 2660	− 1592	− 995	− 636
	$A_{2\cdot0}(n+2.-n)$	− 704	− 3638	(− 663)	− 739	− 182	− 26	+ 11	+ 12	+ 2	− 7	
	$A_{2\cdot0}(n.-n)$	+ 7174	[+24769]	(+ 5394)	+ 27491	(+ 14935)	− 24850	− 5135	− 945	+ 355	+ 747	
	$A_{2\cdot0}(n-2.-n)$	− 704	− 2339	(+ 2529)	[+ 2273]	[(+ 17387)]	+130497	+ 51861	[−185993]	−56801	−27367	
	$A_{1\cdot1}(n+1.-n+1)$	+ 4160	+ 4144	+ 2244	+ 106	− 273	− 223	− 92	+ 23	+ 102	+ 131	
	$A_{1\cdot1}(n-1.-n+1)$	−15378	−14729	−15288	[+ 2242]	− 32553	− 18687	− 13229	− 9805	− 7337	− 5496	
	$A_{1\cdot1}(n+1.-n-1)$	−15378	(−12427)	−20596	(− 16304).	[+ 38243]	+ 9152	+ 2510	+ 206	− 644	− 893	
	$A_{1\cdot1}(n-1.-n-1)$	+ 4160	(− 7349)	−23672	(− 30562)	−323464	−152767	+531007	+162883	+79370	+45798	
	$A_{0\cdot2}(n.-n+2)$	+ 1566	− 6307	− 2185	− 875	− 556	− 505	− 499	− 480	− 439	− 390	
	$A_{0\cdot2}(n.-n)$	+ 8522	+27803	+ 534	+ 28647	+ 20067	+ 15674	+ 12376	+ 9708	+ 7530	+ 5770	
	$A_{0\cdot2}(n.-n-2)$	+ 1566	+26142	+ 8363	+192682	+111614	−377281	−116178	− 57314	−33545	−21282	
	$A_{0\cdot0}(n+1.-n+1)+\sigma$	− 3231	− 4485	− 3071	− 2131	− 1487	− 1038	− 724	− 502	− 318	− 239	
	$A_{0\cdot0}(n-1.-n-1)-\sigma$	− 3231	+ 1914	+10207	+ 5633	+ 15590	+ 5433	− 14561	− 3854	− 1625	− 810	
	$A_{0\cdot0}(n+1.-n-1)+\delta$	− 7865	− 6127	+20397	+ 7050	+ 3532	+ 2026	+ 1239	+ 786	+ 508	+ 333	
	$A_{0\cdot0}(n-1.-n+1)-\delta$	− 7865	− 8522	−14815	− 4453	+ 13454	+ 4420	+ 2111	+ 1161	+ 685	+ 122	
Factor w^2	$A_{0\cdot0}(n.-n)$	− 342	− 1400	− 1414	+ 9275	+ 2217	+ 949	+ 498	+ 288	+ 176	+ 111	+ 72
	$A_{1\cdot0}(n+1.-n)$	+ 1164	− 4860	(+ 606)	− 12988	− 3280	− 1250	− 495	− 176	− 28	+ 35	
	$A_{1\cdot0}(n-1.-n)$	+ 1164	+ 9147	(+ 5277)	+ 77265	+ 23677	[−158227]	− 35751	− 15361	− 8283	− 4977	
	$A_{0\cdot1}(n.-n+1)$	− 5126	− 1504	+ 3211	− 338	− 1089	− 1189	− 1077	− 916	− 743	− 584	
	$A_{0\cdot1}(n.-n-1)$	− 5126	− 5952	−95930	− 31075	+214716	+ 47967	+ 20749	+ 11345	+ 6922	+ 4480	

[1] The terms enclosed by ( ) contain quantities which are functions of $W_2'+[W_2]$. See *Z* 69.

TABLE XXVIII.

$[(1-e\cos\varepsilon)(\overline{W}_2''+\overline{W}_3''+W_4'')]$ Unit = 4th decimal of a radian.

	Cos	w^0	w	w^2
		− 4.1829	+ 12.406	− 16.58
η^2		− 68.61	+ 347.9	
η'^2		− 83.96	+ 413.1	
j^2		+ 83.96	− 413.1	
$\eta\eta'$	Δ	+134.0	− 714.2	
η	$2\theta+2\Delta$	+ 70.842	[− 165.57]	[+255.8]
η'	$2\theta+\Delta$	− 42.107	+ 147.38	−288.6
η^2	$4\theta+4\Delta$	−345.88	[+ 843.0]	
$\eta\eta'$	$4\theta+3\Delta$	+876.64	[−1481.7]	
η'^2	$4\theta+2\Delta$	−514.54	+ 405.5	
j^2	$4\theta+3\Delta-\Sigma$	− 61.87	+ 273.1	
		m'		

TABLE XXVIIIa.

$2[T_4]$ Unit = 4th decimal of a radian.

	Sin	w^{-2}: w^0	w^{-2}: w	w^0: w^0	w^0: w
	$\psi+2\theta+2\Delta$	−0.00005	+0.00073	−0.0682	+0.4056
η	$2\theta+2\Delta$			−0.3324	+2.1665
η	ψ			+0.3381	−2.5547
η	$\psi+4\theta+4\Delta$			+1.0220	−7.370
η'	$2\theta+\Delta$			+0.2654	−1.846
η'	$\psi+\Delta$			−0.3622	+2.472
η'	$\psi+4\theta+3\Delta$			−1.2106	+8.472
		m'^3		m'^2	

TABLE XXVIII*b*.

W_3

Unit=4th decimal of a radian.

	Cos	w^{-3}		w^{-1}			w	
		w^0	w	w^0	w	w^2	w^0	w
	$-\varepsilon+\psi$	−0. 00004	+0. 00038	−0. 0032	−0. 0005	−0. 4803		
	$\varepsilon+2\theta+2\varDelta$			+0. 0237	−0. 15101		+ 13. 16	− 30. 86
	$2\varepsilon-\psi+2\theta+2\varDelta$	0. 00000	+0. 00004	+0. 0050	−0. 0318		− 0. 81	+ 1. 31
	$\psi+2\theta+2\varDelta$	+0. 00001	−0. 00023	+0. 0726	−0. 4507			
	$2\varepsilon+4\theta+4\varDelta$			+0. 0153	−0. 0770		+ 3. 88	− 14. 38
	$\varepsilon+\psi+4\theta+4\varDelta$	+0. 00004	−0. 00038	+0. 0181	−0. 0814		− 16. 23	+ 61. 80
	$2\varepsilon+\psi+6\theta+6\varDelta$			−0. 0088	+0. 0576		− 3. 0	+ 15. 7
η	$2\theta+2\varDelta$			+0. 5242	−3. 3539		+ 13. 16	− 30. 86
η	$\varepsilon-\psi+2\theta+2\varDelta$			+0. 1384	−0. 7747		+ 14. 86	− 11. 30
η	$-\varepsilon+\psi+2\theta+2\varDelta$			−0. 0508	+0. 4660		+ 58. 25	− 170. 9
η	ε			+0. 0749	−0. 2385			
η	$\varepsilon+4\theta+4\varDelta$			+0. 1723	−0. 8380		−154. 6	+ 589. 2
η	$\psi+4\theta+4\varDelta$			−0. 5378	+4. 082		− 16. 23	+ 61. 80
η	$\varepsilon+\psi+2\theta+2\varDelta$			−0. 1801	+1. 1267		− 7. 71	− 6. 66
η	$\varepsilon+\psi+6\theta+6\varDelta$			−0. 3275	+2. 032		+178. 4	− 933. 0
η'	$2\theta+\varDelta$			−0. 6099	+3. 634			
η'	$\varepsilon-\psi+2\theta+\varDelta$			−0. 1412	+0. 6843		+ 10. 77	− 49. 04
η'	$-\varepsilon+\psi+2\theta+\varDelta$			+0. 1220	−0. 8554		− 31. 34	+ 118. 9
η'	$\varepsilon+\varDelta$			+0. 0524	−0. 4182			
η'	$\varepsilon+4\theta+3\varDelta$			−0. 0411	+0. 2314		+221. 0	− 694. 3
η'	$\psi+4\theta+3\varDelta$			+0. 7660	−5. 430			
η'	$\varepsilon+\psi+2\theta+3\varDelta$			+0. 0460	−0. 3060		+ 14. 12	− 26. 01
η'	$\varepsilon+\psi+6\theta+5\varDelta$			+0. 3718	−2. 0745		−287. 1	+1309. 3
	$(\theta-\theta_0)$ sin							
η	$2\theta+2\varDelta$			+0. 0490	−0. 2949			
η	$\varepsilon-\psi+2\theta+2\varDelta$			+0. 0144	−0. 0705			
η	$-\varepsilon+\psi+2\theta+2\varDelta$			−0. 0542	+0. 3582			
η	ψ			+0. 5810	−4. 7017			
η	$\varepsilon+4\theta+4\varDelta$			−0. 0618	+0. 4650			
η	$\psi+4\theta+4\varDelta$			−0. 0453	+0. 3389			
η	$\varepsilon+\psi+2\theta+2\varDelta$			−0. 0010	+0. 0290			
η'	$2\theta+\varDelta$			−0. 0364	+0. 2398			
η'	$\varepsilon-\psi+2\theta+\varDelta$			−0. 0107	+0. 0585			
η'	$-\varepsilon+\psi+2\theta+\varDelta$			+0. 0402	−0. 2889			
η'	$\psi+\varDelta$			−0. 7670	+5. 4890			
η'	$\varepsilon+4\theta+3\varDelta$			+0. 0459	−0. 3715			
η'	$\psi+4\theta+3\varDelta$			+0. 0336	−0. 2709			
η'	$\varepsilon+\psi+2\theta+3\varDelta$			+0. 0007	−0. 0220			
		m'^3		m'^2			m'	

TABLE XXVIIIc.

$[(1-e\cos\varepsilon)\overline{W}_3'']$

Unit=4th decimal of a radian.

	Cos	w	
		w^0	w
η	$2\theta+2\Delta$	+60.76	−152.6
η'	$2\theta+\Delta$	−20.57	+69.8
		m'	

TABLE XXIX.

$[(1-e\cos\varepsilon)(\overline{W}_3-\overline{W}_3'')]$

Unit=4th decimal of a radian.

	Cos	w^{-2}		w^{-1}		w	
		w^0	w	w^0	w	w^0	w
		−0.00004	+0.00038	−0.0032	−0.0005		
η	$2\theta+2\Delta$			+0.5106	−3.0290	+13.16	−30.9
η'	$2\theta+\Delta$			−0.6292	+3.463		
	$(\theta-\theta_0)$ sin						
η	$2\theta+2\Delta$			+0.0092	−0.0072		
η'	$2\theta+\Delta$			−0.0069	+0.0094		
		m'^3		m'^2		m'	

These developments cover the function W within the extent of our tables. This does not mean that W is always inclusive of all these terms, but that these terms occur in one or more of the tables. With the exception of $[(1-e\cos\varepsilon)\overline{W}]$, which contains $\overline{W}_3-\overline{W}_3''$, W is to be understood to mean

$$W=W_1+W_2'+[W_2]+(W_2''+W_3''+W_4'')$$

and

$$\overline{W}=\overline{W}_1+\overline{W}_2'+[\overline{W}_2]+(W_2''+\overline{W}_3''+\overline{W}_4'').$$

The ascending powers of w, η, η', j^2 are selected independently in each function.

To avoid a long series which is analogous in construction to T_2, the function $W_2''+W_3''+W_4''$ is not tabulated. The sum of this function and Tables XVII, XVIII, XIX, XXII*a* gives W. Since W is so long and we only need $\overline{W}$, it is not tabulated. The function

$$\overline{W}=W_{\psi=\varepsilon}$$

is given in Table XXIX*a*.

It is convenient to collect here $[(1-e\cos\varepsilon)\overline{W}]$, which is required later. The function is given by the sum of Tables XVI, XX, XXI, XXVIII, and XXIX, and is tabulated in Table XXIX*b*.

We shall also need the function Ξ

$$\Xi=x+2\eta y=\Xi_1+\Xi_2'+[\Xi_2]+(\Xi_2''+\Xi_3''+\Xi_4'')$$

Evidently Ξ can be written by inspection if W is tabulated. If the double headings are retained in the construction of Ξ the mass-factors and ranks are explicit as in the construction of W. If W is not given, we can write by inspection Ξ_1 (previously required in the computation), Ξ_2' and $[\Xi_2]$ from W_1, W_2', and $[W_2]$, respectively. The remainder, namely, $\Xi_2''+\Xi_3''+\Xi_4''$, can be written from $W_2''+W_3''+W_4''$, i. e., by inspection of Tables XXIII, XXIV, XXV. The function $\frac{1}{3}\Xi$ is given in Table XXIX*c*.

TABLE XXIXa.

$\overline{W}$.

Unit = 1″.

	Cos	w^{-1}				w^{-3}			
		w^0	w	w^2	w^3	w^0	w	w^2	w^3
			− 86.28	+ 255.90	− 342.2		+ 0.2108	− 1.059	+ 2.38
	$\varepsilon+2\theta+2\Delta$	+ 294.89	− 978.6	+ 1571	− 1415		− 0.6139	+ 4.059	− 10.3
	$2\varepsilon+4\theta+4\Delta$		+ 102.7	− 646.5	+ 2209.8		− 0.2108	+ 1.059	− 2.38
η	$2\theta+2\Delta$	+1179.6	− 2306.1	+ 1735	+ 3127		− 6.330	+41.24	
	ε		+ 278	− 961	+ 1645		+ 0.738	− 3.71	
	$\varepsilon+4\theta+4\Delta$	− 839.5	+ 4784	− 14212	+25911	−0.316	+ 3.54	−22.8	
	$2\varepsilon+2\theta+2\Delta$		+ 190	− 449	− 58		+ 0.274	− 1.70	
	$2\varepsilon+6\theta+6\Delta$		− 772	+ 7102	−35536		+ 1.800	−11.66	
η'	$2\theta+\Delta$	− 318.2	+ 212	+ 1487	− 5950		+ 8.285	−48.64	
	$\varepsilon+\Delta$		− 186	+ 766	− 1504		− 0.227	+ 1.34	
	$\varepsilon+4\theta+3\Delta$	+1229.8	− 6059	+ 16395	−31079	+0.114	− 5.00	+34.3	
	$2\varepsilon+2\theta+3\Delta$		− 177	+ 546	− 339		− 0.204	+ 1.38	
	$2\varepsilon+6\theta+5\Delta$		+ 1211	− 10221	+47956		− 2.637	+15.35	
η^2			− 753	+ 4802		−0.95	+ 9.4	XX	
	$4\theta+4\Delta$	−3358	+10660	− 25394		−1.27	+30.3		
	$\varepsilon+2\theta+2\Delta$	+ 978	− 5286	+ 19864		+4.42	−30.4		
	$\varepsilon+6\theta+6\Delta$	+2940	−23666	+102458		+1.80	− 4.5		
	$-\varepsilon+2\theta+2\Delta$	+ 396	+ 489	− 6259		−2.62	+12.6		
	2ε		+ 66	− 959			− 0.2		
	$2\varepsilon+4\theta+4\Delta$		− 2316	+ 13688			− 1.5		
	$2\varepsilon+8\theta+8\Delta$		+ 4626	− 55029			−11.8		
$\eta\eta'$	Δ		+ 2293	− 12753		+0.68	−15.0		
	$4\theta+3\Delta$	+8609	−17128	+ 35165		+0.79	−75.8		
	$\varepsilon+2\theta+\Delta$	−2280	+12028	− 34202		−5.57	+36.0		
	$\varepsilon+2\theta+3\Delta$	+1492	− 2192	− 9774		−1.91	+14.0		
	$\varepsilon+6\theta+5\Delta$	−8658	+63021	−255722		−3.95	+ 7.1		
	$-\varepsilon+2\theta+\Delta$	+2068	− 9588	− 43388		+6.18	−25.5		
	$2\varepsilon+\Delta$		− 637	+ 3699			+ 0.1		
	$2\varepsilon+4\theta+3\Delta$		+ 2798	− 13717			+ 4.3		
	$2\varepsilon+4\theta+5\Delta$		+ 2791	− 18175			− 1.6		
	$2\varepsilon+8\theta+7\Delta$		−14329	+160224			+34.8		
η'^2			− 1732	+ 8521		−0.12	+ 6.8		
	$4\theta+2\Delta$	−5341	+ 3562	− 7700		−0.12	+50.3		
	$\varepsilon+2\theta+2\Delta$	− 861	− 2864	+ 22660		+2.12	−15.3		
	$\varepsilon+6\theta+4\Delta$	+6349	−41206	+156926		+1.90	− 2.2		

XX second degree terms in this column were used.

η'^2	$-\varepsilon+2\theta$	−1634	+ 6264	− 13066		−4.04	+13.8	In the construction of Table	
	$2\varepsilon+2\Delta$		+ 376	− 2185					
	$2\varepsilon+4\theta+4\Delta$		− 3544	+ 20067			+ 0.4		
	$2\varepsilon+8\theta+6\Delta$		+11104	−116175			−25.5		
j^2			+ 1732	− 8521			− 3.8		
	$4\theta+3\Delta-\Sigma$	− 304	+ 203	+ 2250			+ 1.1		
	$\varepsilon+2\theta+2\Delta$	−2677	+14606	− 41509					
	$\varepsilon+6\theta+5\Delta-\Sigma$	+ 260	− 2257	+ 10534		+0.07	− 0.1		
	$-\varepsilon+2\theta+\Delta-\Sigma$	− 866	+ 3827	− 9074		−0.22	+ 1.2		
	$2\varepsilon+\Delta+\Sigma$		+ 1025	− 4486					
	$2\varepsilon+4\theta+4\Delta$		− 1387	+ 11469			+ 3.8		
	$2\varepsilon+8\theta+7\Delta-\Sigma$		+ 300	− 3854			− 0.7		
		m'				m'^2			

[Continued on next page.

TABLE XXIXa—Continued.

IV.

Unit=1''.

	Cos	w^{-1}: w^0	w^{-1}: w	w^{-1}: w^2	w^{-2}: w^0	w^{-2}: w	w^{-2}: w^2
η^3	$\varepsilon+4\theta+4\Delta$	+ 2549	+ 2164		−11.9	+ 89	
	$-\ \varepsilon+4\theta+4\Delta$	− 3089	+ 8155		+ 3.9	− 25	
	$\varepsilon+8\theta+8\Delta$	−11300	+ 76250		− 8.9	+ 70	
$\eta^2\eta'$	$\varepsilon+4\theta+5\Delta$	−11449	+ 42212		+ 1.9	− 23	
	$\varepsilon+4\theta+3\Delta$	− 2661	− 27530		+36.4	−241	
	$-\ \varepsilon+4\theta+3\Delta$	+ 6865	− 4540		−20.3	+118	
	$\varepsilon+8\theta+7\Delta$	+50005	−304611		+33.8	−248	
$\eta\,\eta'^2$	$\varepsilon+4\theta+4\Delta$	+26091	− 71730		−10.1	+ 83	
	$\varepsilon+4\theta+2\Delta$	− 1356	+ 30293		−25.5	+153	
	$-\ \varepsilon+4\theta+2\Delta$	− 2204	− 20846		+28.0	−153	
	$\varepsilon+8\theta+6\Delta$	−73583	+400009		−41.9	+284	
η'^3	$\varepsilon+4\theta+3\Delta$	−13756	+ 22165		+10.1	− 65	
	$-\ \varepsilon+4\theta+\ \Delta$	− 3317	+ 18452		−12.4	+ 64	
	$\varepsilon+8\theta+5\Delta$	+36006	−172164		+16.6	−104	
$j^2\eta$	$\varepsilon+4\theta+3\Delta-\Sigma$	− 2011	+ 14604		− 1.9	+ 14	
	$-\ \varepsilon+4\theta+3\Delta-\Sigma$	+ 1808	− 13617		+ 1.9	− 14	
	$\varepsilon+8\theta+7\Delta-\Sigma$	− 2381	+ 18919		− 1.1	+ 10	
	$\varepsilon+4\theta+4\Delta$	+14204	− 88026		+ 5.7	− 42	
$j^2\ \eta'$	$\varepsilon+4\theta+4\Delta-\Sigma$	− 554	+ 140		+ 0.5	− 4	
	$-\ \varepsilon+4\theta+2\Delta-\Sigma$	− 3545	+ 22886		− 1.8	+ 14	
	$\varepsilon+8\theta+6\Delta-\Sigma$	+ 3827	− 27870		+ 1.3	− 11	
	$\varepsilon+4\theta+3\Delta$	−17503	+ 99584		− 3.7	+ 28	
	$(\theta-\theta_0)$ sin						
η	$2\theta+2\Delta$					− 2.19	+13.6
	ε	+ 767.7	− 2820.9	+ 5210	+ 1.265	− 5.34	+22.7
	$\varepsilon+4\theta+4\Delta$					+ 0.78	− 6.0
	$2\varepsilon+2\theta+2\Delta$					− 0.55	+ 3.4
η'	$2\theta+\ \Delta$					+ 1.63	−11.0
	$\varepsilon+\ \ \Delta$	− 570.0	+ 2421.1	− 4950	− 0.455	+ 5.94	−37.3
	$\varepsilon+4\theta+3\Delta$					− 0.58	+ 4.8
	$2\varepsilon+2\theta+3\Delta$					+ 0.41	− 2.8
η^2	$4\theta+4\Delta$					+ 10.93	
	$\varepsilon+2\theta+2\Delta$					− 2.19	
	$-\ \varepsilon+2\theta+2\Delta$					− 1.92	
	$2\varepsilon+4\theta+4\Delta$					+ 3.12	
$\eta\,\eta'$	Δ	− 570.0	+ 2421.1	− 4950	− 0.455	+ 5.94	− 7.2
	$4\theta+3\Delta$					− 23.00	
η^3	ε	+ 6624	− 47448		+23.8	−221.9	
$\eta^2\eta'$	$\varepsilon+\ \ \Delta$	−18540	+123024		−73.4	+572.4	
	$-\ \varepsilon+\ \ \Delta$	+ 8414	− 57880		+36.0	−282.2	
$\eta\,\eta'^2$	ε	+25564	−157424		+87.3	−652.8	
	$\varepsilon+\ \ 2\Delta$	+10478	− 70250		+55.2	−374.8	
η'^3	$\varepsilon+\ \ \Delta$	−15678	+ 94846		−69.9	+438.6	
$j^2\eta$	ε	−25564	+157424	−511232	−23.1	+165.0	
	$\varepsilon+\ \ \Delta+\Sigma$	+22012	−121258	+359162	+ 9.9	− 77.0	
$j^2\ \eta'$	$\varepsilon+\ \ \Delta$	+23524	−150306	+498328	+14.8	−112.0	
	$\varepsilon+\ \ \Sigma$	−12048	+ 76364	−251640	− 5.2	+ 45.8	
	$(\theta-\theta_0)^2$cos						
η	ε					− 0.356	+ 2.623
η'	$\varepsilon+\ \ \Delta$					+ 0.266	− 2.100
		m'			m'^2		

TABLE XXIXa—Continued.

$\bar{W}$.

Unit=1″

	Cos	w^0	w	w^2
	$\frac{1}{2}\varepsilon+\theta+\Delta$	− 293.4	+ 913.5	− 1400.1
	$\frac{3}{2}\varepsilon+3\theta+3\Delta$	+ 338.1	− 2315	+ 9277
	$\frac{5}{2}\varepsilon+5\theta+5\Delta$	+ 42.9	− 284.3	+ 948.2
	$\frac{7}{2}\varepsilon+7\theta+7\Delta$	+ 10.5	− 79.2	+ 288.5
η	$\frac{1}{2}\varepsilon+3\theta+3\Delta$	+ 6172.8	− 20580	+ 86549
	$-\frac{1}{2}\varepsilon+\theta+\Delta$	+ 511.2	− 2834	+ 7746
	$\frac{3}{2}\varepsilon+\theta+\Delta$	− 467.9	+ 2335	− 6259
	$\frac{3}{2}\varepsilon+5\theta+5\Delta$	− 2217.1	+ 23971	−157308
	$\frac{5}{2}\varepsilon+3\theta+3\Delta$	− 5.8	+ 539	− 3713
	$\frac{5}{2}\varepsilon+7\theta+7\Delta$	− 364.3	+ 3259	− 15083
η'	$\frac{1}{2}\varepsilon+3\theta+2\Delta$	− 8375.5	+ 20591	− 95913
	$-\frac{1}{2}\varepsilon+\theta$	− 1023.4	+ 4443	− 10251
	$\frac{3}{2}\varepsilon+\theta+2\Delta$	− 92.3	− 444	+ 3212
	$\frac{3}{2}\varepsilon+5\theta+4\Delta$	+ 3383.4	− 34097	+214736
	$\frac{5}{2}\varepsilon+3\theta+4\Delta$	− 138.6	+ 608	− 1089
	$\frac{5}{2}\varepsilon+7\theta+6\Delta$	+ 583.3	− 4805	+ 20748
η^2	$\frac{1}{2}\varepsilon+\theta+\Delta$	− 5022	+ 24269	
	$\frac{1}{2}\varepsilon+5\theta+5\Delta$	−31492	+154465	
	$-\frac{1}{2}\varepsilon+3\theta+3\Delta$	+ 8169	− 18309	
	$\frac{3}{2}\varepsilon+3\theta+3\Delta$	− 59	+ 7449	
	$\frac{5}{2}\varepsilon+7\theta+7\Delta$	+12392	−182737	
	$-\frac{3}{2}\varepsilon+\theta+\Delta$	+ 1133	− 5174	
	$\frac{5}{2}\varepsilon+9\theta+9\Delta$	+ 2342	− 25879	
$\eta\,\eta'$	$\frac{1}{2}\varepsilon+\theta$	+ 6153	− 26311	
	$\frac{1}{2}\varepsilon+\theta+2\Delta$	+ 988	− 15732	
	$\frac{1}{2}\varepsilon+5\theta+4\Delta$	+88784	−357566	
	$-\frac{1}{2}\varepsilon+3\theta+2\Delta$	−14498	− 3083	
	$\frac{3}{2}\varepsilon+3\theta+2\Delta$	− 1309	− 5	
	$\frac{5}{2}\varepsilon+3\theta+4\Delta$	+ 4878	− 31947	
	$\frac{5}{2}\varepsilon+7\theta+6\Delta$	−37540	+526187	
	$-\frac{3}{2}\varepsilon+\theta$	− 3487	+ 12764	
	$\frac{5}{2}\varepsilon+9\theta+8\Delta$	− 7382	+ 77025	
η'^2	$\frac{1}{2}\varepsilon+\theta+\Delta$	− 5966	+ 27801	
	$\frac{1}{2}\varepsilon+5\theta+3\Delta$	−61877	+192684	
	$-\frac{1}{2}\varepsilon+3\theta+\Delta$	− 1709	+ 26144	
	$\frac{3}{2}\varepsilon-\theta+\Delta$	+ 1693	− 6306	
	$\frac{5}{2}\varepsilon+3\theta+3\Delta$	− 5297	+ 28649	
	$\frac{3}{2}\varepsilon+7\theta+5\Delta$	+28418	−377278	
j^2	$\frac{1}{2}\varepsilon+\theta+\Delta$	+ 6846	− 30542	
	$\frac{1}{2}\varepsilon+5\theta+4\Delta-\Sigma$	− 3191	+ 15590	
	$-\frac{1}{2}\varepsilon+3\theta+2\Delta-\Sigma$	− 806	+ 10210	
	$\frac{3}{2}\varepsilon+3\theta+3\Delta$	− 3829	+ 33852	
	$\frac{5}{2}\varepsilon+7\theta+6\Delta-\Sigma$	+ 932	− 14562	
	$-\frac{3}{2}\varepsilon+\theta-\Sigma$	+ 1762	− 6460	
		m'		

TABLE XXIXb.

$[(1-e\cos\epsilon)\ \overline{W}]$

Unit = 4th decimal of a radian.

	Cos	w^{-5}	w^{-4}	w^{-3}	w^{-2}	w^{-1}	w^{0}	w	w^{2}
				−0.00004	+0.01060	− 0.054	− 4.0682	+ 12.406	− 16.59
η	$2\theta+2\Delta$		−0.000055	+0.00041	−0.2771	+ 45.203	− 72.26	+ 21.0	+164.8
η'	$2\theta+\Delta$		+0.000020	−0.00017	+0.4017	− 18.414	+ 19.67	+ 72.1	−288.5
η^2			+0.00081	−0.0533	+0.423	− 2.32	− 50.0	+ 279.4	+126.2
	$4\theta+4\Delta$		+0.00011	−0.0470	+1.294	− 131.22	+ 293.6	− 542	+111
$\eta\ \eta'$	Δ		−0.00145	+0.0457	−0.717	+ 4.83	+ 120.2	− 655.4	−120.0
	$4\theta+3\Delta$		−0.00003	+0.0336	−3.432	+ 381.92	− 561.9	+ 910	−162
η'^2			+0.00042	−0.0099	+0.333	− 2.572	− 83.96	+ 413.1	
	$4\theta+2\Delta$		−0.00027	−0.0043	+2.441	− 274.98	+ 192.5	− 373.5	
j^2					−0.186	+ 1.345	+ 79.15	− 413.1	
	$4\theta+3\Delta-\Sigma$		−0.00003	+0.0003	+0.052	− 15.295	+ 10.3	+ 109	
η^3	$2\theta+2\Delta$	+0.00030	−0.0022	+0.262	−1.70	+ 28.2	− 433		
	$6\theta+6\Delta$	+0.0001	−0.0008	+0.262	−1.70	+ 428	− 2295		
$\eta^2\eta'$	$2\theta+\Delta$	−0.00021	+0.0018	−0.767	+4.46	− 316.1	+ 1591.5		
	$2\theta+3\Delta$	−0.00011	+0.0009	−0.094	+0.69	+ 108.5	− 49		
	$6\theta+5\Delta$	−0.0001	+0.001	−0.86	+5.2	−1889	+ 8900		
$\eta\ \eta'^2$	2θ	+0.00004	−0.0004	+0.555	−2.87	+ 237.6	− 1030		
	$2\theta+2\Delta$	+0.00008	−0.0009	+0.276	−1.85	− 125.3	− 552		
	$6\theta+4\Delta$	+0.00004		+0.83	−4.7	+2770	−11240		
η'^3	$2\theta+\Delta$	−0.00001		−0.200	+1.21	− 168.7	+ 867		
	$6\theta+3\Delta$	−0.00005		−0.200	+1.21	−1346	+ 4580		
$j^2\eta$	$2\theta+\Delta-\Sigma$			+0.032	−0.23	+ 126.0	− 620		
	$2\theta+2\Delta$					− 389.4	+ 1844	−4500	
	$6\theta+5\Delta-\Sigma$			+0.032	−0.23	+ 113	− 730		
$j^2\ \eta'$	$2\theta+\Delta$					+ 362.4	− 1749		
	$2\theta+2\Delta-\Sigma$			−0.011	+0.09	− 7.7	+ 144		
	$6\theta+4\Delta-\Sigma$			−0.011	+0.09	− 187	+ 1080		
	$(\theta-\theta_0)$ sin								
η	$2\theta+2\Delta$		−0.000176	+0.00132	−0.1098	+ 0.6675	− 1.920		
η'	$2\theta+\Delta$		+0.000063	−0.00053	+0.0805	− 0.5404	+ 1.692		
η^2	$4\theta+4\Delta$		+0.000100	−0.00890	+0.492	− 3.760	+ 8.04		

$\eta\eta'$	Δ		-0.0000	-0.00001	-0.000	$-$ 0.39	$-$ 5.97		
	$4\theta+3\Delta$		-0.00168	$+0.01469$	-1.086	$+$ 8.037	$-$ 17.28		
η'^2	$4\theta+2\Delta$		$+0.00048$	-0.00444	$+0.536$	$-$ 4.046	$+$ 9.024		
	$(\theta-\theta_0)^2\cos$								
η^2			$+0.000171$	-0.00149	$+0.0520$	$-$ 0.382	$+$ 1.055		
$\eta\eta'$	Δ		-0.000188	$+0.00177$	-0.0771	$+$ 0.610	$-$ 1.798		
η'^2			$+0.000046$	-0.000464	$+0.0286$	$-$ 0.243	$+$ 0.765		
		m'^3	m'^3	m'^3, m'^2	m'^3, m'^2	m'^2, m'	m'^2, m'	m'	m'

TABLE XXIXc.

$\frac{1}{3}\Xi$

Unit = 1″

	Cos	w^{-1}				w^{-3}		
		w^0	w	w^2	w^3	w^0	w	w^2
	$\epsilon+2\theta+2\Delta$		− 90.5	+ 302.7	− 478.2			
	$2\epsilon+4\theta+4\Delta$		− 26.6	+ 125.5	− 270.3			
η	$2\theta+2\Delta$	+ 589.8	− 1571.9	+ 1680			− 1.82	+12.00
	ϵ						+ 0.42	− 2.10
	$\epsilon+4\theta+4\Delta$		+ 616	− 3638	+11175		− 0.42	+ 2.10
	$2\epsilon+2\theta+2\Delta$		+ 23	− 161	+ 439			
	$2\epsilon+6\theta+6\Delta$		+ 219	− 1451	+ 4616			
η'	$2\theta+\Delta$	− 106.1	+ 360	− 517			+ 1.81	−10.64
	$\epsilon+\Delta$		− 43	+ 161	− 269		− 0.08	+ 0.45
	$\epsilon+4\theta+3\Delta$		− 760	+ 3906	−10778		+ 0.08	− 0.45
	$2\epsilon+2\theta+3\Delta$		+ 52	− 143	+ 87			
	$2\epsilon+6\theta+5\Delta$		− 314	+ 1874	− 5403			
η^2						−0.317	+ 1.63	
	$4\theta+4\Delta$	−1679	+ 7272	−13527		−0.633	+10.02	
	$\epsilon+2\theta+2\Delta$		+ 274	− 63			− 1.20	
	$\epsilon+6\theta+6\Delta$		− 3474	+29267			+ 3.60	
	$-\epsilon+2\theta+2\Delta$		+ 1156	− 2171			− 2.40	
	2ϵ						− 0.21	
	$2\epsilon+4\theta+4\Delta$		+ 180	+ 113			+ 0.21	
	$2\epsilon+8\theta+8\Delta$		− 1375	+11897				
$\eta\eta'$	Δ					+0.227	− 1.30	
	$4\theta+3\Delta$	+3690	−12966	+19401		+0.340	−19.92	
	$\epsilon+2\theta+\Delta$		+ 222	− 1234			+ 1.96	
	$\epsilon+2\theta+3\Delta$		− 769	+ 2197			+ 0.01	
	$\epsilon+6\theta+5\Delta$		+ 9240	−70866			− 7.25	
	$-\epsilon+2\theta+\Delta$		− 646	+ 1806			+ 5.27	
	$2\epsilon+\Delta$		+ 99	− 444			+ 0.04	
	$2\epsilon+4\theta+3\Delta$		− 109	− 922			− 0.04	
	$2\epsilon+4\theta+5\Delta$		− 846	+ 4256				
	$2\epsilon+8\theta+7\Delta$		+ 4012	−31827				
η'^2						−0.039	+ 0.24	
	$4\theta+2\Delta$	−1780	+ 4725	− 5354		−0.039	+10.42	
	$\epsilon+2\theta+2\Delta$		+ 499	− 649			− 0.32	
	$\epsilon+6\theta+4\Delta$		− 5930	+40905			+ 2.86	
	$-\epsilon+2\theta$						− 2.54	
	$2\epsilon+2\Delta$		− 55	+ 285				
	$2\epsilon+4\theta+4\Delta$		+ 980	− 4150				
	$2\epsilon+8\theta+6\Delta$		− 2890	+20791				
j^2	$4\theta+3\Delta-\Sigma$	− 101	+ 493	− 1128			+ 0.11	
	$\epsilon+2\theta+2\Delta$		+ 587	− 2983				
	$\epsilon+6\theta+5\Delta-\Sigma$		− 193	+ 1759			+ 0.14	
	$-\epsilon+2\theta+\Delta-\Sigma$						− 0.14	
	$2\epsilon+\Delta+\Sigma$		− 192	+ 705				
	$2\epsilon+4\theta+4\Delta$		+ 298	− 1876				
	$2\epsilon+8\theta+7\Delta-\Sigma$		− 65	+ 616				
		m'				m'^2		

TABLE XXIXc—Continued.

$\frac{1}{3}\Xi$

Unit = 1″

	Cos	w^0	w^1	w^2
	$\frac{1}{2}\varepsilon+\theta+\Delta$	-31.4	$+131.0$	-255
	$\frac{3}{2}\varepsilon+3\theta+3\Delta$	-48.0	$+193.6$	-360
	$\frac{5}{2}\varepsilon+5\theta+5\Delta$	-15.2	$+81.7$	-201
	$\frac{7}{2}\varepsilon+7\theta+7\Delta$	-5.2	$+34.7$	-107
η	$\frac{1}{2}\varepsilon+3\theta+3\Delta$	$+1304$	-8173	$+30282$
	$-\frac{1}{2}\varepsilon+\theta+\Delta$	-196	$+506$	-598
	$\frac{3}{2}\varepsilon+\theta+\Delta$	$+34$	-146	$+292$
	$\frac{3}{2}\varepsilon+5\theta+5\Delta$	$+356$	-2212	$+6781$
	$\frac{5}{2}\varepsilon+3\theta+3\Delta$	$+1$	-67	$+294$
	$\frac{5}{2}\varepsilon+7\theta+7\Delta$	$+138$	-999	$+3437$
η'	$\frac{1}{2}\varepsilon+3\theta+2\Delta$	-1361	$+7468$	-25691
	$-\frac{1}{2}\varepsilon+\theta$			
	$\frac{3}{2}\varepsilon+\theta+2\Delta$	$+29$	-12	-155
	$\frac{3}{2}\varepsilon+5\theta+4\Delta$	-482	$+2635$	-7209
	$\frac{5}{2}\varepsilon+3\theta+4\Delta$	$+52$	-197	$+280$
	$\frac{5}{2}\varepsilon+7\theta+6\Delta$	-207	$+1348$	-4176
η^2	$\frac{1}{2}\varepsilon+\theta+\Delta$	-625	$+3058$	
	$\frac{1}{2}\varepsilon+5\theta+5\Delta$	-7151	$+70387$	
	$-\frac{1}{2}\varepsilon+3\theta+3\Delta$	$+5478$	-5874	
	$\frac{3}{2}\varepsilon+3\theta+3\Delta$	$+18$	$+1924$	
	$\frac{3}{2}\varepsilon+7\theta+7\Delta$	-2111	$+17665$	
	$-\frac{3}{2}\varepsilon+\theta+\Delta$	$+187$	-590	
	$\frac{5}{2}\varepsilon+9\theta+9\Delta$	-771	$+6931$	
$\eta\eta'$	$\frac{1}{2}\varepsilon+\theta+2\Delta$	-231	-1142	
	$\frac{1}{2}\varepsilon+5\theta+4\Delta$	$+17640$	-159928	
	$-\frac{1}{2}\varepsilon+3\theta+2\Delta$	-9842	$+1346$	
	$\frac{3}{2}\varepsilon+3\theta+4\Delta$	-892	$+3699$	
	$\frac{3}{2}\varepsilon+3\theta+2\Delta$	$+106$	-2494	
	$\frac{3}{2}\varepsilon+7\theta+6\Delta$	$+5918$	-45149	
	$-\frac{3}{2}\varepsilon+\theta$			
	$\frac{5}{2}\varepsilon+9\theta+8\Delta$	$+2513$	-20914	
η'^2	$\frac{1}{2}\varepsilon+\theta+\Delta$	-507	$+2729$	
	$\frac{1}{2}\varepsilon+5\theta+3\Delta$	-10202	$+84314$	
	$-\frac{1}{2}\varepsilon+3\theta+\Delta$	$+1055$	-678	
	$\frac{3}{2}\varepsilon-\theta+\Delta$	-100	$+387$	
	$\frac{3}{2}\varepsilon+3\theta+3\Delta$	$+871$	-2817	
	$\frac{3}{2}\varepsilon+7\theta+5\Delta$	-4065	$+27951$	
j^2	$\frac{1}{2}\varepsilon+\theta+\Delta$	$+601$	-3122	
	$\frac{1}{2}\varepsilon+5\theta+4\Delta-\Sigma$	-423	$+4435$	
	$-\frac{1}{2}\varepsilon+3\theta+2\Delta-\Sigma$	$+285$	-356	
	$\frac{3}{2}\varepsilon+3\theta+3\Delta$	$+426$	-2410	
	$\frac{3}{2}\varepsilon+7\theta+6\Delta-\Sigma$	-108	$+988$	
	$-\frac{3}{2}\varepsilon+\theta-\Sigma$	-106	$+402$	
		m'		

TABLE XXIXc—Continued.

$$\frac{1}{3}\Xi$$

Unit=1″

	Cos	w^{-1}			w^{-3}		
		w^0	w	w^2	w^0	w	w^2
η^3	$2\theta+2\varDelta$	+ 1568	− 8912		+3.6	−23.3	
	$6\theta+6\varDelta$	+ 5879	− 31559		+3.6	−23.3	
$\eta^2\eta'$	$2\theta+\varDelta$	− 2385	+ 11662		−4.7	+26.6	
	$2\theta+3\varDelta$	+ 2238	− 1015		−2.6	+18.4	
	$6\theta+5\varDelta$	−21644	+102003		−9.9	+59.1	
$\eta\,\eta'^2$	2θ				−0.2	+ 1.7	
	$2\theta+2\varDelta$	− 1723	− 7588		+4.0	−27.1	
	$6\theta+4\varDelta$	+25396	−103013		+7.6	−43.2	
η'^3	$2\theta+\varDelta$	− 1160	+ 5960		−1.4	+ 8.3	
	$6\theta+3\varDelta$	− 9257	+ 31500		−1.4	+ 8.3	
$j^2\eta$	$6\theta+5\varDelta-\Sigma$	+ 1040	− 6697		+0.3	− 2.1	
	$2\theta+2\varDelta$	− 5354	+ 25370	−61855			
$j^2\ \eta'$	$2\theta+\varDelta$	+ 2492	− 12023				
	$2\theta+2\varDelta-\Sigma$	− 53	+ 989		−0.1	+ 0.6	
	$6\theta+4\varDelta-\Sigma$	− 1285	+ 7413		−0.1	+ 0.6	
	$(\theta-\theta_0)$ sin						
η	$2\theta+2\varDelta$					− 0.55	+3.40
η'	$2\theta+\varDelta$					+ 0.41	−2.74
η^2	$4\theta+4\varDelta$					− 3.12	
	$\varepsilon+2\theta+2\varDelta$					− 1.10	
	$-\varepsilon+2\theta+2\varDelta$					− 1.10	
$\eta\,\eta'$	$\varDelta$	− 569.95	+ 2421.1	− 4950	+0.45	+ 5.94	
	$4\theta+3\varDelta$					− 5.75	
	$\varepsilon+2\theta+\varDelta$					+ 0.20	
	$\varepsilon+2\theta+3\varDelta$					+ 0.82	
	$-\varepsilon+2\theta+\varDelta$					+ 1.01	
η'^2	$4\theta+2\varDelta$					+ 2.55	
	$\varepsilon+2\theta+2\varDelta$					− 0.15	
	$-\varepsilon+2\theta$					− 0.15	
	$(\theta-\theta_0)^2$ cos						
$\eta\,\eta'$	$\varDelta$					− 0.26	
η'^2						+ 0.20	
		m'			m'^2		

COMPARISON OF TABLES.

As a computer would discover in constructing tables, and as will be evident from an application of the method to a planet, the coefficients in Table II and others of the same form are given with unnecessary accuracy. Although so many digits would never be required, except in a much more exhaustive development, they are given, for completeness, as they resulted from computation.

In all the tables whose constructions involve the multiplication of trigonometric series, the errors are difficult or impossible to determine. Although v. Zeipel's manuscript, which the author generously furnished for comparison, is of assistance, the computations are not entirely parallel, and comparison is not always possible. Many of the computations are so long and

complicated that the origin of certain discrepancies is obscure. Aside from possible errors of calculation, differences are due to the independent adoption of the highest powers of m', w, η, η', j^2, and the number of arguments in a given series or product of series. In most cases our series are more complete than v. Zeipel's. Whether or not the extension of the tables increases the accuracy of the result remains to be seen from future applications of the theory.

Tables II–XV.—The discrepancies seem to be due to v. Zeipel's errors of calculation and to their subsequent effects. The larger number of these errors have been traced in the manuscript.

Tables XVI, XVII check satisfactorily.

Table XVIII.—The bracketed quantities in the first three columns are in error through previous discrepancies. We did not discover the source of the general disagreement in terms of the third degree, second order in the mass. These terms do not affect v. Zeipel's subsequent tables, since they are of order higher than have been included.

Tables XIX, XX agree satisfactorily.

Table XXI.—The discrepancies are numerous and their origin is obscure because of the very long computation involved. In addition to performing a complete duplicate computation, the terms of first degree and a part of the computation of second degree terms were checked by the solution of the differential equation in the form given in Z 64. With the exception of three or four single instances, the discrepancies occur in two groups, having the following probable explanations. The neglect of the term

$$\frac{3}{4}[W_1{}^2]\frac{d\phi_1}{d\theta}$$

in Z 65, eq. (109), by v. Zeipel accounts for one group of differences. The other group can be fairly well explained by an error in the addition of second order terms in $+\frac{w}{2}\phi_1$ to $\phi_2-\frac{w}{2}\phi_1$. Assuming that for these terms he added $-w\phi$, and, correcting his table, three discrepancies are removed and two others are improved.

Table XXII.—Considering the disagreements in Table XXI, Table XXII checks satisfactorily.

Table XXIII–XXVII.—These tables, like II–XV, are simple in construction, and the discrepancies are due to errors of calculation, or they are the result of previous ones, with the exception that some quantities have different numerical values because they are more inclusive. The latter have been indicated by (   ).

Table XXVIII.—The discrepancies arise from the quantities in parentheses in Table XXVII. The omission of the term depending upon the inclination is justifiable in view of its magnitude.

Table XXIX.—The discrepancies are numerous and striking, but, since v. Zeipel does not give the formulæ of computation, they remain unexplained. The remark is made (Z 77), "Die Berechnung der Funktion $[(1-e\cos\varepsilon)(\overline{W}_3-\overline{W}_3'')]$, welche eine sehr komplicirte war, wird hier nicht im Einzelnen mitgetheilt." For this reason the development of the formulæ which we used has been included and the auxiliary functions $2[T_4]$, W_3, $[(1-e\cos\varepsilon)\ \overline{W}_3'']$ have been tabulated. The differences are not serious because of the high rank of the function. Our table is deficient in certain terms whose computation would be long and the omission of which is justifiable in view of their magnitude.

PERTURBATIONS OF THE MEAN ANOMALY.

For clearness some of v. Zeipel's developments will be amplified and repeated in an order which we found more convenient.

The determination of the disturbed mean anomaly is accomplished with the integration of Z 9, eq. (47), (which implicitly contains Z 8, eq. (38)). By Z 9, eq. (43),

$$\theta=\tfrac{1}{2}(\varepsilon-e\sin\varepsilon)-g'=\tfrac{1}{2}g-g'$$

The differential equation is repeated in Z 78, eq. (124), in which is emphasized the fact that the arguments are functions of both ε and θ, as is the case for $\frac{dW}{d\varepsilon}$.

If we observe the character of θ as it is expressed in the definition and recall that we have admitted trigonometric terms in θ, multiplied by t, it is evident that this argument, which is a function of the disturbed positions of the planet and Jupiter, is not periodic, but varies continuously with the time. In the foregoing equation g and g' can not be regarded as angles which are always less than 360°. θ contains, therefore, a nontrigonometric secular part in ε and a periodic part in θ and ε.

If we write

$$\theta=(\theta-[\theta])+[\theta]$$

$\theta-[\theta]$ contains the secular term in ε as well as periodic terms. The segregation of terms of different type can be made explicit by the introduction of

$$\theta=\vartheta+\theta_1(\vartheta,\varepsilon)+\theta_2(\vartheta,\varepsilon)+\theta_3(\vartheta,\varepsilon)+\cdot\cdot\cdot\cdot\cdot\cdot \qquad Z\ 78,\ \text{eq. } (125)$$

where ϑ is a function of ε and $\theta_1, \theta_2, \theta_3 \cdot\cdot\cdot$ are the periodic parts of $\theta-[\theta]$, i. e., they are entirely trigonometric functions of ε. This covers the condition that θ_i can not include trigonometric secular terms in ε. By definition of ϑ and θ_i

$$\frac{d\vartheta}{d\varepsilon}=\left[\frac{d\theta}{d\varepsilon}\right]=[F(\theta,\varepsilon)]-\frac{d[n'\delta z']}{d\varepsilon}$$

$$\Sigma\frac{d\theta_i}{d\varepsilon}=(F(\theta,\varepsilon)-[F(\theta,\varepsilon)])-\frac{d}{d\varepsilon}(n'\delta z'-[n'\delta z'])$$

where $[n'\delta z']$ is the long period term between Jupiter and Saturn.

The derivative of (125) is

$$\frac{d\theta}{d\varepsilon}=\frac{\partial\theta}{\partial\varepsilon}+\frac{\partial\theta}{\partial\vartheta}\frac{d\vartheta}{d\varepsilon}$$

$$=\left(\frac{\partial\theta_1}{\partial\varepsilon}+\frac{\partial\theta_2}{\partial\varepsilon}+\frac{\partial\theta_3}{\partial\varepsilon}+\cdot\cdot\cdot\cdot\cdot\cdot\right)+\left(1+\frac{\partial\theta_1}{\partial\vartheta}+\frac{\partial\theta_2}{\partial\vartheta}+\frac{\partial\theta_3}{\partial\vartheta}+\cdot\cdot\cdot\cdot\cdot\right)\frac{d\vartheta}{d\varepsilon}$$

Expanding $F(\theta,\varepsilon)$, eq. (124) in a Taylor's series in ascending powers of θ_i and making the above substitution for $\frac{d\theta}{d\varepsilon}$, (124) becomes (126), in which

$$F(\vartheta,\ \varepsilon)=\frac{1}{2}(1-e\cos\varepsilon)\left\{w+(1-w)\,W-\frac{3}{4}(1-w)\left(W-\frac{1}{3}\Xi\right)\left(W+\frac{1}{9}\Xi\right)+\cdot\cdot\cdot\cdot\right\};\qquad \theta=\vartheta$$

From the Taylor's series $\frac{d\vartheta}{d\varepsilon}$ is written in (127). This is the differential equation for ϑ, the right-hand side of which can be computed.

Substituting $\frac{d\vartheta}{d\varepsilon}$ in (126) and equating functions of equal rank, we have the differential equations (128_1-128_3) for θ_i, which can be integrated in succession.

Before integration we convert eqs. (128) into differential equations for $n\delta z$ as follows:

Let

$$n\delta z=(n\delta z\ -[n\delta z])+[n\delta z]$$
$$=n\delta z_1+n\delta z_2\ +n\delta z_3+\cdot\cdot\cdot\cdot+[n\delta z] \qquad Z\ 88,\ \text{eq. } (144),$$

where $n\delta z_i$ is not only a function of first and higher orders in m', in which the lowest rank is i, but is entirely trigonometric or periodic. Then

Z 9, eq.(46) gives $n\delta z-[n\delta z]=\frac{2}{1-w}\left\{\theta_1(\vartheta,\varepsilon)+\theta_2(\vartheta,\varepsilon)+\theta_3(\vartheta,\varepsilon)+\cdot\cdot\cdot\cdot\cdot\cdot+w\eta\sin\varepsilon+(n'\delta z'-[n'\delta z'])\right\}$

and

$$[n\delta z]=\frac{2}{1-w}\left\{\vartheta-\frac{w}{2}\varepsilon+[n'\delta z']+c'-\mu c\right\} \qquad Z\ 88,\ \text{eq. } (145),$$

where it is to be noticed that $[n\delta z]$, unlike $[W]$, is not free from terms in ε. Subdividing the first of these two equations according to rank, we have Z 79, eqs. (130), in which $-n'\delta z'+[n'\delta z']$ can be neglected.

Differentiating eqs. (130) partially with respect to ε, substituting in eqs. (128), evaluating the right-hand sides of eqs. (128), we have eqs. (131_1-131_3), in which the superscript indicates that only terms of first order in the mass are included, and where the argument ϑ replaces the argument θ.

For purposes of calculation, the integrations are arranged as follows:

In

$$\overline{W}=\overline{W}_1+\overline{W}_2'+[\overline{W}_2]+(\overline{W}_2''+\overline{W}_3''+\overline{W}_4'')$$

consider first only $\overline{W}_2''+\overline{W}_3''+\overline{W}_4''$ in the integration of eqs. (131). The integrations will concern only part of the terms of first order in $n\delta z_1+n\delta z_2+n\delta z_3$. It is shown by v. Zeipel that the integration for all three ranks can be performed conveniently at the same time. Let this part of the function be indicated by enclosing it in ( ). The integral

$$(n\delta z_1^{(1)})+(n\delta z_2^{(1)})+(n\delta z_3^{(1)})$$

which is a trigonometric series, is given by Z 80, eq. (135), in which the coefficients $\overline{L}_{p\cdot q}$ are defined by (136) and are easily derived from Table XXVII. The coefficients $L_{p\cdot q}$ are tabulated in Table XXX.

The remaining terms of rank one which are of first order only, namely, $n\delta z_1^{(1)}-(n\delta z_1^{(1)})$, are given by the first of Z 81, eqs. (137), in which $\overline{W}_1$, $\overline{W}_2$, $[\overline{W}_2]$, can be written by inspection from Tables XVII, XVIII, XIX, XXII*a*. The function is tabulated in Table XXXI.

The remaining terms of first order in $n\delta z_2$ and $n\delta z_3$ are given by the sum of Z 82, eqs. (139) and (140). The function

$$n\delta z_2^{(1)}-(n\delta z_2^{(1)})+n\delta z_3^{(1)}-(n\delta z_3^{(1)})$$

is given in Table XXXII.

These developments complete $n\delta z^{(1)}$ within the limits of the tables, and we next consider $n\delta z^{(2)}$. We shall limit ourselves to functions in which the lowest rank is the first or second. Consequently, $n\delta z_3$ contributes nothing.

Any function of second order in the mass and first rank must contain the factor $\frac{m'^2}{w^3}$ and in the given $F(\vartheta, \varepsilon)$ this factor occurs only in $\overline{W}_1^{(2)}$. We have, therefore, by Z 80, eq. (131_1), for one part of $n\delta z_1^{(2)}$, indicated by parentheses,

$$(n\delta z_1^{(2)})=\int\{(1-e\cos\varepsilon)\,\overline{W}_1^{(2)}-[(1-e\cos\varepsilon)\,\overline{W}_1^{(2)}]\}d\varepsilon$$

This function is tabulated in Table XXXIII.

TABLE XXX.

Unit = 1″

n	0	1	2	3	4	5	6	7	8	9	10
$\bar{L}_{0\cdot0}(n.-n)$		− 586.66	− 237.94	+ 225.33	+ 51.36	+ 17.14	[+ 6.81]	+ 2.99	+ 1.38	+ 0.70	+ 0.36
$\bar{L}_{1\cdot0}(n+1.-n)$	+ 225.4	+ 116.5	+ 66.4	− 137.5	− 28.3	− 6.5	− 0.8	+ 0.6	+ 0.8	+ 0.6	+ 0.5
$\bar{L}_{1\cdot0}(n-1.-n)$	− 225.4	− 1608.8		+ 11669.6	+ 1185.8	+ 1506.7	+ 396.2	− 149.9	− 66.5	− 32.3	− 16.5
$\bar{L}_{0\cdot1}(n.-n+1)$	− 1023.6	− 185.7	− 61.5	− 88.4	− 55.4	− 32.5	− 18.9	− 11.1	− 6.5	− 3.9	− 2.4
$\bar{L}_{0\cdot1}(n.-n-1)$	+ 1023.6		− 16751.2	− 1989.6	+ 2255.6	+ 605.5	+ 233.3	+ 105.2	+ 51.8	+ 26.9	+ 14.5
$\bar{L}_{2\cdot0}(n+2.-n)$	+ 43	+ 258	+ 50	+ 6	− 2	− 2	− 1	0	0	0	0
$\bar{L}_{2\cdot0}(n.-n)$		−10130	− 1514	− 4151	− 972	+ 626	+ 75	− 18	− 32	− 28	− 21
$\bar{L}_{2\cdot0}(n-2.-n)$	− 43	− 415	+ 776	− 3992		− 58550	− 5885	+ 8504	+ 2410	+ 981	+466
$\bar{L}_{1\cdot1}(n+1.-n+1)$	− 821	− 368	− 64	+ 27	+ 21	+ 8	+ 1	− 2	− 4	− 3	− 3
$\bar{L}_{1\cdot1}(n-1.-n+1)$	− 7178		+ 2160	− 1047	+ 3345	+ 1444	+ 767	+ 440	+ 262	+ 160	+ 99
$\bar{L}_{1\cdot1}(n+1.-n-1)$	+ 7178	+ 2559	+ 4711	+ 1164	− 1096	− 191	− 10	+ 30	+ 34	+ 29	+ 22
$\bar{L}_{1\cdot1}(n-1.-n-1)$	+ 821	− 1797	+ 12244		+ 170804	+ 19042	− 25415	− 7322	− 3025	− 1459	−765
$\bar{L}_{0\cdot2}(n.-n+2)$	− 408	+ 1129	+ 188	+ 56	+ 33	+ 27	+ 21	+ 16	+ 12	+ 9	+ 6
$\bar{L}_{0\cdot2}(n.-n)$		−11930	+ 930	− 3531	− 1772	− 1018	− 614	− 379	− 238	− 151	− 96
$\bar{L}_{0\cdot2}(n.-n-2)$	+ 408	+ 3418		− 123754	− 15453	+ 18945	+ 5553	+ 2333	+ 1142	+ 608	+340
$\bar{L}_{0\cdot0}(n+1.-n+1)+\sigma$	+ 587	+ 512	+ 237	+ 121	+ 65	+ 36	+ 20	+ 12	+ 7	+ 4	+ 2
$\bar{L}_{0\cdot0}(n-1.-n-1)-\sigma$	− 587	+ 433	+ 1612		− 6382	− 582	+ 621	+ 150	+ 53	+ 22	+ 10
$\bar{L}_{0\cdot0}(n+1.-n-1)+\delta$	+ 3570	+ 1201	− 1615	− 450	− 177	− 81	− 40	− 21	− 12	− 7	− 4
$\bar{L}_{0\cdot0}(n-1.-n+1)-\delta$	− 3570		+ 6552	+ 725	− 939	− 244	− 91	− 40	− 19	− 10	− 5
Factor w											
$\bar{L}_{0\cdot0}(n.-n)$		+ 2413.6	+ 1075.1	− 1768.5	− 379.8	− 130.8	− 54.8	− 25.6	− 12.8	− 6.7	− 3.6
$\bar{L}_{1\cdot0}(n+1.-n)$	− 737	+ 1084	− (187)	+ 1021	+ 200	+ 46	+ 4	− 7	− 8	− 7	− 5
$\bar{L}_{1\cdot0}(n-1.-n)$	+ 737	+ 6180		− 70526	− 9499	+ 18752	+ 4232	+ 1558	+ 701	+ 351	+187
$\bar{L}_{0\cdot1}(n.-n+1)$	+ 3420	+ 766	− 275	+ 317	+ 275	+ 193	+ 128	+ 83	+ 54	+ 35	+ 22
$\bar{L}_{0\cdot1}(n.-n-1)$	− 3420		+ 91438	+ 14703	− 26491	− 6018	− 2248	− 1027	− 521	− 282	−159
$\bar{L}_{2\cdot0}(n+2.-n)$	− 352	− 1472	− (282)	− 66	− 2	+ 8	+ 5	0	− 2	− 4	
$\bar{L}_{2\cdot0}(n.-n)$		+58689	[(+ 5580)]	+ 16299	+ (7168)	− 8328	− 998	[+ 126]	+ 324	+ 288	
$\bar{L}_{2\cdot0}(n-2.-n)$	+ 352	+ 1153	[(− 3214)]	− 34026		+542444	+ 67140	−144713	−33619	−12916	
$\bar{L}_{1\cdot1}(n+1.-n+1)$	+ 2329	+ 2072	+ 929	+ 85	− 27	− 11	+ 16	+ 32	+ 40	+ 36	
$\bar{L}_{1\cdot1}(n-1.-n+1)$	+37422		− 32828	+ 3112	[− 25186]	− 10886	− 6134	− 3752	− 2386	− 1553	
$\bar{L}_{1\cdot1}(n+1.-n-1)$	−37422	[(−14117)]	− 10066	− (7326)	+ 13010	+ 2031	+ 144	− 295	− 358	− 315	
$\bar{L}_{1\cdot1}(n-1.-n-1)$	− 2329	[(+ 6421)]	+109204		−1484032	−206260	+414666	+ 96717	+37531	+17877	

	$\bar{L}_{0\cdot2}(n.-n+2)$	+ 1159	− 3828	− 1093	− 361	− 196	− 156	− 135	− 116	− 95	− 77	
	$\bar{L}_{0\cdot2}(n.-n)$		+67536	− 396	+ 22629	+ 11805	+ 7288	+ 4739	+ 3153	+ 2120	+ 1433	
	$L_{0\cdot2}(n.-n-2)$	− 1159	−42030		+1004134	+ 157973	−295725	− 69195	− 27124	−13085	− 7034	
	$\bar{L}_{0\cdot0}(n+1.-n+1)+\sigma$	− 1958	− 2242	− 1276	− 751	− 453	− 278	− 172	− 108	− 68	− 42	
	$L_{0\cdot0}(n-1.-n-1)-\sigma$	+ 1958	− 1481	− 15578		+ 63070	+ 7179	− 11156	− 2227	− 745	− 307	
	$L_{0\cdot0}(n+1.-n-1)+\partial$	−19300	− 7328	+ 15213	+ 3975	+ 1590	+ 756	+ 400	+ 218	+ 125	+ 74	
	$L_{0\cdot0}(n-1.-n+1)-\partial$	+19300		− 36180	− 5178	+ 9908	+ 2454	+ 935	+ 427	+ 215	+ 116	
Factor n^2	$L_{0\cdot0}(n.-n)$		− 5214	− 2489	+ 7952	+ 1488	+ 510	+ 221	+ 108	+ 56	+ 31	+ 18
	$L_{1\cdot0}(n+1.-n)$	+ 1164	− 3601	+ (397)	− 5808	− 1227	− 390	− 127	− 34	+ 1	+ 12	
	$L_{1\cdot0}(n-1.-n)$	− 1164	−12114		+ 366108	+ 42675	[−136738]	− 24224	− 8325	− 3696	− 1873	
	$\bar{L}_{0\cdot1}(n.-n+1)$	− 6831	− 1504	+ 2232	− 328	− 601	− 525	− 399	− 291	− 207	− 145	
	$\bar{L}_{0\cdot1}(n.-n-1)$	+ 6831		−466174	− 60481	+ 187295	+ 33010	+ 11447	+ 5151	+ 2648	+ 1472	

The coefficients in parentheses are functions of the coefficients in parentheses in Table XXVII.

TABLE XXXI.

$n\delta z_1^{(1)} - (n\delta z_1^{(1)})$ Unit=1″

	Sin	v^{-1}		
		u^0	u	u^2
	$\varepsilon+2\vartheta+2\Delta$	+294.89	−740.6	+734
η	ε			
	$\varepsilon+4\vartheta+4\Delta$	−839.5	+3495	−6224
	$2\varepsilon+2\vartheta+2\Delta$	−147.4	+517	−737
η'	$\varepsilon+\Delta$			
	$\varepsilon+4\vartheta+3\Delta$	+1229.8	−4069	+5671
η^2	$-\ \varepsilon+2\vartheta+2\Delta$	+784	(−3570)	(+10522)
	$\varepsilon+2\vartheta+2\Delta$	−202	(−1657)	(+13183)
	$\varepsilon+6\vartheta+6\Delta$	+2940	(−17009)	[(+43527)]
	$2\varepsilon+4\vartheta+4\Delta$	+415	(−2587)	(+6440)
	2ε		−192	+705
$\eta\,\eta'$	$-\ \varepsilon+2\vartheta+\Delta$	−2386	(+11567)	(+37527)
	$\varepsilon+2\vartheta+3\Delta$	+1492	(−968)	(−12562)
	$\varepsilon+2\vartheta+\Delta$	−1962	(+9257)	(−23263)
	$\varepsilon+6\vartheta+5\Delta$	−8658	(+42767)	(−92732)
	$2\varepsilon+4\vartheta+3\Delta$	−615	(+3264)	(−6905)
	$2\varepsilon+\Delta$	.	+142	−605
η'^2	$-\ \varepsilon+2\vartheta$	+1634	−7081	+16199
	$\varepsilon+2\vartheta+2\Delta$	−861	−3794	+22127
	$\varepsilon+6\vartheta+4\Delta$	+6349	−25753	+45318
j^2	$-\ \varepsilon+2\vartheta+\Delta-\Sigma$	+866	−4260	+10988
	$\varepsilon+6\vartheta+5\Delta-\Sigma$	+260	−1674	+5101
	$\varepsilon+2\vartheta+2\Delta$	−2677	+12681	−30930
η^3	$\varepsilon+4\vartheta+4\Delta$	+5907	−11149	
	$-\ \varepsilon+4\vartheta+4\Delta$	−269	+5158	
	$\varepsilon+8\vartheta+8\Delta$	−11300	+76249	
$\eta^2\eta'$	$\varepsilon+4\vartheta+5\Delta$	−11449	+42212	
	$\varepsilon+4\vartheta+3\Delta$	−11270	+951	
	$-\ \varepsilon+4\vartheta+3\Delta$	+1744	−23941	
	$\varepsilon+8\vartheta+7\Delta$	+50005	−304611	
$\eta\,\eta'^2$	$\varepsilon+4\vartheta+4\Delta$	+26091	−71730	
	$\varepsilon+4\vartheta+2\Delta$	+3985	+16118	
	$-\ \varepsilon+4\vartheta+2\Delta$	−3137	+35021	
	$\varepsilon+8\vartheta+6\Delta$	−73583	+400009	
η'^3	$\varepsilon+4\vartheta+3\Delta$	−13756	+22165	
	$-\ \varepsilon+4\vartheta+\Delta$	+3317	−18452	
	$\varepsilon+8\vartheta+5\Delta$	+36006	−172164	
$j^2\eta$	$\varepsilon+4\vartheta+3\Delta-\Sigma$	−1707	+13125	
	$-\ \varepsilon+4\vartheta+3\Delta-\Sigma$	−2112	+15096	
	$\varepsilon+8\vartheta+7\Delta-\Sigma$	−2381	+18919	
	$\varepsilon+4\vartheta+4\Delta$	+14204	−88026	
$j^2\ \eta'$	$\varepsilon+4\vartheta+4\Delta-\Sigma$	−554	+140	
	$-\ \varepsilon+4\vartheta+2\Delta-\Sigma$	+3545	−22885	
	$\varepsilon+8\vartheta+6\Delta-\Sigma$	+3827	−27870	
	$\varepsilon+4\vartheta+3\Delta$	−17503	+99584	
	$+(\vartheta-\vartheta_0)$ cos			
η	ε	−767.7	+2821	−5210
η'	$\varepsilon+\Delta$	+570.0	−2421	+4950
η^2	2ε	−384	+1410	−2605
$\eta\,\eta'$	$2\varepsilon+\Delta$	+285	−1211	+2475
η^3	ε	−6624	+47448	
$\eta^2\eta'$	$\varepsilon+\Delta$	[+17970]	[−120603]	
	$-\ \varepsilon+\Delta$	+8984	−60301	
$\eta\,\eta'^2$	$\varepsilon+2\Delta$	−10478	+70250	
	ε	−25564	+157424	
η'^3	$\varepsilon+\Delta$	+15678	−94846	
$j^2\eta$	$\varepsilon+\Delta+\Sigma$	−22012	+121258	−359162
	ε	+25565	−157424	[+511232]
$j^2\ \eta'$	$\varepsilon+\Sigma$	+12048	−76364	+251640
	$\varepsilon+\Delta$	−23524	+150306	−498328
			m'	

Table XXXII.

$n\delta z_2^{(1)} - (n\delta z_2^{(1)})$

$+\, n\delta z_3^{(1)} - (n\delta z_3^{(1)})$ Unit=1″.

	Sin	w^0	w
	$\varepsilon+2\vartheta+2\Delta$	$-$ 294.9	$+$ 1036
η	ε	$+$ 384	$-$ 1410
	$\varepsilon+4\vartheta+4\Delta$	$+$ 1679	$-$ 10348
	$2\varepsilon+2\vartheta+2\Delta$	$-$ 74	$+$ 149
η'	$\varepsilon+\Delta$	$-$ 285	$+$ 1211
	$\varepsilon+4\vartheta+3\Delta$	$-$ 2460	$+$ 13057
η^2	$-\varepsilon+2\vartheta+2\Delta$	$-$ 101	$-$ 883
	$\varepsilon+2\vartheta+2\Delta$	$-$ 978	$+$ 6459
	$\varepsilon+6\vartheta+6\Delta$	$-$ 8820	($+$ 77487)
	$2\varepsilon+4\vartheta+4\Delta$	$+$ 424	$-$ 1332
	2ε	[$+$ 96]	[$-$ 352]
$\eta\,\eta'$	$-\varepsilon+2\vartheta+\Delta$	$-$ 2068	$+$ 8418
	$\varepsilon+2\vartheta+3\Delta$	$-$ 1492	$+$ 2460
	$\varepsilon+2\vartheta+\Delta$	$+$ 2280	$-$ 12618
	$\varepsilon+6\vartheta+5\Delta$	$+$ 25974	($-$206223)
	$2\varepsilon+4\vartheta+3\Delta$	$-$ 615	$+$ 1420
	$2\varepsilon+\Delta$	[$-$ 71]	[$+$ 303]
η'^2	$-\varepsilon+2\vartheta$	$+$ 1634	$-$ 5447
	$\varepsilon+2\vartheta+2\Delta$	$+$ 861	$+$ 2933
	$\varepsilon+6\vartheta+4\Delta$	$-$ 19047	[$+$134400]
j^2	$-\varepsilon+2\vartheta+\Delta-\Sigma$	$+$ 866	$-$ 3394
	$\varepsilon+6\vartheta+5\Delta-\Sigma$	$-$ 780	$+$ 7362
	$\varepsilon+2\vartheta+2\Delta$	$+$ 2677	$-$ 15358
η^3	$\varepsilon+4\vartheta+4\Delta$	$-$ 5098	
	$-\varepsilon+4\vartheta+4\Delta$	$+$ 4499	
	$\varepsilon+8\vartheta+8\Delta$	$+$ 45200	
$\eta^2\eta'$	$\varepsilon+4\vartheta+5\Delta$	$+$ 22898	
	$\varepsilon+4\vartheta+3\Delta$	$+$ 5322	
	$-\varepsilon+4\vartheta+3\Delta$	$-$ 11270	
	$\varepsilon+8\vartheta+7\Delta$	$-$200020	
$\eta\,\eta'^2$	$\varepsilon+4\vartheta+4\Delta$	$-$ 52182	
	$\varepsilon+4\vartheta+2\Delta$	$+$ 2712	
	$-\varepsilon+4\vartheta+2\Delta$	$+$ 4408	
	$\varepsilon+8\vartheta+6\Delta$	$+$294332	
η'^3	$\varepsilon+4\vartheta+3\Delta$	$+$ 27512	
	$-\varepsilon+4\vartheta+\Delta$	$+$ 6634	
	$\varepsilon+8\vartheta+5\Delta$	$-$144024	
$j^2\eta$	$\varepsilon+4\vartheta+3\Delta-\Sigma$	$+$ 4022	
	$-\varepsilon+4\vartheta+3\Delta-\Sigma$	$-$ 3616	
	$\varepsilon+8\vartheta+7\Delta-\Sigma$	$+$ 9524	
	$\varepsilon+4\vartheta+4\Delta$	$-$ 28408	
$j^2\,\eta'$	$\varepsilon+4\vartheta+4\Delta-\Sigma$	$+$ 1108	
	$-\varepsilon+4\vartheta+2\Delta-\Sigma$	$+$ 7090	
	$\varepsilon+8\vartheta+6\Delta-\Sigma$	$-$ 15308	
	$\varepsilon+4\vartheta+3\Delta$	$+$ 35006	
		m'	

The coefficients in parentheses differ from v. Zeipel's values because they contain additional terms. See p. 134.

The remaining terms in the differential equation for $n\delta z_1^{(2)}$ are, by eq. (143),

$$\frac{\partial}{\partial\varepsilon}\{n\delta z_1^{(2)}-(n\delta z_1^{(2)})\}=(1-e\cos\varepsilon)\left\{\overline{W}_2'^{(2)}+[\overline{W}_2]^{(2)}-\frac{3}{4}\left(\overline{W}_1^{(1)}-\frac{1}{3}\Xi_1^{(1)}\right)\left(\overline{W}_1^{(1)}+\frac{1}{9}\Xi_1^{(1)}\right)\right\}$$
$$-\left[(1-e\cos\varepsilon)\left\{\overline{W}_2'^{(2)}+[\overline{W}_2]^{(2)}-\frac{3}{4}\left(\overline{W}_1^{(1)}-\frac{1}{3}\Xi_1^{(1)}\right)\left(\overline{W}_1^{(1)}+\frac{1}{9}\Xi_1^{(1)}\right)\right\}\right]$$

all the terms of which are of the second order whose lowest rank is the second. They therefore contain the factor $\frac{m'^2}{w^2}$.

To obtain $n\delta z_2^{(2)}$ it is necessary to return to eqs. (124)–(130) and make developments for terms of the second order similar to those for first order. The resulting differential equation is:

$$\frac{\partial}{\partial\varepsilon}n\delta z_2^{(2)}=\frac{(1-e\cos\varepsilon)}{2}\{n\delta z_1^{(1)}-(n\delta z_1^{(1)})\}\frac{\partial}{\partial\vartheta}\overline{W}_1^{(1)}-(1-e\cos\varepsilon)\eta\ w\ \sin\varepsilon\frac{\partial}{\partial\vartheta}\overline{W}_1^{(2)}$$
$$-\left[\frac{(1-e\cos\varepsilon)}{2}\{n\delta z_1^{(1)}-(n\delta z_1^{(1)})\}\frac{\partial}{\partial\vartheta}\overline{W}_1^{(1)}-(1-e\cos\varepsilon)\eta\ w\ \sin\varepsilon\frac{\partial}{\partial\vartheta}\overline{W}_1^{(2)}\right]$$
$$-\frac{1}{2}[(1-e\cos\varepsilon)\overline{W}_1^{(1)}]\frac{\partial}{\partial\vartheta}\{n\delta z_1-(n\delta z_1^{(1)})\}-\frac{w}{2}\frac{\partial}{\partial\vartheta}(n\delta z_1^{(2)}).$$

The sum of the last two equations, when integrated, gives the terms of second order having the factor $\frac{m'^2}{w^2}$. It has been shown by v. Zeipel through computation and we have shown analytically that

$$\overline{W}_2'^{(2)}+\frac{1}{2}\{n\delta z_1^{(1)}-(n\delta z_1^{(1)})\}\frac{\partial}{\partial\vartheta}\overline{W}_1^{(1)}=\eta\ w\ \sin\varepsilon\frac{\partial}{\partial\vartheta}\overline{W}_1^{(2)}$$

and

$$[(1-e\cos\varepsilon)\overline{W}_1^{(1)}]\frac{\partial}{\partial\vartheta}\{n\delta z_1^{(1)}-(n\delta z_1^{(1)})\}+w\frac{\partial}{\partial\vartheta}(n\delta z_1^{(2)})=0.$$

Therefore,

$$\frac{\partial}{\partial\varepsilon}\{n\delta z_1^{(2)}-(n\delta z_1^{(2)})+n\delta z_2^{(2)}\}=(1-e\cos\varepsilon)\left\{[\overline{W}_2]^{(2)}-\frac{3}{4}\left(\overline{W}_1^{(1)}-\frac{1}{3}\Xi_1^{(1)}\right)\left(\overline{W}_1^{(1)}+\frac{1}{9}\Xi_1^{(1)}\right)\right\}$$
$$-\left[(1-e\cos\varepsilon)\left\{[\overline{W}_2]^{(2)}-\frac{3}{4}\left(\overline{W}_1^{(1)}-\frac{1}{3}\Xi_1^{(1)}\right)\left(\overline{W}_1^{(1)}+\frac{1}{9}\Xi_1^{(1)}\right)\right\}\right]$$

The integral is tabulated in Table XXXIV.

Summarizing, we have included first order terms in

$$n\delta z_1+n\delta z_2+n\delta z_3$$

given by tables XXX, XXXI, XXXII and second order terms in

$$n\delta z_1+n\delta z_2$$

given by Tables XXXIII and XXXIV. The addition of Tables XXX–XXXIV gives the short period terms in $n\delta z$, or, the function

$$n\delta z-[n\delta z]$$

which is tabulated in Table XXXV.

Returning now to the differential equation for ϑ, the evaluation of $F(\vartheta,\varepsilon)$ and its derivatives in Z 78, eq. (127) gives Z 91, eq. (146). The variable does not appear; $\frac{d\vartheta}{d\varepsilon}$ is a function of ϑ alone Therefore the function is of long period. The integration is one step in the determination of $[n\delta z]$, the long period terms in the perturbations of the mean anomaly.

The function $[(1-e\cos\varepsilon)\overline{W}]$ is tabulated in Table XXIX*b*.

The function $\left[(1-e\cos\varepsilon)\left(\overline{W}-\frac{1}{3}\Xi\right)\left(\overline{W}+\frac{1}{9}\Xi\right)\right]$, computed from Tables XXIX*a* and XXIX*c*, is given in Table XXXVI.

The function $\left[(1-e\cos\varepsilon)(\theta_1+\theta_2+\theta_3)\dfrac{\partial W}{\partial\vartheta}\right]$ is computed as follows:

First,
$$\theta-\vartheta=\theta_1+\theta_2+\theta_3$$
is given by Z 93, eq. (150) by means of Table XXXV, and $\dfrac{\partial W}{\partial\vartheta}$ is readily written by inspection of Table XXIX*a*. Performing the indicated multiplications and retaining only the terms which are independent of ε, we have the required function as tabulated in Table XXXVII.

By eq. (146), the sum of Tables XXIX*b*, XXXVI, and XXXVII, multiplied by the factor $\dfrac{1-w}{w}$, gives $\Phi(\vartheta)$, tabulated in Table XXXVIII.

TABLE XXXIII.

$(n\delta z_1^{(2)})$ Unit=1″

	Sin	w^{-3}		
		w^0	w	w^2
η	$\varepsilon+4\vartheta+4\Delta$	−0.316	+1.59	−3.6
η'	$\varepsilon+4\vartheta+3\Delta$	+0.114	−0.67	+1.8
η^2	$-\varepsilon+2\vartheta+2\Delta$	+2.62	−16.8	
	$\varepsilon+2\vartheta+2\Delta$	+4.42	−28.4	
	$\varepsilon+6\vartheta+6\Delta$	+1.80	−11.7	
	$2\varepsilon+4\vartheta+4\Delta$	+0.16	−0.8	+1.8
$\eta\eta'$	$-\varepsilon+2\vartheta+\Delta$	−6.18	+36.9	
	$\varepsilon+2\vartheta+3\Delta$	−1.90	+13.6	
	$\varepsilon+2\vartheta+\Delta$	−5.57	+32.8	
	$\varepsilon+6\vartheta+5\Delta$	−3.95	+23.6	
	$2\varepsilon+4\vartheta+3\Delta$	−0.06	+0.3	−0.9
η'^2	$-\varepsilon+2\vartheta$	+4.04	−21.4	
	$\varepsilon+2\vartheta+2\Delta$	+2.12	−14.4	
	$\varepsilon+6\vartheta+4\Delta$	+1.90	−10.8	
j^2	$-\varepsilon+2\vartheta+\Delta-\Sigma$	+0.22	−1.6	
	$\varepsilon+6\vartheta+5\Delta-\Sigma$	+0.07	−0.5	
	$+(\vartheta-\vartheta_0)$ cos			
η	ε	−1.265	+6.35	−14.3
η'	$\varepsilon+\Delta$	+0.455	−2.69	+7.2
η^2	2ε	+0.63	−3.2	+7.2
$\eta\eta'$	$2\varepsilon+\Delta$	−0.23	+1.3	−3.6
η^3	ε	−23.8	+222	
$\eta^2\eta'$	$\varepsilon+\Delta$	+72.9	−569	
	$-\varepsilon+\Delta$	+36.5	−285	
$\eta\eta'^2$	$\varepsilon+2\Delta$	−55.2	+375	
	ε	−87.3	+653	
η'^3	$\varepsilon+\Delta$	+69.9	−439	
$j^2\eta$	$\varepsilon+\Delta+\Sigma$	−9.9	+77	
	ε	+23.1	−166	
$j^2\eta'$	$\varepsilon+\Sigma$	+5.2	−45	
	$\varepsilon+\Delta$	−14.8	+112	
		m'^2		

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TABLE XXXIV.

$\{n\delta z_1^{(2)} - (n\delta z_1^{(2)})\} + n\delta z_2^{(2)}$

Unit = 1

	Sin	m'^2		
		w^{-1}	w	w^2
	ε+2ϑ+2⅃	−0.614	+4.06	−10.3
	2ε+4ϑ+4⅃	−0.079	+0.40	
η	ε	−0.74	+3.7	
	ε+4ϑ+4⅃	+1.74	−18.1	
	2ε+2ϑ+2⅃	+0.31	−2.0	
	2ε+6ϑ+6⅃	+0.45	−2.9	
η′	ε+ ⅃	+0.30	−1.8	
	ε+4ϑ+3⅃	−4.26	+32.0	
	2ε+6ϑ+5⅃	−0.66	+3.8	
η^2	− ε+2ϑ+2⅃	−6.4		
	ε+2ϑ+2⅃	+6.4		
	ε+6ϑ+6⅃	+5.1		
	2ε+4ϑ+4⅃	−1.4		
	2ε+8ϑ+8⅃	−2.2		
η η′	− ε+2ϑ+ ⅃	+12.0		
	ε+2ϑ+3⅃	−0.9		
	ε+2ϑ+ ⅃	−8.5		
	ε+6ϑ+5⅃	−11.8		
	2ε+4ϑ+3⅃	+3.4		
	2ε+4ϑ+5⅃	−0.8		
	2ε+8ϑ+7⅃	+6.5		
η'^2	− ε+2ϑ	−5.1		
	ε+2ϑ+2⅃	+1.3		
	ε+6ϑ+4⅃	+6.4		
j^2	− ε+2ϑ+ ⅃−Σ	−0.3		
	ε+6ϑ+5⅃−Σ	+0.3		
	2ε+4ϑ+4⅃	+1.4		
	$+(\vartheta-\vartheta_0)$ cos			
η	ε	−1.02	−8.4	
	ε+4ϑ+4⅃	−0.78	+6.0	
	2ε+2ϑ+2⅃	+0.41	−2.5	
η′	ε + ⅃	−3.25	+30.1	
	ε+4ϑ+3⅃	+0.58	−4.8	
	2ε+2ϑ+3⅃	−0.31	+2.1	
η^2	− ε+2ϑ+2⅃	+3.6		
	ε+2ϑ+2⅃	+1.1		
	2ε	+0.5		
	2ε+4ϑ+4⅃	−0.8		
η η′	− ε+2ϑ+ ⅃	−3.4		
	2ε+ ⅃	+1.6		
	$(\vartheta-\vartheta_0)^2$ sin			
η	ε	−0.36	+2.6	
η′	ε+ ⅃	+0.27	−2.1	
		m'^2		

TABLE XXXV.

Logarithmic. $n\delta z-[n\delta z]$ Unit=1″.

	Sin	w^{-3}	w^{-2}	w^{-1}	w^0	w	w^2
$\eta\,\eta'$	$\frac{1}{2}\varepsilon+\vartheta$				4.1570	4.8741_n	
	$\frac{1}{2}\varepsilon+\vartheta+\Delta$				2.7684_n	3.3827	3.7172_n
η^2	$\frac{1}{2}\varepsilon+\vartheta+\Delta$				4.0056_n	4.7686	
η'^2	$\frac{1}{2}\varepsilon+\vartheta+\Delta$				4.0766_n	4.8295	
j^2	$\frac{1}{2}\varepsilon+\vartheta+\Delta$				4.1365	4.8738_n	
$\eta\,\eta'$	$\frac{1}{2}\varepsilon+\vartheta+2\Delta$				3.3345	4.5162_n	
η'	$\frac{1}{2}\varepsilon+3\vartheta+2\Delta$				4.2240_n	4.9611	5.6685_n
η	$\frac{1}{2}\varepsilon+3\vartheta+3\Delta$				4.0671	4.8483_n	5.5636
η'^2	$\frac{1}{2}\varepsilon+5\vartheta+3\Delta$				5.0926_n	6.0018	
$\eta\,\eta'$	$\frac{1}{2}\varepsilon+5\vartheta+4\Delta$				5.2325	6.1714_n	
η^2	$\frac{1}{2}\varepsilon+5\vartheta+5\Delta$				4.7675_n	5.7341	
j^2	$\frac{1}{2}\varepsilon+5\vartheta+4\Delta-\Sigma$				3.8050_n	4.7998	
η'	$-\frac{1}{2}\varepsilon+\vartheta$				3.3112	3.8350_n	4.1355
η	$-\frac{1}{2}\varepsilon+\vartheta+\Delta$				3.2065_n	3.7910	4.0833_n
η'^2	$-\frac{1}{2}\varepsilon+3\vartheta+\Delta$				3.5338	4.6236_n	
$\eta\,\eta'$	$-\frac{1}{2}\varepsilon+3\vartheta+2\Delta$				4.0879	5.0382	
η^2	$-\frac{1}{2}\varepsilon+3\vartheta+3\Delta$				3.6012_n	4.5318_n	
j^2	$-\frac{1}{2}\varepsilon+3\vartheta+2\Delta-\Sigma$				3.2074	4.1925_n	
η	ε		9.868_n	0.5689	2.922	3.4600_n	3.3670
η'	$\varepsilon+\Delta$		9.482	0.2533_n	2.673_n	3.2959	3.1772_n
$\eta\,\eta'$	$\varepsilon+2\vartheta+\Delta$	0.746_n	[1.384]	3.2927_n	[4.14906]	[4.6990_n]	
	$\varepsilon+2\vartheta+2\Delta$		9.788_n	2.47560	3.10847_n	3.4540	[3.3960_n]
η^2	$\varepsilon+2\vartheta+2\Delta$	0.645	[1.342_n]	2.305_n	[3.6179_n]	[4.4018]	
η'^2	$\varepsilon+2\vartheta+2\Delta$	0.326	1.119_n	2.935_n	3.3017_n	[4.39206]	
j^2	$\varepsilon+2\vartheta+2\Delta$			3.4276_n	4.23764	4.76933_n	
$\eta\,\eta'$	$\varepsilon+2\vartheta+3\Delta$	0.28_n	1.102	3.1738	[3.5449_n]	[3.8446_n]	
$\eta\,\eta'^2$	$\varepsilon+4\vartheta+2\Delta$			3.6004	4.27485		
η'	$\varepsilon+4\vartheta+3\Delta$	9.057	0.692_n	3.10161	3.9302_n	4.52415	[4.78162_n]
$\eta^2\eta'$	$\varepsilon+4\vartheta+3\Delta$			4.0519_n	3.7975		
η'^3	$\varepsilon+4\vartheta+3\Delta$			4.1385_n	4.6961		
$j^2\ \eta'$	$\varepsilon+4\vartheta+3\Delta$			4.2431_n	5.1290		
η	$\varepsilon+4\vartheta+4\Delta$	9.500_n	0.522	2.9351_n	3.8035	4.41616_n	4.63017
η^3	$\varepsilon+4\vartheta+4\Delta$			3.7714	4.2108_n		
$\eta\,\eta'^2$	$\varepsilon+4\vartheta+4\Delta$			4.4165	5.0931_n		
$j^2\eta$	$\varepsilon+4\vartheta+4\Delta$			4.1524	5.0661_n		
$\eta^2\eta'$	$\varepsilon+4\vartheta+5\Delta$			4.0588_n	4.8136		
$j^2\eta$	$\varepsilon+4\vartheta+3\Delta-\Sigma$			3.2322_n	4.2342		
$j^2\ \eta'$	$\varepsilon+4\vartheta+4\Delta-\Sigma$			2.744_n	[3.0962]		
η'^2	$\varepsilon+6\vartheta+4\Delta$	0.28	[0.64_n]	3.8027	4.77998_n	[5.52852]	
$\eta\,\eta'$	$\varepsilon+6\vartheta+5\Delta$	0.596_n	[1.070]	3.9374_n	[4.94342]	[5.70347_n]	
η^2	$\varepsilon+6\vartheta+6\Delta$	0.255	[0.8_n]	3.4684	[4.50125_n]	[5.27451]	
j^2	$\varepsilon+6\vartheta+5\Delta-\Sigma$	8.8	[9.3_n]	2.415	3.4823_n	[4.2931]	
η'^3	$\varepsilon+8\vartheta+5\Delta$			4.5564	5.4999_n		
$\eta\,\eta'^2$	$\varepsilon+8\vartheta+6\Delta$			4.8668_n	5.8416		
$\eta^2\eta'$	$\varepsilon+8\vartheta+7\Delta$			4.6990	5.7030_n		
η^3	$\varepsilon+8\vartheta+8\Delta$			4.0531_n	5.0844		
$j^2\ \eta'$	$\varepsilon+8\vartheta+6\Delta-\Sigma$			3.5829	4.6352_n		
$j^2\eta$	$\varepsilon+8\vartheta+7\Delta-\Sigma$			3.3768_n	4.4540		
η'^2	$-\varepsilon+2\vartheta$	0.606	[1.422_n]	3.2132	3.6657_n	3.9260	
$\eta\,\eta'$	$-\varepsilon+2\vartheta+\Delta$	0.791_n	[1.690]	3.3777_n	3.8866	4.72168	
η^2	$-\varepsilon+2\vartheta+2\Delta$	0.418	[1.365_n]	2.894	[3.4616_n]	3.8078	
j^2	$-\varepsilon+2\vartheta+\Delta-\Sigma$	9.34	0.28_n	2.938	3.4714_n	3.7862	
η'^3	$-\varepsilon+4\vartheta+\Delta$			3.5208	4.07255		
$\eta\,\eta'^2$	$-\varepsilon+4\vartheta+2\Delta$			3.4965_n	4.59582		
$\eta^2\eta'$	$-\varepsilon+4\vartheta+3\Delta$			3.2416	4.5467_n		
η^3	$-\varepsilon+4\vartheta+4\Delta$			2.430_n	3.9848		
$j^2\ \eta'$	$-\varepsilon+4\vartheta+2\Delta-\Sigma$			3.5496	4.19852_n		
$j^2\eta$	$-\varepsilon+4\vartheta+3\Delta-\Sigma$			3.3247_n	4.05994		

TABLE XXXV—Continued.

Logarithmic. $n\delta z-[n\delta z]$ Unit=1″.

	Sin	w^{-3}	w^{-2}	w^{-1}	w^0	w	w^2
$\eta\ \eta'$	$\frac{3}{2}\varepsilon+3\vartheta+2\varDelta$				3.6731	4.0029_n	
	$\frac{3}{2}\varepsilon+3\vartheta+3\varDelta$				2.3528	3.2475_n	3.9005
η^2	$\frac{3}{2}\varepsilon+3\vartheta+3\varDelta$				3.6181_n	4.2122	
j^2	$\frac{3}{2}\varepsilon+3\vartheta+3\varDelta$				3.4072_n	4.4000	
$\eta\ \eta'$	$\frac{3}{2}\varepsilon+3\vartheta+4\varDelta$				3.5244	4.4012_n	
η'	$\frac{3}{2}\varepsilon+5\vartheta+4\varDelta$				3.3533	4.4231_n	5.2725
η	$\frac{3}{2}\varepsilon+5\vartheta+5\varDelta$				3.1780_n	4.2730	5.1359_n
η'^2	$\frac{3}{2}\varepsilon+7\vartheta+5\varDelta$				4.2775	5.4708_n	
$\eta\ \eta'$	$\frac{3}{2}\varepsilon+7\vartheta+6\varDelta$				4.4051_n	5.6177	
η^2	$\frac{3}{2}\varepsilon+7\vartheta+7\varDelta$				3.9296	5.1605_n	
η	$2\varepsilon+2\vartheta+2\varDelta$		9.486	2.1744_n	2.708	$[2.889_n]$	2.599_n
η'	$2\varepsilon+2\vartheta+3\varDelta$				1.946_n	2.501	2.516_n
$\eta\ \eta'$	$2\varepsilon+4\vartheta+3\varDelta$	8.8_n	$[0.561]$	2.789_n	$[3.5813]$	$[4.1074_n]$	
	$2\varepsilon+4\vartheta+4\varDelta$		8.90_n	9.599	1.711	2.5795_n	3.1726
η^2	$2\varepsilon+4\vartheta+4\varDelta$	9.2	$[0.34_n]$	2.618	$[3.4962_n]$	$[4.0890]$	
η'	$2\varepsilon+6\vartheta+5\varDelta$		9.819_n	0.5840	2.7821	3.7794_n	4.51865
η	$2\varepsilon+6\vartheta+6\varDelta$		9.653	0.4645_n	2.5979_n	3.6265	4.38424_n
	$\frac{5}{2}\varepsilon+5\vartheta+5\varDelta$				1.2340	2.1166_n	2.7076
η'	$\frac{5}{2}\varepsilon+7\vartheta+6\varDelta$				2.3679	3.3518_n	4.0587
η	$\frac{5}{2}\varepsilon+7\vartheta+7\varDelta$				2.1758_n	3.1926	3.9204_n
	$(\vartheta-\vartheta_0)$ cos						
η	ε	0.1021_n	0.728	2.8978_n	3.4504	3.7168_n	
η^3	ε	1.377_n	$[2.346]$	3.8211_n	4.6762		
$\eta\ \eta'^2$	ε	1.941_n	2.815	4.4076_n	5.1971		
$j^2\eta$	ε	1.364	2.220_n	4.4076	5.1971_n	5.7086	
η'	$\varepsilon+\varDelta$	9.658	0.774_n	2.7836	3.3840_n	3.6946	
$\eta^2\eta'$	$\varepsilon+\varDelta$	1.863	2.755_n	4.2546	5.0814_n		
η'^3	$\varepsilon+\varDelta$	1.844	2.642_n	4.1953	4.9770_n		
$j^2\ \eta'$	$\varepsilon+\varDelta$	1.170_n	2.049	4.3715_n	5.1770	5.6975_n	
$\eta\ \eta'^2$	$\varepsilon+2\varDelta$	1.742_n	2.574	4.0203_n	4.8466		
$j^2\ \eta'$	$\varepsilon+\Sigma$	0.716	1.65_n	4.0809	4.8829_n	5.4008	
$j^2\eta$	$\varepsilon+\varDelta+\Sigma$	1.00_n	1.89	4.3427_n	5.0837	5.5553_n	
$\eta^2\eta'$	$-\varepsilon+\varDelta$	1.562	2.455_n	3.9535	4.7803_n		
η^2	2ε	9.801	$[0.43_n]$	2.5842	3.1493_n	3.4158	
$\eta\ \eta'$	$2\varepsilon+\varDelta$	9.357_n	$[0.473]$	2.4548_n	3.0830	3.3936_n	
	$(\vartheta-\vartheta_0)^2$ sin						
η	ε		9.56_n	0.42			
η'	$\varepsilon+\varDelta$		9.43	0.32_n			

$n\delta z-[n\delta z]=\Sigma w^s\eta^p\eta'^q j^{2t}C_1 \sin \text{Arg.}+(\vartheta-\vartheta_0)\Sigma w^s\eta^p\eta'^q j^{2t}C_2 \cos \text{Arg.}+(\vartheta-\vartheta_0)^2\Sigma w^s\eta^p\eta'^q j^{2t}C_3 \sin \text{Arg.}$
where C_1, C_2, C_3, represent the respective coefficients.

TABLE XXXVI.

$$-\frac{3}{4}\left[(1-e\cos\varepsilon)\left(\mathrm{W}-\frac{1}{3}\,\Xi\right)\left(\mathrm{W}+\frac{1}{9}\,\Xi\right)\right]$$

Unit=4th decimal of a radian.

	Cos	w^{-4}	w^{-3}	w^{-2}	w^{-1}	w^{0}	w
			+0.000032	−0.0080	+0.0493	− 0.176	+0.52
η^2		−0.00028	+0.0037	−0.133	+1.10	− 8.8	
η'^2		−0.00014	+0.0026	−0.095	+1.27	−14.4	
j^2			−0.0003	+0.139	−1.20	+ 5.9	
$\eta\ \eta'$	Δ	+0.00047	−0.0070	+0.252	−2.51	+22.8	
η	$2\vartheta+2\Delta$	+0.000017	−0.00042	+0.0437	−0.366	+ 2.10	
η'	$2\vartheta+\Delta$	−0.000006	+0.00045	−0.0639	+0.508	− 2.79	
η^2	$4\vartheta+4\Delta$	+0.00004	+0.0006	−0.194	+1.64	−11.4	
$\eta\ \eta'$	$4\vartheta+3\Delta$	−0.00012	−0.0012	+0.372	−3.59	+32.2	
η'^2	$4\vartheta+2\Delta$	+0.00011	+0.0003	−0.252	+2.40	−19.8	
j^2	$4\vartheta+3\Delta-\Sigma$	+0.00001	−0.0001	+0.032	−0.19		
	$+(\vartheta-\vartheta_0)\sin$						
$\eta\ \eta'$	Δ		−0.00004		+0.010	− 0.08	
η	$2\vartheta+2\Delta$	+0.000066	−0.00060	+0.0399	−0.275	+ 0.94	
η'	$2\vartheta+\Delta$	−0.000024	+0.00047	−0.0296	+0.221	− 0.81	
η^2	$4\vartheta+4\Delta$	−0.00023	+0.0028	−0.114	+1.02	− 4.7	
$\eta\ \eta'$	$4\vartheta+3\Delta$	+0.00039	−0.0053	+0.251	−2.20	+ 9.9	
η'^2	$4\vartheta+2\Delta$	−0.00011	+0.0024	−0.124	+1.11	− 5.1	
	$(\vartheta-\vartheta_0)^2\cos$						
η^2		−0.00017	+0.0014	−0.052	+0.38	− 1.4	
$\eta\ \eta'$	Δ	+0.00019	−0.0021	+0.077	−0.61	+ 2.4	
η'^2		−0.00005	+0.0008	−0.029	+0.24	− 1.0	
		m'^3	m'^3	m'^3, m'^2	m'^2	m'^2	m'^2

TABLE XXXVII.

$$\left[(\theta-\vartheta)(1-e\cos\varepsilon)\frac{\partial\overline{\mathrm{W}}}{\delta\vartheta}\right]$$

Unit=4th decimal of a radian.

	Cos	w^{-4}	w^{-3}	w^{-2}	w^{-1}	w^{0}	w	w^{2}
			+0.000042	−0.01071	+ 0.0883	− 0.402	+ 1.31	− 3.9
η^2		−0.00043	+0.0056	−0.189	+ 2.73	− 51.3	+ 299	
η'^2		−0.00021	+0.0048	−0.296	+ 4.47	− 59.8	+ 416	
j^2			−0.0004	+0.186	− 2.00	+ 11.7	− 40	
$\eta\ \eta'$	Δ	+0.00076	−0.0110	+0.530	− 7.59	+104.2	− 682	
η	$2\vartheta+2\Delta$	+0.000055	−0.00086	+0.1005	− 1.153	+ 21.86	− 81.5	+217
η'	$2\vartheta+\Delta$	−0.000020	+0.00090	−0.1377	+ 1.463	− 9.50	+ 44.2	−176
η^2	$4\vartheta+4\Delta$	−0.00031	+0.0041	−0.477	+ 6.49	−133.8	+ 708	
$\eta\ \eta'$	$4\vartheta+3\Delta$	+0.00068	−0.0084	+1.295	−17.43	+261.3	−1266	
η'^2	$4\vartheta+2\Delta$	−0.00030	+0.0041	−0.921	+11.58	− 95.2	+ 452	
j^2	$4\vartheta+3\Delta-\Sigma$	−0.00001	+0.0001	−0.036	+ 0.58	− 5.3	+ 25	
	$+(\vartheta-\vartheta_0)\sin$							
$\eta\ \eta'$	Δ	0.00000	−0.0004	+0.052	− 0.44	+ 2.0		
η	$2\vartheta+2\Delta$	+0.000044	−0.00052	+0.0266	− 0.212	+ 0.83		
η'	$2\vartheta+\Delta$	−0.000016	+0.00038	−0.0197	+ 0.170	− 0.70		
η^2	$4\vartheta+4\Delta$	−0.00031	+0.0037	−0.153	+ 1.62	− 7.9		
$\eta\ \eta'$	$4\vartheta+3\Delta$	+0.00052	−0.0072	+0.335	− 3.40	+ 16.9		
η'^2	$4\vartheta+2\Delta$	−0.00015	+0.0032	−0.165	+ 1.68	− 8.6		
		m'^2	m'^3	m'^3, m'^2	m'^2	m'^2, m'	m'^2, m'	m'^2, m'

TABLE XXXVIII.

Logarithmic. $\Phi(\vartheta)$ Unit=1 radian.

	Cos	w^{-6}	w^{-5}	w^{-4}	w^{-3}	w^{-2}	w^{-1}	w^{0}	w	w^{2}
				1.5	[3.909$_n$]	4.960	6.6748$_n$	7.2764	7.540$_n$	7.31
η^2			2.0	[4.644$_n$]	[5.160]	[6.150]	[8.048$_n$]	8.838	8.655$_n$	8.100$_n$
η'^2			1.9	[3.41$_n$]	[4.75$_n$]	[6.509]	[8.2077$_n$]	[8.994]	8.919$_n$	
j^2				2.83$_n$	5.146	[6.299$_n$]	[7.994]	8.740$_n$	[8.656]	
$\eta\eta'$	⅃		2.34$_n$	[4.446]	[4.57]	[6.728$_n$]	[8.4022]	9.1999$_n$	9.0854	8.079
$\eta\eta'^2$	2ϑ	1.6	[2.6$_n$]	5.744	6.535$_n$	8.3811	9.1031$_n$	9.0128		
η'	$2\vartheta+$ ⅃		0.8$_n$	[3.068]	[5.2988]	7.2212$_n$	[7.3772]	[8.0372]	[8.764$_n$]	8.668
$\eta^2\eta'$	$2\vartheta+$ ⅃	2.32$_n$	3.30	5.886$_n$	6.718	8.5059$_n$	9.2804	9.2017$_n$		
η'^3	$2\vartheta+$ ⅃			5.301$_n$	6.149	8.2302$_n$	9.0154	8.938$_n$		
$j^2\ \eta'$	$2\vartheta+$ ⅃					8.5592	9.3245$_n$	9.2428		
η^3	$2\vartheta+2$⅃	2.48	3.40$_n$	5.422	6.292$_n$	7.476	8.664$_n$	8.636		
η	$2\vartheta+2$⅃		1.22	[2.94$_n$]	[5.1206$_n$]	7.6416	[7.9638$_n$]	[7.083$_n$]	[8.645]	8.582$_n$
$\eta\eta'^2$	$2\vartheta+2$⅃	1.9	[3.0$_n$]	5.442	6.328$_n$	8.0915$_n$	8.630$_n$	8.742		
$j^2\eta$	$2\vartheta+2$⅃					8.5901$_n$	9.3489	9.8024$_n$	9.6532	
$\eta^2\eta'$	$2\vartheta+3$⅃	2.04$_n$	3.00	4.98$_n$	5.89	8.0326	8.1973$_n$	7.69		
$j^2\eta$	$2\vartheta+$ ⅃ $-\Sigma$			4.51	5.42$_n$	8.1011	8.873$_n$	8.792		
$j^2\ \eta'$	$2\vartheta+2$⅃ $-\Sigma$			4.04$_n$	5.00	6.89$_n$	8.182	8.158$_n$		
η'^2	$4\vartheta+2$⅃		[2.66$_n$]	[2.7]	6.1031	[8.4188$_n$]	[8.5297]	[6.0]	7.90$_n$	
$\eta\eta'$	$4\vartheta+3$⅃		[2.72]	[4.369]	6.2526$_n$	8.5594	[8.7988$_n$]	[7.94$_n$]	8.287	8.210
η^2	$4\vartheta+4$⅃		[2.20$_n$]	[4.624$_n$]	[5.824]	[8.0924$_n$]	[8.4338]	[7.24]	7.74$_n$	8.044$_n$
j^2	$4\vartheta+3$⅃ $-\Sigma$		1.5$_n$	2.45	4.68	7.1747$_n$	7.301	8.111	8.127$_n$	
η'^3	$6\vartheta+3$⅃			5.301$_n$	6.149	9.1294$_n$	9.7728	9.6609$_n$		
$\eta\eta'^2$	$6\vartheta+4$⅃			5.92	6.74$_n$	9.4432	0.14644$_n$	0.05077		
$\eta^2\eta'$	$6\vartheta+5$⅃	2.0$_n$	3.0	5.93$_n$	6.79	9.2774$_n$	0.03298	9.9494$_n$		
η^3	$6\vartheta+6$⅃	2.0	3.0$_n$	5.420	6.292$_n$	8.634	9.4351$_n$	9.3608		
$j^2\ \eta'$	$6\vartheta+4$⅃ $-\Sigma$			4.04$_n$	5.00	8.272$_n$	9.1028	9.0334$_n$		
$j^2\eta$	$6\vartheta+5$⅃ $-\Sigma$			4.51	5.42$_n$	8.0554	8.926$_n$	8.864		
	$(\vartheta-\vartheta_0)$ sin									
$\eta\eta'$	⅃			[2.60$_n$]	4.71	5.94$_n$	6.507$_n$	6.606		
η'	$2\vartheta+$ ⅃		1.36	[2.48]	4.49	[5.255$_n$]	[5.51]	5.25$_n$		
η	$2\vartheta+2$⅃		1.82$_n$	[2.42]	4.64$_n$	5.350	[5.51$_n$]	[5.16]		
η'^2	$4\vartheta+2$⅃		2.34	[3.00]	5.392	[6.179$_n$]	6.528$_n$	6.665		
$\eta\eta'$	$4\vartheta+3$⅃		2.89$_n$	[3.46]	5.702$_n$	[6.467]	6.851	6.979$_n$		
η^2	$4\vartheta+4$⅃		2.66	[3.459$_n$]	[5.357]	[6.127$_n$]	6.530$_n$	6.653		
	$(\vartheta-\vartheta_0)^2$ cos									
η^2				2.08$_n$	2.08		5.546$_n$	5.546		
η'^2				2.54	2.54$_n$		5.396$_n$	5.396		
$\eta\eta'$	⅃			2.5$_n$	2.5		5.776	5.776$_n$		
		m'^3	m'^3	m'^3, m'^2	m'^3, m'^2	m'^2, m'	m'^2, m'	m'^2, m'	m'^2, m'	m'^2, m'

$$\Phi(\vartheta)=\Sigma w^s\eta^p\eta'^q j^{2t}C'_1 \cos \text{Arg.} +(\vartheta-\vartheta_0)\Sigma w^s\eta^p\eta'^q j^{2t}C'_2 \sin \text{Arg.} \quad +(\vartheta-\vartheta_0)^2\Sigma w^s\eta^p\eta'^q j^{2t}C'_3 \cos \text{Arg.}$$

where C'_1, C'_2, C'_3, represent the respective coefficients.

COMPARISON OF TABLES.

Table XXX.—With the aid of the manuscript the source of all the discrepancies indicated by brackets has been traced. Coefficients in parentheses are functions of coefficients in parentheses in Table XXVII.

Table XXXI.—The function was computed by the first of Z 81, eqs. (137), which is more rigid than the one following it, which v. Zeipel used. Aside from the addition of omitted terms, the bracketed coefficients are more accurate by reason of the errors in v. Zeipel's Table XVIII.

Table XXXII.—The computation was performed according to Z 82, eqs. (139) and (140), in place of eq. (141) which is less rigid. Besides the discrepancies due to the addition of omitted terms, four bracketed coefficients are of opposite sign. These discrepancies may be due either

to a numerical error or to the number of terms included. The remaining discrepancy is due to slight inaccuracies of v. Zeipel's computation.

Table XXXIII.—The discrepancy in this table follows from one in Table XVIII. Third degree terms in Table XVIII were not integrated because, in the aggregate, they amount to very little.

Table XXXIV.—Our table is more extensive. Second degree terms are, however, not complete, for they do not include second degree terms in

$$[y_2] \cos \varepsilon + [z_2] \sin \varepsilon$$

The discrepancies are of no importance.

The integration of eq. (146) is best performed individually for each planet. The analytical developments are as follows:

The differential equation can be written

$$\frac{d\vartheta}{d\varepsilon} = \Phi(\vartheta)\frac{d}{d\varepsilon}\left(\frac{w}{2}\varepsilon\right) + \frac{d}{d\varepsilon}\left(\frac{w}{2}\varepsilon - [n'\partial z']\right)$$

By a change of variable

$$\frac{d\vartheta}{d\left(\frac{w}{2}\varepsilon - [n'\partial z']\right)} = 1 + \Phi(\vartheta)\frac{d\left(\frac{w}{2}\varepsilon\right)}{d\left(\frac{w}{2}\varepsilon - [n'\partial z']\right)}$$

Writing

$$\frac{w}{2}\varepsilon = \left(\frac{w}{2}\varepsilon - [n'\partial z']\right) + [n'\partial z']$$

we have *Z* 96, eq. (152), in which the last term can be neglected.

For a given planet the factors w, η, j^2 and the argument $\varDelta$ are known constants. Therefore $1+\Phi\ (\vartheta)$ can be expressed as in eq. (153), as a Fourier series of sines and cosines of multiples of 2ϑ, in which the nontrigonometrical term is designated by σ.

Expressing eq. (153) in terms of exponentials and solving for $d\left(\frac{w}{2}\varepsilon - [n'\partial z']\right)$ by the expansion of $\{1+\Phi(\vartheta)\}^{-1}$, and reintroducing the trigonometric functions, we have the equation following eq. (153), in which the nontrigonometrical part is taken outside the brackets as a common factor. The brackets in this equation do not have the special significance which they have had previously.

The variables ε and ϑ are now separate and the integration can be performed. Transferring the common factor to the left-hand side of the equation, performing the integration and adding

$$\frac{n'}{n}c - c'$$

as the constant of integration, we have the argument ζ expressed as a function of ϑ in eq. (154), where ζ is defined by eq. (155).

The reversion of the series gives ϑ as a function of ζ. We have by eq. (154)

$$\vartheta = \zeta + \Sigma C \begin{matrix}\sin\\ \cos\end{matrix}\, n\vartheta$$

where ΣC is a small quantity. Given

$$z = w + \alpha\Phi(z), \text{ where } \alpha \text{ is small,}$$

we have, by a theorem of Lagrange,

$$F(z) = F(w) + \alpha\Phi(w)\ F'(w) + \frac{\alpha^2}{1\cdot 2}\ \frac{\partial}{\partial w}[\{\Phi(w)\}^2 F'\ (w)] + \cdot\ \cdot\ \cdot\ \frac{\alpha^{n+1}}{n+1}\frac{\partial^n}{\partial w^n}[\{\phi(w)\}^{n+1}F'(w)] + \cdot\ \cdot\ \cdot$$

By means of this theorem eqs. (156), (157) can be derived, where it is to be noticed that $(\zeta - \frac{1}{2}c + c')$ is an approximation for $(\zeta - \zeta_0)$. In our developments we have used $(\zeta - \zeta_0)$.

If in Z eq. (155) we add and subtract$\left(\frac{w}{2}\varepsilon-[n'\delta z']\right)$

$$\zeta=\frac{\sigma-\frac{1}{2}(A_2{}^2+B_2{}^2)}{1+\frac{1}{2}(A_2{}^2+B_2{}^2)}\left(\frac{w}{2}\varepsilon-[n'\delta z']\right)+\frac{n'}{n}c-c'+\left(\frac{w}{2}\varepsilon-[n'\delta z']\right)$$

Substituting this value of ζ in eq. (156),

$$\vartheta-\frac{w}{2}\varepsilon+[n'\delta z']-\frac{n'}{n}c+c'=\frac{\sigma-\frac{1}{2}(A_2{}^2+B_2{}^2)}{1+\frac{1}{2}(A_2{}^2+B_2{}^2)}\left(\frac{w}{2}\varepsilon-[n'\delta z']\right)+\text{Series}$$

Substituting the last equation in eq. (145), we obtain *Z* 98, eqs. (159), (160), and (161). In eq. (160) the factor $(\varepsilon-c)$ is an approximation for $\frac{2}{w}(\zeta-\zeta_0)$; in our work we have used the latter.

Since $[n\delta z]_1$ is the series in eq. (156) multiplied by the factor $\frac{2}{1-w}$,

$$\vartheta=\frac{1-w}{2}[n\delta z]_1+\zeta$$

Table XXXV.—With the exception of the two coefficients under the heading w^2, all the bracketed quantities are functions of other coefficients in parentheses or brackets, or they are functions of additional terms. The two coefficients excepted seem to be in disagreement through some numerical error by v. Zeipel.

Table XXXVI.—Since the mass factors have not been kept explicit, it may be well to remark that only the zero degree term of third order has been included under the heading w^{-2}.

The bracketed quantities are numerous. Aside from the accumulation of discrepancies already discussed, the disagreements are to be attributed, in general, to the relative extent of the computations. It is found from computation that as the number of terms included in a product is increased the resulting coefficient for a given argument is numerically larger. For the most part our values are larger than v. Zeipel's. Hence the discrepancies are explained by assuming that our computation is more extensive. On the other hand, the function is computed much more accurately than is necessary, and many of our disagreements are less important than they appear to be.

Table XXXVII.—The comparison of Tables XXXVII is similar to that for Tables XXXVI with the exception that our values are not, in general, numerically larger. Some are larger and some are smaller. Below are brief tables showing to what extent we used the necessary series. The 0, 1, 2 signify the degrees of the terms included.

$$\frac{1-w}{2}\left\{n\delta z-[n\delta z]\right\}$$

w^{-1}				w^{-2}		
w^0	w	w^2	w^3	w^0	w	w^2
0 1 2	0 1 2	0 1 2	0 1	0 1 2	0 1 2	0
m'				m'^2		

$$\frac{\partial}{\partial \vartheta}\left\{(1-e\cos\varepsilon)\overline{W}-[(1-e\cos\varepsilon)\overline{W}]\right\}$$

w^{-1}				w^{-3}			w^{-5}			
w^0	w	w^2	w^3	w^0	w	w^2	w^0	w	w^2	w^3
0	0	0	0	0	0	0		0	0	0
1	1	1	1	1	1			1	1	1
2	2	2	2	2	2					
m'				m'^2			m'^3			

Table XXXVIII.—All the bracketed quantities probably contain only the accumulation of the discrepancies in Tables XXIX*b*, XXXVI, and XXXVII. This is a very important table, and it is from differences in Φ (ϑ) that the perturbations may be expected to differ most.

PERTURBATIONS OF THE RADIUS VECTOR.

If $\overline{W}$ and $\frac{1}{3}\Xi$ are tabulated and the computation is performed in duplicate, it is not necessary to make the long developments and the auxiliary tables in *Z* §6, 99–114. For this reason the formulae in §6 have not been checked and the list of errata does not cover this section.

The essential formulae are given in *Z* 99. By *Z* 7, eq. (36),

$$\nu=-\frac{1}{2}W-\frac{1}{6}\Xi+\frac{1}{4}\overline{W}^2+\ .\ .\ .\ .\ .$$

In order to parallel the form of $n\delta z$, we write

$$\nu=f(\theta)=f(\vartheta+\theta_1+\theta_2+\theta_3+\ .\ .\ .\ .)=f(\vartheta)+\frac{\delta f(\vartheta)}{\delta\vartheta}\ (\theta_1+\theta_2+\theta_3+\ .\ .\ .\ )+\ .\ .\ .\ .$$

where $(\theta_1+\theta_2+\theta_3)$ is given by *Z* 93, eq. (150).

Hence the computation proceeds as follows: the perturbation is computed by eq. (36), the argument θ is replaced by ϑ, and a corrective term which is the product of $(\theta_1+\theta_2+\theta_3)$ and the derivative of the function with respect to ϑ is added. The perturbation ν is then expressed as a function of ϑ. It is tabulated in Table XLIII.

Table XLIII.—If there are no errors of calculation in the construction of the table, all the discrepancies are due to the accumulation of other discrepancies previously discussed.

The perturbation $\nu=f(\theta)$ includes

w^{-1}				w^{-3}		
w^0	w	w^2	w^3	w^0	w	w^2
0	0	0	0	0	0	0
1	1	1	1	1	1	1
2	2	2		2	2	
3	3			3	3	
m'				m'^2		

where the tabulated numbers signify the degrees of the terms included and where only $\overline{W}_1$ and Ξ_1 are inclusive of third degree.

TABLE XLIII.

Logarithmic. ι Unit=1″.

	Cos	w^{-3}	w^{-2}	w^{-1}	w^0	w	w^2
			8.72	[9.88_n]	1.6349	2.1070_n	2.2333
η^2		9.80	[0.212_n]		2.759	3.4922_n	
η'^2		8.9	9.23		2.937	3.6295_n	
j^2					2.937_n	3.6295	
$\eta\eta'$	Δ	9.66_n	9.78		3.1136_n	3.8440	
$\eta\eta'^2$	2ϑ	0.556_n	1.204	3.2111_n	3.7970		
η'	$2\vartheta+\Delta$		0.504_n	2.3472	2.456_n	2.686_n	3.4735
$\eta^2\eta'$	$2\vartheta+\Delta$	0.997	1.711_n	3.6559	4.3103_n		
η'^3	$2\vartheta+\Delta$	0.438	1.220_n	3.3654	4.0763_n		
$j^2\eta'$	$2\vartheta+\Delta$			3.6975_n	4.3810		
η	$2\vartheta+2\Delta$		0.438	2.952_n	3.2529	[3.0689_n]	3.3979_n
η^3	$2\vartheta+2\Delta$	0.732_n	1.497	3.2410_n	4.0643		
$\eta\eta'^2$	$2\vartheta+2\Delta$	0.772_n	1.589	3.4136	4.0723		
$j^2\eta$	$2\vartheta+2\Delta$			3.9048	[1.5649_n]	4.9303	
$\eta^2\eta'$	$2\vartheta+3\Delta$	0.505	1.344_n	3.4757_n	[2.783]		
$j^2\eta$	$2\vartheta+\Delta-\Sigma$	9.33_n	0.15	2.938_n	[3.5830]		
$j^2\eta'$	$2\vartheta+2\Delta-\Sigma$	9.20	0.10_n	2.0251	3.2961_n		
η'^2	$4\vartheta+2\Delta$	8.9	1.2819_n	3.5514	3.6173_n	3.8147	
$\eta\eta'$	$4\vartheta+3\Delta$	9.75_n	[1.5024]	3.7885_n	[4.1394]	4.3110_n	
η^2	$4\vartheta+4\Delta$	9.98	[1.1342_n]	3.4007	[3.9091_n]	[4.1480]	
j^2	$4\vartheta+3\Delta-\Sigma$		9.64_n	2.305	2.542_n	[2.749_n]	
η'^3	$6\vartheta+3\Delta$	0.438	1.220_n	4.2675	4.7993_n		
$\eta\eta'^2$	$6\vartheta+4\Delta$	1.125_n	1.862	4.6479_n	[5.2324]		
$\eta^2\eta'$	$6\vartheta+5\Delta$	1.198	1.947_n	4.5397	[5.1768_n]		
η^3	$6\vartheta+6\Delta$	0.732_n	1.508	3.9457_n	[4.6328]		
$j^2\eta'$	$6\vartheta+4\Delta-\Sigma$	9.20	0.10_n	3.4099	4.1710_n		
$j^2\eta$	$6\vartheta+5\Delta-\Sigma$	9.70_n	0.56	3.2601_n	[4.0542]		
$\eta\eta'$	$\frac{1}{2}\varepsilon+\vartheta$				3.4878_n	4.1106	
	$\frac{1}{2}\varepsilon+\vartheta+\Delta$			8.3_n	2.2106	2.7179_n	2.919
j^2	$\frac{1}{2}\varepsilon+\vartheta+\Delta$				3.5709_n	4.2261	
η^2	$\frac{1}{2}\varepsilon+\vartheta+\Delta$				3.4507	4.1296_n	
η'^2	$\frac{1}{2}\varepsilon+\vartheta+\Delta$				3.5100	4.1837_n	
$\eta\eta'$	$\frac{1}{2}\varepsilon+\vartheta+2\Delta$				2.579_n	3.9270	
η'	$\frac{1}{2}\varepsilon+3\vartheta+2\Delta$			0.08	3.6873	4.1471_n	4.7839
η	$\frac{1}{2}\varepsilon+3\vartheta+3\Delta$			9.5	3.5727_n	4.1511	4.7545_n
η'^2	$\frac{1}{2}\varepsilon+5\vartheta+3\Delta$				4.5568	5.1414_n	
$\eta\eta'$	$\frac{1}{2}\varepsilon+5\vartheta+4\Delta$				4.7261_n	[5.4067]	
η^2	$\frac{1}{2}\varepsilon+5\vartheta+5\Delta$				4.2862	[5.0418_n]	
j^2	$\frac{1}{2}\varepsilon+5\vartheta+4\Delta-\Sigma$				3.2570	4.0005_n	
η'	$-\frac{1}{2}\varepsilon+\vartheta$			1.086_n	2.7090	3.3467_n	3.7098
η	$-\frac{1}{2}\varepsilon+\vartheta+\Delta$			0.88	2.1967_n	3.0952	3.5836_n
η'^2	$-\frac{1}{2}\varepsilon+3\vartheta+\Delta$				2.514	4.1049_n	
$\eta\eta'$	$-\frac{1}{2}\varepsilon+3\vartheta+2\Delta$				4.0853	[3.9122]	
η^2	$-\frac{1}{2}\varepsilon+3\vartheta+3\Delta$				3.8341_n	[3.8118]	
j^2	$-\frac{1}{2}\varepsilon+3\vartheta+2\Delta-\Sigma$				2.416	3.6926_n	
η	ε		9.62	0.58_n	2.143_n	2.682	2.9151_n
η'	$\varepsilon+\Delta$		9.04_n	9.9	2.061	2.666_n	2.9477
$\eta\eta'$	$\varepsilon+2\vartheta+\Delta$	0.444	1.1661_n	3.0588	3.8025_n	[4.2554]	
	$\varepsilon+2\vartheta+2\Delta$		9.487	2.1744_n	2.7280	2.972_n	2.976
η^2	$\varepsilon+2\vartheta+2\Delta$	0.344_n	1.1143	2.692_n	[3.5334]	[4.0772_n]	
η'^2	$\varepsilon+2\vartheta+2\Delta$	0.025_n	0.828	2.634	3.0726	4.0416_n	
j^2	$\varepsilon+2\vartheta+2\Delta$			3.1265	3.8806_n	4.3473	
$\eta\eta'$	$\varepsilon+2\vartheta+3\Delta$	9.98	0.811_n	2.873_n	3.1697	[3.5856]	
$\eta\eta'^2$	$\varepsilon+4\vartheta+2\Delta$	1.105	1.89_n	2.864	4.3477_n		
η'	$\varepsilon+4\vartheta+3\Delta$	8.8_n	0.398	2.8000_n	3.5327	4.0065_n	4.3207
$\eta^2\eta'$	$\varepsilon+4\vartheta+3\Delta$	1.260_n	2.083	3.0931	4.4160		
η'^3	$\varepsilon+4\vartheta+3\Delta$			3.8375	4.0446_n		
$j^2\eta'$	$\varepsilon+4\vartheta+3\Delta$	0.267	1.15_n	3.9421	4.6972_n		
η	$\varepsilon+4\vartheta+4\Delta$	9.19	0.248_n	[2.6356]	3.4317_n	3.9469	4.2558_n
η^3	$\varepsilon+4\vartheta+4\Delta$	0.774	1.66_n	3.0934_n	[3.7866_n]		
$\eta\eta'^2$	$\varepsilon+4\vartheta+4\Delta$			4.1154_n	4.5547		
$j^2\eta$	$\varepsilon+4\vartheta+4\Delta$	0.455_n	1.32	3.8518_n	4.6436		
$\eta^2\eta'$	$\varepsilon+4\vartheta+5\Delta$			3.7579	4.3244_n		

TABLE XLIII Continued.

Logarithmic. ν Unit=1″

	Cos	w^{-3}	w^{-2}	w^{-1}	w^{0}	w	w^{2}
$j^2\eta$	$\varepsilon+4\vartheta+3J-\Sigma$			3.0030	3.8860_n		
$j^2\ \eta'$	$\varepsilon+4\vartheta+4J-\Sigma$			2.1425	1.85_n		
η'^2	$\varepsilon+6\vartheta+4J$	9.98_n	0.480	3.5016_n	4.3723	4.9952_n	
$\eta\eta'$	$\varepsilon+6\vartheta+5J$	0.296	0.823_n	3.6369	$[4.5582_n]$	5.2093	
η^2	$\varepsilon+6\vartheta+6J$	9.95_n	0.538	3.1685_n	[4.4334]	$[1.8434_n]$	
j^2	$\varepsilon+6\vartheta+5J-\Sigma$	8.5_n	[9.15]	2.144_n	3.0881	$[3.7886_n]$	
η'^3	$\varepsilon+8\vartheta+5J$			4.2554_n	4.9349		
$\eta\eta'^2$	$\varepsilon+8\vartheta+6J$	1.320	2.152_n	4.5657	5.3010_n		
$\eta^2\eta'$	$\varepsilon+8\vartheta+7J$	1.228_n	2.093	4.3995_n	5.1827		
η^3	$\varepsilon+8\vartheta+8J$	0.648	1.54_n	3.7543	4.5812_n		
$j^2\ \eta'$	$\varepsilon+8\vartheta+6J-\Sigma$			3.2818_n	4.1412		
$j^2\eta$	$\varepsilon+8\vartheta+7J-\Sigma$			3.0763	3.9759_n		
η'^2	$-\varepsilon+2\vartheta$	0.305	$[1.1007_n]$	2.912	3.4958_n	[3.8151]	
$\eta\eta'$	$-\varepsilon+2\vartheta+J$	0.490_n	[1.3330]	3.0166_n	[3.7273]	[4.3119]	
η^2	$-\varepsilon+2\vartheta+2J$	0.117	0.982_n	2.288_n	$[3.2375_n]$	[3.7892]	
j^2	$-\varepsilon+2\vartheta+J-\Sigma$	9.04	9.96_n	2.636	3.2817_n	[3.6568]	
η'^3	$-\varepsilon+4\vartheta+J$			3.2197	3.9650_n		
$\eta\eta'^2$	$-\varepsilon+4\vartheta+2J$	1.146_n	1.89	3.0204	4.2441		
$\eta^2\eta'$	$-\varepsilon+4\vartheta+3J$	1.005	1.78_n	3.5247_n	4.0012_n		
η^3	$-\varepsilon+4\vartheta+4J$	0.290_n	1.15	3.1793	2.932		
$j^2\ \eta'$	$-\varepsilon+4\vartheta+2J-\Sigma$			3.2486	4.0585_n		
$j^2\eta$	$-\varepsilon+4\vartheta+3J-\Sigma$	9.98_n	0.8	2.957_n	3.8580		
η	$\frac{3}{2}\varepsilon+\vartheta+J$			9.0	2.3363	$[3.0704_n]$	[3.5111]
η'	$\frac{3}{2}\varepsilon+\vartheta+2J$			9.5	1.500	2.3585	3.1842_n
$\eta\eta'$	$\frac{3}{2}\varepsilon+3\vartheta+2J$				2.779	3.7820_n	
	$\frac{3}{2}\varepsilon+3\vartheta+3J$			9.28	2.1614_n	3.0257	3.6491_n
η^2	$\frac{3}{2}\varepsilon+3\vartheta+3J$				1.32	2.966	
η'^2	$\frac{3}{2}\varepsilon+3\vartheta+3J$				3.3450	4.1111_n	
j^2	$\frac{3}{2}\varepsilon+3\vartheta+3J$				3.2309	4.1965_n	
$\eta\eta'$	$\frac{3}{2}\varepsilon+3\vartheta+4J$				3.2994_n	4.1520	
η'	$\frac{3}{2}\varepsilon+5\vartheta+4J$			1.017	3.1617_n	4.1967	5.0160_n
η	$\frac{3}{2}\varepsilon+5\vartheta+5J$			0.88_n	2.9688	4.0380_n	4.8781
η'^2	$\frac{3}{2}\varepsilon+7\vartheta+5J$				4.0855_n	5.2422	
$\eta\eta'$	$\frac{3}{2}\varepsilon+7\vartheta+6J$				4.1991	5.3823_n	
η^2	$\frac{3}{2}\varepsilon+7\vartheta+7J$				3.7114_n	4.9188	
j^2	$\frac{3}{2}\varepsilon+7\vartheta+6J-\Sigma$				2.615_n	3.8317	
$\eta\eta'$	$-\frac{3}{2}\varepsilon+\vartheta$				3.2411	3.7872_n	
η^2	$-\frac{3}{2}\varepsilon+\vartheta+J$				2.819_n	3.4476	
j^2	$-\frac{3}{2}\varepsilon+\vartheta-\Sigma$				2.9181_n	3.4813	
η^2	2ε				2.364_n	3.0737	
$\eta\eta'$	$2\varepsilon+J$				2.624	3.3489_n	
η'^2	$2\varepsilon+2J$				2.207_n	2.978	
j^2	$2\varepsilon+J+\Sigma$				2.620_n	3.2765	
η	$2\varepsilon+2\vartheta+2J$			9.8_n	1.63	2.362_n	2.873
η'	$2\varepsilon+2\vartheta+3J$			9.5	1.796	2.303_n	2.1007
$\eta\eta'$	$2\varepsilon+4\vartheta+3J$				1.92_n	2.700	
	$2\varepsilon+4\vartheta+4J$		8.7	[8.8]	1.5802_n	2.4158	2.9867_n
η^2	$2\varepsilon+4\vartheta+4J$				2.330	3.1764_n	
η'^2	$2\varepsilon+4\vartheta+4J$				3.1079	3.9008_n	
j^2	$2\varepsilon+4\vartheta+4J$				2.736	3.6809_n	
$\eta\eta'$	$2\varepsilon+4\vartheta+5J$				2.9881_n	3.8425	
η'	$2\varepsilon+6\vartheta+5J$		9.64	[0.53]	2.652_n	3.6204	4.3279_n
η	$2\varepsilon+6\vartheta+6J$		9.48_n	$[0.36_n]$	2.4419	3.4512_n	4.1892
η'^2	$2\varepsilon+8\vartheta+6J$				3.6135_n	4.6784	
$\eta\eta'$	$2\varepsilon+8\vartheta+7J$				3.7124	4.8075_n	
η^2	$2\varepsilon+8\vartheta+8J$				3.2109_n	4.3338	
j^2	$2\varepsilon+8\vartheta+7J-\Sigma$				2.068_n	3.2092	
	$\frac{5}{2}\varepsilon+5\vartheta+5J$			9.3_n	1.140_n	2.0056	2.5727_n
η'	$\frac{5}{2}\varepsilon+7\vartheta+6J$			0.5_n	2.2749_n	3.2377	3.9184_n
η	$\frac{5}{2}\varepsilon+7\vartheta+7J$			0.3	2.0542	3.0565_n	3.7710
	$\frac{7}{2}\varepsilon+7\vartheta+7J$			8.1	0.43_n	1.346	1.959_n

TABLE XLIII—Continued.

Logarithmic. ν Unit$=1''$

	Cos	w^{-3}	w^{-2}	w^{-1}	w^{0}	w	w^{2}
	$(\vartheta-\vartheta_0)$ sin						
$\eta\ \eta'$	⅃	9.66	0.810_n	2.7559	3.3840_n	3.6946	
η'	$2\vartheta+$ ⅃		9.79_n	0.54			
η	$2\vartheta+2$⅃		9.92	$[0.63_n]$			
η	ε	9.801_n	0.425	2.5970_n	3.1493	3.4158_n	
η^3	ε	1.075_n	2.045	3.5201_n	4.3751		
$\eta\ \eta'^2$	ε	1.640_n	2.514	4.1066_n	4.8961		
$j^2\eta$	ε	1.063	1.916_n	4.1066	4.8961_n	5.4076	
η'	$\varepsilon+$ ⅃	9.36	0.471_n	2.4824	3.0830_n	3.3936	
$\eta^2\eta'$	$\varepsilon+$ ⅃	1.565	2.456_n	3.9671	4.7890_n		
η'^3	$\varepsilon+$ ⅃	1.543	2.341_n	3.8942	4.6760_n		
$j^2\ \eta'$	$\varepsilon+$ ⅃	0.87_n	1.75	4.0705_n	4.8759	5.3965_n	
$\eta\ \eta'^2$	$\varepsilon+$ 2⅃	1.441_n	2.273	3.7192_n	4.5456		
$j^2\ \eta'$	$\varepsilon+$ Σ	0.42	1.36_n	3.7799	4.5819_n	5.0998	
$j^2\eta$	$\varepsilon+$ ⅃$+\Sigma$	0.695_n	1.585	4.0417_n	4.7827	5.2543_n	
η^2	$\varepsilon+4\vartheta+4$⅃		9.59_n	0.45			
η'	$\varepsilon+4\vartheta+3$⅃		9.46	0.34_n			
η	$2\varepsilon+2\vartheta+2$⅃		9.45	$[0.11_n]$			
η'	$2\varepsilon+2\vartheta+3$⅃		9.32_n	[0.04]			
$\eta^2\eta'$	$-\ \varepsilon+$ ⅃	1.255_n	2.149	3.6240_n	4.4615		
	$(\vartheta-\vartheta_0)^2$ cos						
η	ε		9.25	0.117_n			
η'	$\varepsilon+$ ⅃		9.12_n	0.02			
		m'^2	m'^2	m'^2, m'	m'	m'	m'

$$\tau=\Sigma w^s\eta^p\eta'^q j^{2t}C_1 \cos \text{Arg.}+(\vartheta-\vartheta_0)\Sigma w^s\eta^p\eta'^q j^{2t}C_2 \sin \text{Arg.}+(\vartheta-\vartheta_0)^2\Sigma w^s\eta^p\eta'^q j^{2t}C_3 \cos \text{Arg.}$$

where C_1, C_2, C_3 represent the respective coefficients.

PERTURBATIONS OF THE THIRD COORDINATE.

For the third coordinate the developments are limited to perturbations of the first order and of the first degree with the exception of some secular terms of second degree. We can therefore use osculating elements in this section, and use θ and ϑ without distinction.

By *Z* 8 eq. (39), 41, eq. (83) and 8, eq. (41) the equations *Z* 115, (192) are given, in which Σ is defined.

Since

$$\frac{dS}{d\varepsilon}=\frac{\partial S}{\partial\varepsilon}+\frac{\partial S}{\partial\theta}\frac{d\theta}{d\varepsilon}=\Sigma$$

By Z 9, eq. (45) we have, with sufficient accuracy, *Z* 115, eqs. (193). Within these limits,

$$\frac{d\theta}{d\varepsilon}=\frac{w}{2}(1-e\cos\varepsilon).$$

Substituting this relation in the above equation and in eq. (192) in turn, the differential equation to be integrated is (194).

Since *F*, *G*, *H* are power series in *w*, it is evident from eqs. (192) that

$$\frac{dS}{d\varepsilon}=\Sigma_0+\Sigma_1 w+\Sigma_2 w^2+\ .\ .\ .\ .\ .$$

where

$$-\Sigma_i=F_{i.p.q}+G_{i.p.q}+H_{i.p.q}$$

Therefore, eq. (194) becomes

$$\frac{\partial S}{\partial \varepsilon}+\frac{w}{2}(1-e\cos\varepsilon)\frac{\partial S}{\partial \theta}=\Sigma_0+\Sigma_1 w+\Sigma_2 w^2+\ .\ .\ .\ .\ .$$

Comparing the coefficients of like powers of w on either side of the equation, it is evident that the integral must be of the form

$$S=S_{-1}\frac{1}{w}+S_0+S_1 w+S_2 w^2+\ .\ .\ .\ .\ .$$

Substituting this relation in the preceding equation and equating like powers of w, the system of equations $(195_{-1})-(195_1)$ follows.

Within the extent of the following developments one more equation should be written by analogy.

This system of equations is integrated in a manner similar to that for $\frac{dW}{d\varepsilon}$ (see p. 81). Each equation is broken up into two equations, one a function of ε and one independent of ε. The differential equation (194) is then replaced by eight differential equations, the integrals of which can be obtained in the order,

$$S_{-1},\ (S_0-[S_0]),\ [S_0],\ (S_1-[S_1]),\ [S_1],$$

As in the case of $\frac{dW}{d\varepsilon}$, the condition is imposed that

$$\left[\frac{\partial S_i}{\partial \varepsilon}\right]=0$$

The equivalent equations are (196)–(200).

A comparison of the differential equations for $(S_i-[S_i])$ with the expressions for $\frac{dW_2''}{d\varepsilon}$ $\frac{dW_3''}{d\varepsilon}$ leads to an analogous form of integration for certain terms. Within the extent of our developments,

$$(\Sigma_0-[\Sigma_0])+w(\Sigma_1-[\Sigma_1])$$

and

$$-\frac{1}{2}\,(1-e\cos\varepsilon)\frac{\partial}{\partial\theta}\int(\Sigma_0-[\Sigma_0])d\varepsilon-\left[(1-e\cos\varepsilon)\frac{\partial}{\partial\theta}\int(\Sigma_0-[\Sigma_0])d\varepsilon\right]$$

take the place of $\frac{\partial W_2''}{\partial\varepsilon}$ and $\frac{\partial W_3''}{\partial\varepsilon}$, respectively. Without change of notation for the third coordinate, $(S-[S])$ is given by eqs. (201), (202), where $\tilde{F}$, $\tilde{G}$, $\tilde{H}$ are computed from F, G, H in Tables XII–XIV, by means of eqs. (118) and (119). The coefficients $\tilde{F}$, $\tilde{G}$, $\tilde{H}$ are tabulated in Tables L to LII.

The function $[S]$ is obtained from the integration of eq. (203). A constant of integration is added, which is the same in form as Hansen's constant of integration for the perturbation of the third coordinate, namely,

$$c_1(\cos\psi-e)+c_2\sin\psi \qquad \text{Z eq. (204)}$$

where c_1 and c_2 are undetermined.

By eqs. (192), the pertubation $\frac{u}{\cos i}$ is derived from

$$\frac{u}{\iota\cos i}=\bar{S}$$

The perturbation comprises the computed value of eq. (202), the trigonometric sine series given by Tables L to LII (which can be written by inspection with the aid of Table XV*b*), the series forming Table LIII, and the constant of integration (204), in all of which

$$\psi=\varepsilon$$

TABLE L. Unit=1″.

	n	0	1	2	3	4	5
	$\tilde{F}_{1\cdot0}(n+1.-n+1)+\pi'$	+ 52.7	+ 96.0	+ 57.0	+ 33.8	+ 20.1	+ 12.0
	$\tilde{F}_{1\cdot0}(n-1.-n+1)+\pi'$	+158.2		− 285.0	−101.4	− 47.0	− 24.0
	$\tilde{F}_{1\cdot0}(n+1.-n-1)-\pi'$	−158.2	−191.9	− 95.0	− 50.7	− 28.2	− 16.0
	$\tilde{F}_{1\cdot0}(n-1.-n-1)-\pi'$	− 52.7	−191.9	− 285.0		+ 140.9	+ 48.1
Factor w	$\tilde{F}_{1\cdot0}(n+1.-n+1)+\pi'$	−201	−352	− 253	−176	− 119	− 80
	$\tilde{F}_{1\cdot0}(n-1.-n+1)+\pi'$	−812		+1495	+594	+ 305	+172
	$\tilde{F}_{1\cdot0}(n+1.-n-1)-\pi'$	+812	+897	+ 498	+297	+ 183	+114
	$\tilde{F}_{1\cdot0}(n-1.-n-1)-\pi'$	+201	+513	+ 355		−1473	−439

TABLE LI. Unit=1″.

		0	1	2	3	4	5
	$\tilde{G}_{0\cdot0}(n.-n+1)+\pi'$	− 26.37	− 47.98	− 28.50	− 16.91	− 10.06	− 6.02
	$\tilde{G}_{0\cdot0}(n.-n-1)-\pi'$	+ 79.10	+ 95.96	+ 47.50	+ 25.36	+ 14.09	+ 8.02
	$\tilde{G}_{1\cdot0}(n+1.-n+1)+\pi'$	+ 90.3	+ 112.3	+ 58.5	+ 29.0	+ 13.6	+ 5.8
	$\tilde{G}_{1\cdot0}(n-1.-n+1)+\pi'$	+ 530.8	+ 720.8	+ 468.9	+ 311.7	+ 207.7	+ 138.0
	$\tilde{G}_{1\cdot0}(n+1.-n-1)-\pi'$	− 124.2	− 120.5	− 53.3	− 21.8	− 7.4	− 1.2
	$\tilde{G}_{1\cdot0}(n-1.-n-1)-\pi'$	+ 609.9		− 1549.1	− 674.1	− 369.6	− 219.0
	$\tilde{G}_{0\cdot1}(n.-n+2)+\pi'$	− 162.4	− 211.6	− 103.8	− 47.7	− 19.7	− 6.4
	$\tilde{G}_{0\cdot1}(n.-n)+\pi'$	− 166.5	− 352.6	− 298.2	− 229.0	− 167.2	− 118.3
	$\tilde{G}_{0\cdot1}(n.-n)-\pi'$	+ 166.5	+ 96.7	+ 13.2	− 14.4	− 20.7	− 19.3
	$\tilde{G}_{0\cdot1}(n.-n-2)-\pi'$		+ 1825.5	+ 881.4	+ 516.6	+ 321.2	+ 204.1
Factor w	$\tilde{G}_{0\cdot0}(n.-n+1)+\pi'$	+ 100.4	+ 176.3	+ 126.8	+ 87.8	+ 59.6	+ 39.9
	$\tilde{G}_{0\cdot0}(n.-n-1)-\pi'$	− 406.6	− 448.6	− 249.2	− 148.6	− 91.5	− 57.2
	$\tilde{G}_{1\cdot0}(n+1.-n+1)+\pi'$	− 432	− 592	− 370	− 218	− 122	− 64
	$\tilde{G}_{1\cdot0}(n-1.-n+1)+\pi'$	−2047	− 3188	− 2412	−1811	−1342	− 982
	$\tilde{G}_{1\cdot0}(n+1.-n-1)-\pi'$	+ 718	+ 821	+ 440	+ 225	+ 107	+ 44
	$\tilde{G}_{1\cdot0}(n-1.-n-1)-\pi'$	−2401		+12134	+4939	+2788	+1744
	$\tilde{G}_{0\cdot1}(n.-n+2)+\pi'$	+ 693	+ 951	+ 568	+ 314	+ 158	+ 68
	$\tilde{G}_{0\cdot1}(n.-n)+\pi'$	+ 893	+ 1773	+ 1607	+1356	+1089	+ 844
	$\tilde{G}_{0\cdot1}(n.-n)-\pi'$	− 893	− 747	− 254	− 27	+ 68	+ 98
	$\tilde{G}_{0\cdot1}(n.-n-2)-\pi'$		−13263	− 5889	−3549	−2336	−1586

TABLE LII. Unit=1″.

		0	1	2	3	4	5
	$\tilde{H}_{0\cdot0}(n.-n+1)+\pi'$	− 79.10		+ 142.49	+ 50.72	+ 23.48	+ 12.03
	$\tilde{H}_{0\cdot0}(n.-n-1)-\pi'$	+ 26.37	+ 95.96	+ 142.49		− 70.45	− 24.07
	$\tilde{H}_{1\cdot0}(n+1.-n+1)+\pi'$	− 609.9	− 528.9	− 231.4	− 108.8	− 52.7	− 25.7
	$\tilde{H}_{1\cdot0}(n-1.-n+1)+\pi'$	+ 124.2	+ 528.9	+1121.6		− 897.4	− 365.9
	$\tilde{H}_{1\cdot0}(n+1.-n-1)-\pi'$	− 530.8		+ 551.7	+ 166.9	+ 64.3	+ 26.5
	$\tilde{H}_{1\cdot0}(n-1.-n-1)-\pi'$	− 90.3	− 312.4	− 421.4	− 572.6	− 967.8	
	$\tilde{H}_{0\cdot1}(n.-n+2)+\pi'$		+1057.8	+ 311.5	+ 111.3	+ 39.4	+ 11.6
	$\tilde{H}_{0\cdot1}(n.-n)+\pi'$	− 166.5	−1057.8		+1145.2	+ 501.5	+ 276.0
	$\tilde{H}_{0\cdot1}(n.-n)-\pi'$	+ 166.5	+ 290.1		+ 71.9	+ 62.1	+ 44.9
	$\tilde{H}_{0\cdot1}(n.-n-2)-\pi'$	+ 162.4	+ 608.5	+ 881.4	+1551.0		− 1020.6
Factor w	$\tilde{H}_{0\cdot0}(n.-n+1)+\pi'$	+ 406.6		− 747.8	− 297.2	− 152.4	− 85.8
	$\tilde{H}_{0\cdot0}(n.-n-1)-\pi'$	− 100.4	− 256.6	− 177.8		+ 739.1	+ 219.8
	$\tilde{H}_{1\cdot0}(n+1.-n+1)+\pi'$	+2402	+2483	+1362	+ 740	+ 406	+ 222
	$\tilde{H}_{1\cdot0}(n-1.-n+1)+\pi'$	− 717	−2183	−4550		+8048	+ 3120
	$\tilde{H}_{1\cdot0}(n+1.-n-1)-\pi'$	+2046		−4336	−1408	− 697	− 298
	$\tilde{H}_{1\cdot0}(n-1.-n-1)-\pi'$	+ 432	+1214	+1481	+1901	+1186	
	$\tilde{H}_{0\cdot1}(n.-n+2)+\pi'$		−3908	−1705	− 753	− 325	− 126
	$\tilde{H}_{0\cdot1}(n.-n)+\pi'$	+ 893	+3908		−9529	−3936	− 2233
	$\tilde{H}_{0\cdot1}(n.-n)-\pi'$	− 893	−1855		− 39	− 287	− 270
	$\tilde{H}_{0\cdot1}(n.-n-2)-\pi'$	− 693	−1987	−2363	− 310		+13643

TABLE LIII.

$[S]-\{e_1(\cos\psi-e)+e_2\sin\psi\}$ Unit=1″.

	Sin	w^{-1}	w^0	w
	$\psi+4\theta+3J-\Pi'$	$-\ \ 25.36$	$+\ \ 123.2$	$-\ 281.8$
η	$4\theta+3J-\Pi'$	$+\ \ 50.7$	$-\ \ 246.5$	$+\ 563.6$
	$-\psi+2\theta+\ J-\Pi'$	$-\ 816.8$	$+\ 3636$	-8548
	$\psi+2\theta+3J+\Pi'$	$-\ 521.8$	$+\ 2851$	-7663
	$\psi+2\theta+\ J-\Pi'$	$+\ 432.9$	$-\ 2031$	$+5237$
	$\psi+6\theta+5J-\Pi'$	$+\ 129.9$	$-\ \ 861$	$+2770$
η'	$\psi-2\theta\ \ +\Pi'$	$-\ 649.4$	$+\ 3096$	-7475
	$\psi+2\theta+2J+\Pi'$	$+\ 596.4$	$-\ 2916$	$+7216$
	$\psi+2\theta+2J-\Pi'$	$-\ \ 26.5$	$+\ \ 494$	-2266
	$\psi+6\theta+4J-\Pi'$	$-\ 214$	$+\ 1236$	-3395
	$(\theta-\theta_0)$ cos			
	$\psi+\ \ J+\Pi'$	$+\ 191.93$	$-\ \ 705.2$	$+1302.6$
η	$J+\Pi'$	$-\ 383.8$	$+\ 1410$	-2605
η^2	$\psi+\ \ J+\Pi'$	$+6584$	-40060	
	$\psi-\ \ J-\Pi'$	-5312	$+29610$	
$\eta\ \eta'$	$\psi+\ \ 2J+\Pi'$	-5742	$+36970$	
	$\psi+\ \ \Pi'$	-6024	$+38180$	
	$\psi-\ \ \Pi'$	$+6024$	-38180	
η'^2	$\psi+\ \ J+\Pi'$	$+6584$	-40060	
	$\psi+\ \ J-\Pi'$	-1656	$[+11860]$	
j^2	$\psi+\ \ J+\Pi'$	-3002	$+18970$	
			m'	

and by eq. (193),

$$\theta-\theta_0=\frac{w}{2}nt$$

By inspection it is clear that the periodic part of $\overline{S}$ is of the form

$$\Sigma\, U_{p\cdot q}\eta^p\eta'^q\sin A$$

and the secular terms are of the form

$$\Sigma\, U_{p\cdot q}\eta^p\eta'^q\frac{w}{2}nt\cos\{(A-\varepsilon)+\varepsilon\}+\frac{w}{2}nt.\ \eta\, U_{1\cdot 0}\cos A$$

Expanding $\cos\{(A-\varepsilon)+\varepsilon\}$, and collecting coefficients of $\sin\varepsilon$ and $\cos\varepsilon$, the secular terms can be written

$$nt\{K_1(\cos\varepsilon-e)+K_2\sin\varepsilon\}$$

where

$$K_1=\Sigma\, U_{p\cdot q}\eta^p\eta'^q\frac{w}{2}\cos(A-\varepsilon)-\frac{w}{4}U_{1\cdot 0}\cos A$$

$$K_2=-\Sigma\, U_{p\cdot q}\eta^p\eta'^q\frac{w}{2}\sin(A-\varepsilon)$$

Introducing this notation, the perturbation can be written in the form of eq. (205).

The coefficients $U_{p\cdot q}$ are given in Table LIV. K_1 and K_2, which are constants, are tabulated in Tables $\mathrm{LV_I}$ and $\mathrm{LV_{II}}$, respectively. For a given planet the factors and arguments are known. Therefore K_1 and K_2 reduce each to a single numerical quantity.

Since the Bohlin-v.Zeipel method is based on the fundamental principles of Hansen, the constants of integration are determined by the condition which must be satisfied when the

perturbations are developed on the basis of osculating elements, namely, that the perturbations and their first derivatives shall be zero at the time $t=0$. The relations to be satisfied are

$$\left.\begin{aligned} u &= 0 \\ \frac{du}{dt} &= 0 \end{aligned}\right\} t=0$$

and the following equations are equivalent relations:

$$\left.\begin{aligned} \frac{u}{\iota \cos i} &= 0 \\ \frac{d}{d\varepsilon}\left(\frac{u}{\iota \cos i}\right) &= 0 \end{aligned}\right\} t=0$$

TABLE LIV

Logarithmic. $\Sigma\, U_{p\cdot q}\eta^p\eta'^q \sin \mathrm{Arg.}$ Unit=1″

	Sin	w^{-1}	w^0	w
η	$-\Delta-\Pi'$		3.0621_n	3.7258
η'	$-\Pi'$		2.8235	3.5528_n
	$2\theta+\Delta-\Pi'$		2.2831	2.8483_n
η	$4\theta+3\Delta-\Pi'$	1.705	3.1591_n	3.8608
η'	$4\theta+2\Delta-\Pi'$		3.2462	3.9166_n
η	$\frac{1}{2}\varepsilon+\theta-\Pi'$		3.2112_n	3.8544
η'	$\frac{1}{2}\varepsilon+\theta+\Delta-\Pi'$		2.5875	3.4153_n
	$\frac{1}{2}\varepsilon+3\theta+2\Delta-\Pi'$		2.2787	2.6304_n
η'	$\frac{1}{2}\varepsilon+5\theta+3\Delta-\Pi'$		3.3155	3.5865_n
η	$\frac{1}{2}\varepsilon+5\theta+4\Delta-\Pi'$		3.0779_n	3.3972
η	$-\frac{1}{2}\varepsilon-\theta-2\Delta-\Pi'$		3.1158_n	3.7378
η'	$-\frac{1}{2}\varepsilon-\theta-\Delta-\Pi'$		3.1493	3.7544_n
	$-\frac{1}{2}\varepsilon+\theta-\Pi'$		2.3242	3.0060_n
η'	$-\frac{1}{2}\varepsilon+3\theta+\Delta-\Pi'$		3.3863	4.1833_n
η	$-\frac{1}{2}\varepsilon+3\theta+2\Delta-\Pi'$		3.3532_n	4.1452
η	$\varepsilon+2\theta+\Delta-\Pi'$	2.6364	3.3704_n	3.8423
η'	$\varepsilon+2\theta+2\Delta-\Pi'$	1.423_n	2.706	3.4014_n
	$\varepsilon+4\theta+3\Delta-\Pi'$	1.4042_n	2.1720	2.6339_n
η'	$\varepsilon+6\theta+4\Delta-\Pi'$	2.3306_n	3.1922	3.7582_n
η	$\varepsilon+6\theta+5\Delta-\Pi'$	2.1137	3.0138_n	3.6101
η	$-\varepsilon-2\theta-3\Delta-\Pi'$	2.7175	3.4858_n	3.9484
η'	$-\varepsilon-2\theta-2\Delta-\Pi'$	2.7756_n	3.5070	3.9456_n
	$-\varepsilon-\Delta-\Pi'$		1.6810	2.2463_n
η'	$-\varepsilon+2\theta-\Pi'$	2.8125	3.4427_n	3.7846
η	$-\varepsilon+2\theta+\Delta-\Pi'$	2.9121_n	3.4958	3.8338_n
η	$\frac{3}{2}\varepsilon+3\theta+2\Delta-\Pi'$		2.6058	3.5312_n
η'	$\frac{3}{2}\varepsilon+3\theta+3\Delta-\Pi'$		1.760	1.82_n
	$\frac{3}{2}\varepsilon+5\theta+4\Delta-\Pi'$		1.7510_n	2.8113
η'	$\frac{3}{2}\varepsilon+7\theta+5\Delta-\Pi'$		2.9120_n	4.0813
η	$-\frac{3}{2}\varepsilon-3\theta-4\Delta-\Pi'$		2.8673	3.8458_n
η'	$-\frac{3}{2}\varepsilon-3\theta-3\Delta-\Pi'$		2.9620_n	3.9124
	$-\frac{3}{2}\varepsilon-\theta-2\Delta-\Pi'$		2.0569_n	2.7932
η'	$-\frac{3}{2}\varepsilon+\theta-\Delta-\Pi'$		2.9275_n	3.4708
η	$-\frac{3}{2}\varepsilon+\theta-\Pi'$		2.9702	3.5487_n
η	$2\varepsilon+4\theta+3\Delta-\Pi'$		1.640	2.731_n
η'	$2\varepsilon+4\theta+4\Delta-\Pi'$		1.617	2.340_n
	$2\varepsilon+6\theta+5\Delta-\Pi'$		1.206_n	2.2110
η	$-2\varepsilon-4\theta-5\Delta-\Pi'$		2.4012	3.3634_n
η'	$-2\varepsilon-4\theta-4\Delta-\Pi'$		2.5241_n	3.4544
	$-2\varepsilon-2\theta-3\Delta-\Pi'$		1.5290_n	2.3210
η'	$-2\varepsilon-2\Delta-\Pi'$		2.3174_n	3.0558
η	$-2\varepsilon-\Delta-\Pi'$		2.3514	3.0737_n
		m'		

$$\frac{u}{\iota \cos i}=\Sigma\, U_{p\cdot q}\eta^p\eta'^q \sin A+nt\{K_1(\cos\varepsilon-e)+K_2\sin\varepsilon\}+c_1(\cos\varepsilon-e)+c_2\sin\varepsilon$$

TABLE LV,I.

Logarithmic K_1 Unit=1″.

	Cos	w^0	w	w^2
η'^2	$\Delta-\Pi'$	2.9180_n	3.7732	
	$\Delta+\Pi'$	1.9821	2.5473_n	2.8138
η^2	$\Delta+\Pi'$	2.8035	3.7182_n	
η'^2	$\Delta+\Pi'$	3.5175	4.3017_n	
j^2	$\Delta+\Pi'$	3.1764_n	3.9772	
$\eta\eta'$	$2\Delta+\Pi'$	3.4580_n	4.2668	
			m'	

$$K_1=\Sigma w^s\eta^p\eta'^q j^{2t}\cos \text{Arg}.$$

TABLE LV,II.

Logarithmic K_2 Unit=1″.

	Sin	w^0	w	w^2
η'^2	$\Delta-\Pi'$	2.9180	3.7732_n	
$\eta\eta'$	Π'	3.7799	4.5819_n	
	$\Delta+\Pi'$	1.9821_n	2.5473	2.8138_n
η^2	$\Delta+\Pi'$	3.7744_n	4.5420	
η'^2	$\Delta+\Pi'$	3.5175_n	4.3017	
$\eta\eta'$	$2\Delta+\Pi'$	3.4580	4.2668_n	
j^2	$\Delta+\Pi'$	3.1764	3.9772_n	
			m'	

$$K_2=\Sigma w^s\eta^p\eta'^q j^{2t}\sin \text{Arg}.$$

$$\frac{u}{\iota\cos i}=\Sigma U_{p\cdot q}\eta^p\eta'^q\sin A+nt\Big\{K_1(\cos\varepsilon-e)+K_2\sin\varepsilon\Big\}+c_1(\cos\varepsilon-e)+c_2\sin\varepsilon$$

By eq. (205), at the date of osculation,

$$t=0,\qquad \theta=\theta_0$$

$$\frac{u}{\iota\cos i}=\Sigma U_{p\cdot q}\eta^p\eta'^q\sin A+c_1(\cos\varepsilon-e)+c_2\sin\varepsilon \tag{A}$$

By Hansen,[1]

$$\frac{d}{d\varepsilon}\left(\frac{u}{\iota\cos i}\right)=\overline{\frac{d}{d\psi}\left(\frac{U}{\iota\cos i}\right)}=\overline{\frac{dS}{d\psi}}=0 \tag{B}$$

in which v. Zeipel's notation is adopted.

From the various parts of S, enumerated above, $\overline{\frac{dS}{d\psi}}$ can be computed. Since S contains the constants of integration

$$c_1(\cos\psi-e)+c_2\sin\psi$$

the derivative, $\overline{\frac{dS}{d\psi}}$, contains the constants

$$-c_1\sin\varepsilon+c_2\cos\varepsilon$$

The solution of eqs. (A) and (B) gives c_1 and c_2. But there is a better way of determining the derivative of the perturbation. The exposition of this second method is postponed until a particular example is considered, for the perturbations are not yet in a form which leads to the development of the equations.

[1] Auseinandersetzung einer zweckmässigen Methode zur Berechnung der Absoluten Störungen der kleinen Planeten, Erste Abhandlung, § 5, p. 8

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COMPARISON OF TABLES.

Tables L, LI, LII check satisfactorily.

Table LIII.—With one exception, the agreement is satisfactory. The bracketed coefficient contains a misprint in sign in v. Zeipel's table. That it is a misprint is evident from Table LV_I, in which the correct sign is given to the corresponding coefficient.

The terms included in the last column are computed from the additional tables, $XIIw^2$, $XIIIw^2$, $XIVw^2$ and from first degree terms in Z 116, eq. (200). The latter part, namely,

$$\frac{1}{2}\left[e\cos\varepsilon\left(\frac{\delta}{\delta\theta}\int(\Sigma_1-[\Sigma_1])d\varepsilon-\frac{1}{2}\frac{\delta}{\delta\theta}\int\left\{\frac{\delta}{\delta\theta}\int(\Sigma_0-[\Sigma_0])d\varepsilon\right\}d\varepsilon\right)\right]$$

is added to both eq. (200) and eq. (203).

Table LIV.—Our table is more extensive. The one bracketed quantity includes an additional term from Table LIII.

Tables LV_I, LV_{II} check satisfactorily.

CONSTANTS OF INTEGRATION IN $n\delta z$ AND ν.

The constants in $\frac{u}{\cos i}$ were treated in the preceding section by the familiar Hansen method. It is the purpose of this section to modify the similar treatment of the constants in the perturbations $n\delta z$ and ν so as to incorporate them in the elements a_0, e_0, π_0, ψ_0. Through the constants of integration, the constant elements, which have been used from the beginning without definition, are to be explained.

Since the group method of developing perturbations is built upon the fundamental principles of Hansen, his conditions for the determination of the constants of integration must be fulfilled. These conditions depend upon the choice of initial osculating or mean elements. Osculating elements are used here. The corresponding conditions are that the perturbations and their first derivatives, at the date of osculation, $(t=0)$, shall be zero.

Consider the relation of the constants of integration to the elements. There are two constants in each perturbation since the differential equations are of the second order. The constant added in the first integration is a velocity; the one added in the second integration is a displacement, or, a perturbation. Now, recalling that the position and velocity of a body for any time t can be transformed into the constants which are ordinarily called the elements of the orbit, it is evident, by analogy, that a displacement of the body and the velocity of the displacement can be transformed similarly into changes in the elements. The four constants in $n\delta z$ and ν are related to the four elements, a, e, π, c, defining the shape and size of the orbit and the position in the orbit, and the two constants in the perturbation which is measured perpendicular to the plane of the orbit are related to the elements Ω, i, which determine the position of the plane of the orbit. It is possible therefore to modify all six elements, but it is v. Zeipel's preference to make the transformations only for the first four constants.

It is not necessary to compute

$$\left.\begin{matrix} n\delta z & \nu \\ \frac{dn\delta z}{d\varepsilon} & \frac{d\nu}{d\varepsilon}\end{matrix}\right\} t=0$$

for the following developments perform the transformation analytically, and the changes in the elements can be computed from auxiliary functions.

Let a_0, e_0, π_0, c_0 be osculating elements; let a, e, π, c be the osculating elements modified by the constants of integration in the manner indicated above.

For undisturbed motion,

$$\varepsilon-e_0\sin\varepsilon=c_0+n_0t \qquad tg(v-\pi_0)=\sqrt{\frac{1+e_0}{1-e_0}}\,tg\,\tfrac{1}{2}\,\varepsilon$$

$$r\cos(v-\pi_0)=a\,(\cos\varepsilon-e_0) \qquad r\sin(v-\pi_0)=a_0\sqrt{1-e_0{}^2}\sin\varepsilon$$

Hansen's choice of ideal coordinates demands that the coordinates and their velocities shall have the same form of expression for disturbed and undisturbed motion. The ideal polar

coordinates are designated by $\bar{\varepsilon}$ or $\bar{f}$ and $\bar{r}$. The relations which are analogous to the above are

$$\bar{\varepsilon} - e_0 \sin \bar{\varepsilon} = c_0 + n_0 t + n_0 \delta z = c_0 + n_0 (t + \delta z) \qquad tg(v - \pi_0) = \sqrt{\frac{1 + e_0}{1 - e_0}}\, tg\, \tfrac{1}{2}\bar{\varepsilon}$$

$$\bar{r} \cos \bar{f} = a_0 \,(\cos \bar{\varepsilon} - e_0) \qquad \bar{r} \sin \bar{f} = a_0 \sqrt{1 - e_0{}^2} \sin \bar{\varepsilon}$$

$$\bar{f} = v - \pi_0 \qquad r = \bar{r}(1 + \nu)$$

These are the equations for motion in the orbit based on constant osculating elements and appropriately determined constants of integration.

If, in place of osculating elements and Hansen's $n\delta z$ and ν, v. Zeipel's elements and the corresponding perturbations are used, the equations are the same in form. In v. Zeipel's notation ε and f take the place of $\bar{\varepsilon}$ and $\bar{f}$. The omission of the dash over these variables is permissible, since the physically real values, with which they might be confused, do not occur in the theory except for the date of osculation, where the subscript zero is added. It is to be noted that, through v. Zeipel's choice of elements, the coordinates and the perturbations have values which are numerically different from the Hansen quantities of the same designation.

Let the time be the date of osculation and denote the true coordinates by ε_0, v_0, r_0. Then the preceding equations for undisturbed motion become Z 121, equations (206), (207), and Z 125, equation (230).

Let the disturbed eccentric anomaly and radius vector $(\varepsilon, \bar{r})$ be ε_1 and r_1, respectively. The relations for disturbed motion become Z 121, equation (209), and Z 122, equations (210).

The first derivatives of these expressions are given by equations (208) and (211), respectively, and the time rate of ε is given by the equation following (209).

The solution of the four equations (210), (211), with the aid of all the others, determines the four unknown constant elements, a, e, π, c, or, better, $a - a_0$, $e - e_0$, $\pi - \pi_0$, and c.

The fact that the adoption of the new elements in connection with the perturbations $n\delta z$ and ν, as developed in the preceding sections, is equivalent to the use of osculating elements, follows from the simultaneous solution of the equations for the disturbed coordinates and their velocities and the corresponding equations for undisturbed motion.

The method of calculating c from the equation

$$c = \varepsilon_1 - e \sin \varepsilon_1 - n\delta z$$

is given in the example, page 18.

After many laborious transformations the other three unknowns are expressed in terms of familiar functions in equations (233)–(236). In the verification of these equations slight differences in the numerical coefficients of certain unimportant terms were found. The magnitudes of these coefficients depend upon the number of the terms included in making the transformations. Since it makes little difference whether or not they are included and since v. Zeipel's values present a more symmetrical form of a later auxiliary function, we adopted his coefficients.

In the functions x, y, z the arguments and factors are functions of η, π, ε_1, θ_1, $\varDelta$, $\varSigma$, where

$$\theta_1 = \frac{1}{2}(\varepsilon_1 - e \sin \varepsilon_1) - g'$$

but at the beginning of the computation only η_0, π_0, ε_0, θ_0, $\varDelta_0$, $\varSigma_0$, the corresponding functions of osculating elements are known.[1]

[1] There is a confusion of notation in v. Zeipel's developments. In Z 127, equation (238), θ_0 is defined to be the value of θ at the date of osculation when osculating elements are used for the planet, and θ_1 signifies the argument if the elements a, e, π, etc., are employed, or by Z 9, equation (43),

$$\theta_0 = \frac{1}{2}(\varepsilon_0 - e_0 \sin \varepsilon_0) - g'$$

$$\theta_1 = \frac{1}{2}(\varepsilon_1 - e \sin \varepsilon_1) - g'$$

and their difference is computed by Z 127, equation (238).

In the collection of formulae by Z 133,

$$\theta_0 = \frac{1}{2}\, c_0 - c'$$

This is an approximation for the above equation.

Again, in Z, 60,

$$\theta_0 = \frac{1}{2} c - c'$$

$$= \frac{1}{2}\, (\varepsilon_1 - e \sin \varepsilon_1) - n\delta z - c'$$

If the secular terms are counted from the date of osculation, the factor $(\theta - \theta_0)$ ought to be replaced by $(\theta - \theta_1)$.

By equations Z (43), (235), (236) and the equations preceding (233), the factor η and the arguments Δ, ε_1, θ_1 are given in equations (238) in terms of osculating values and functions of perturbations, inclusive of first order.

To these should be added

$$\Sigma=\Sigma_0-\frac{1}{4\eta_0}z+\frac{1}{2}\eta_0 z+\ .\ .\ .\ .$$

and

$$\Gamma_1=\Gamma+\frac{3}{4}\left(1-\frac{2}{3}\eta_0\cos\varepsilon_0\right)(y\sin\varepsilon_0-z\cos\varepsilon_0)+\frac{1}{2}\eta_0 z+\ .\ .\ .\ .\ .$$

where

$$\Gamma_1=\frac{1}{2}\varepsilon_1+\theta_1+\Delta_1$$

$$\Gamma=\frac{1}{2}\varepsilon_0+\theta_0+\Delta_0$$

The equations (233), (235), (236), and (238) permit the construction of two tables which determine w, n or a, and e and π. From here on the developments differ in form from v. Zeipel's although they are the same in principle. If v. Zeipel's equations (237) and (239) are used, the term $(x_2''-\eta y_2'')$ should read

$$(x_2''+x_3''+x_4'')-\eta(y_2''+y_3''+y_4'')$$

in agreement with Z 91, line 14.

Suppose that $w-w_0$ has been computed by equation (233) and the argument Γ has been introduced. The arguments and factors are unknown.

$$w-w_0=f(\eta,\ \Gamma_1,\ \theta_1,\ \Delta,\ \Sigma)$$

By Taylor's theorem

$$w-w_0=f(\eta_0,\ \Gamma,\ \theta_0,\ \Delta_0,\ \Sigma_0)+\frac{\partial f}{\partial\eta_0}\Delta\eta_0+\frac{\partial f}{\partial\Gamma}\Delta\Gamma_0+\frac{\partial f}{\partial\theta_0}\Delta\theta_0+\frac{\partial f}{\partial\Delta_0}\Delta\Delta_0+\frac{\partial f}{\partial\Sigma_0}\Delta\Sigma_0+\ .\ .\ .\ .$$

Inclusive of second order in m', the differentiation is for first order terms.

Substituting the values of $\Delta\eta$, $\Delta\Gamma$, $\Delta\theta_0$, $\Delta\Delta_0$, $\Delta\Sigma_0$ from equations (238) and the additional equations above,

$$w-w_0=f(\eta_0,\ \Gamma,\ \theta_0,\ \Delta_0,\ \Sigma_0)+\left(\frac{\partial f}{\partial(2\theta_0)}-\frac{\partial f}{\partial\Delta_0}-\frac{\partial f}{\partial\Sigma_0}\right)\frac{1}{4\eta_0}z+\left(\frac{\partial f}{\partial\Delta_0}+\frac{\partial f}{\partial\Gamma}+\frac{\partial f}{\partial\Sigma_0}\right)\frac{1}{2}\eta_0 z$$

$$-\frac{\partial f}{\partial\eta_0}\frac{1}{4}y+\left\{\frac{1}{2}(1-\eta_0\cos\varepsilon_0)\frac{\partial f}{\partial\theta_0}+\frac{3}{4}\left(1-\frac{2}{3}\eta_0\cos\varepsilon_0\right)\frac{\partial f}{\partial\Gamma}\right\}(y\sin\varepsilon_0-z\cos\varepsilon_0)+\ .\ .\ .\ .$$

The order of calculation is: computation of equation (233), in which the arguments and the factors are given the subscript zero, differentiation of first order terms, computation of the second order terms in the above equation, and the additon of these second order terms to the first calculation.

With some foresight the computation can be simplified. The arguments should be arranged in groups like the following:

$$\begin{array}{r}
-n\Gamma+2\theta+2\Delta\\
-(n-1)\Gamma+2\theta+2\Delta\\
-(n-2)\Gamma+2\theta+2\Delta\\
.\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\\
.\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\\
.\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\\
(n-1)\Gamma+2\theta+2\Delta\\
n\Gamma+2\theta+2\Delta
\end{array}$$

Then, for whole groups of arguments,

$$\frac{\partial f}{\partial(2\theta_0)}-\frac{\partial f}{\partial\Delta_0}-\frac{\partial f}{\partial\Sigma_0}=0$$

Also for some particular argument in a group, the condition

$$\frac{\partial f}{\partial\Delta_0}+\frac{\partial f}{\partial\Gamma}+\frac{\partial f}{\partial\Sigma_0}=0$$

may be satisfied.

Finally, by inspection of the arguments, considerable computation can be avoided if

$$\frac{1}{\eta_0}\left(\frac{\partial f}{\partial(2\theta_0)}-\frac{\partial f}{\partial J_0}-\frac{\partial f}{\partial \Sigma_0}\right)z=\frac{\partial f}{\partial \eta_0}\,y$$

The function $w-w_0$ is tabulated in Table LVI. Since it is unavoidably a function of w itself, the determination of w for a given case must be made by successive trials, the first approximation being

$$w=w_0$$

TABLE LVI.

Logarithmic. $w-w_0$ Unit = 1 radian.

	Cos	w^{-3}	w^{-2}	w^{-1}	w^0	w	w^2
			4.360	[5.1966_n]	[5.7767]		
	Γ			4.766	6.6599	7.3732_n	7.7492
	2Γ			4.446	7.1194	7.7572_n	8.0553
	3Γ			4.412	6.8442	7.5458_n	7.9060
	4Γ			4.484	6.5883	7.3450_n	7.7602
	5Γ				6.3437	7.1490_n	7.6136
	7Γ				5.875	6.7632_n	7.3134
η_0	$-5\Gamma+2\theta_0+2J_0$				6.5090	6.6325_n	7.4746_n
	$-4\Gamma+2\theta_0+2J_0$			4.161_n	6.169	7.0658	7.8698_n
	$-3\Gamma+2\theta_0+2J_0$			3.19	6.8821_n	7.6078	7.9975_n
	$-2\Gamma+2\theta_0+2J_0$			3.52	7.0986_n	7.6970	7.9394_n
	$-\Gamma+2\theta_0+2J_0$			5.1420	6.359	7.0722_n	7.4480
	$2\theta_0+2J_0$		4.379	7.6355_n	8.2144	8.4125_n	
	$\Gamma+2\theta_0+2J_0$			4.856_n	8.0894_n	8.9548	9.5668_n
	$2\Gamma+2\theta_0+2J_0$			4.92_n	7.8150_n	8.6561	9.2006_n
	$3\Gamma+2\theta_0+2J_0$			5.5174_n	7.6056_n	8.4650	9.0111_n
	$4\Gamma+2\theta_0+2J_0$			5.4248_n	7.4128_n	[8.2958]	[8.8561_n]
	$5\Gamma+2\theta_0+2J_0$				7.2254_n	8.1426	8.7346_n
	$7\Gamma+2\theta_0+2J_0$				[6.8746_n]	[7.8484]	8.4936_n
η'	$-5\Gamma+2\theta_0+J_0$				6.8776_n	7.5604	7.8425_n
	$-4\Gamma+2\theta_0+J_0$			4.582	6.8815_n	7.4536	7.5238_n
	$-3\Gamma+2\theta_0+J_0$			4.674	6.6271_n	6.7816	7.3174
	$-2\Gamma+2\theta_0+J_0$			4.99	6.7985	7.4732_n	7.7966
	$-\Gamma+2\theta_0+J_0$			5.4623_n			
	$2\theta_0+J_0$		4.605_n	[7.1987]	7.8314_n	8.1061	
	$\Gamma+2\theta_0+J_0$			5.0056	8.2964	9.1086_n	9.6833
	$2\Gamma+2\theta_0+J_0$			4.38	8.0434	8.8316_n	9.3296
	$3\Gamma+2\theta_0+J_0$			5.6251	7.8458	8.6564_n	9.1558
	$4\Gamma+2\theta_0+J_0$			5.5812	7.6603	8.5030_n	9.0248
	$5\Gamma+2\theta_0+J_0$				7.4778	8.3544_n	8.9050
	$7\Gamma+2\theta_0+J_0$				7.1130	8.0545_n	8.6668
η_0^2		4.664	4.71	5.83			
	Γ				7.8102	8.6250_n	
	2Γ				7.7520_n	8.1242	
	3Γ				7.6172_n	6.6043_n	
	4Γ				7.7135_n	8.2308	
η_0^2	$-4\Gamma+4\theta_0+4J_0$				7.1862	7.9072_n	
	$-3\Gamma+4\theta_0+4J_0$				7.1804	7.8679_n	
	$-2\Gamma+4\theta_0+4J_0$				6.817	7.456_n	
	$-\Gamma+4\theta_0+4J_0$				8.4680_n	8.8822	
	$4\theta_0+4J_0$	4.666	[5.807_n]	[8.0913]	8.8270_n	9.2073	
	$\Gamma+4\theta_0+4J_0$				8.7850	9.8236_n	
	$2\Gamma+4\theta_0+4J_0$				[8.5144]	9.4910_n	
	$3\Gamma+4\theta_0+4J_0$				8.3274	9.3006_n	
	$4\Gamma+4\theta_0+4J_0$				8.1627	9.1494_n	
	$5\Gamma+4\theta_0+4J_0$				8.0050	9.0105_n	
$\eta_0\eta'$	$-4\Gamma+4\theta_0+3J_0$				7.354_n	8.1083	
	$-3\Gamma+4\theta_0+3J_0$				7.5708_n	8.2084	
	$-2\Gamma+4\theta_0+3J_0$						
	$-\Gamma+4\theta_0+3J_0$				8.8838	9.0548_n	
	$4\theta_0+3J_0$	4.516_n	[6.2084]	8.5565_n	9.2180	9.5174_n	
	$\Gamma+4\theta_0+3J_0$				9.2783_n	0.2833	
	$2\Gamma+4\theta_0+3J_0$				9.0241_n	9.9635	
	$3\Gamma+4\theta_0+3J_0$				8.8480_n	9.7850	
	$4\Gamma+4\theta_0+3J_0$				8.6916_n	9.6434	
	$5\Gamma+4\theta_0+3J_0$				8.5401_n	9.5128	
		m'^2	m'^2	m'^2, m'	m'^2, m'	m'	m'

TABLE LVI—Continued.

Logarithmic. $w-w_0$ Unit=1 radian.

	Cos	w^{-3}	w^{-2}	w^{-1}	w^0	w	w^2
$\eta_0\eta'$	$-4\Gamma+\Delta_0$				7.7640	7.8364_n	
	$-3\Gamma+\Delta_0$				7.4203	8.3915	
	$-2\Gamma+\Delta_0$				7.8104_n	8.6268	
	$-\Gamma+\Delta_0$				8.0479_n	8.8018	
	Δ_0	4.518_n	$[5.886_n]$	$[5.70_n]$			
	$\Gamma+\Delta_0$				7.1339	7.8500	
	$2\Gamma+\Delta_0$				7.8421	8.4293_n	
	$3\Gamma+\Delta_0$				7.9669	8.6796_n	
	$4\Gamma+\Delta_0$				7.9760	8.7576_n	
η'^2	$-4\Gamma+4\theta_0+2\Delta_0$				6.9002	7.6938_n	
	$-3\Gamma+4\theta_0+2\Delta_0$				7.1638	7.8502_n	
	$-2\Gamma+4\theta_0+2\Delta_0$						
	$-\Gamma+4\theta_0+2\Delta_0$				8.1860_n	8.4016	
	$4\theta_0+2\Delta_0$	3.76	6.0608_n	8.4157	8.9760_n	9.1661	
	$+\Gamma+4\theta_0+2\Delta_0$				9.1714	0.1382_n	
	$+2\Gamma+4\theta_0+2\Delta_0$				8.9358	9.8333_n	
	$+3\Gamma+4\theta_0+2\Delta_0$				8.7718	9.6681_n	
	$+4\Gamma+4\theta_0+2\Delta_0$				8.6236	9.5372_n	
η'^2		3.76	5.7516	4.7			
	Γ				7.8677	8.6727_n	
	2Γ				7.8610_n	8.2228	
	3Γ				8.1026_n	8.7296	
	4Γ				8.1538_n	8.8728	
j^2	Γ				7.9418_n	8.7337	
	2Γ				7.9312_n	8.7154	
	3Γ				7.7920_n	8.6154	
	4Γ				7.639_n	8.5001	
j^2	$-4\Gamma+4\theta_0+3\Delta_0-\Sigma_0$				7.446	8.1156_n	
	$-3\Gamma+4\theta_0+3\Delta_0-\Sigma_0$				7.1858	7.8677_n	
	$-2\Gamma+4\theta_0+3\Delta_0-\Sigma_0$						
	$-\Gamma+4\theta_0+3\Delta_0-\Sigma_0$				7.6176_n	7.9693	
	$4\theta_0+3\Delta_0-\Sigma_0$		4.804_n	7.168	7.9368_n	8.3724	
	$\Gamma+4\theta_0+3\Delta_0-\Sigma_0$				7.7887	8.8492_n	
	$2\Gamma+4\theta_0+3\Delta_0-\Sigma_0$				7.448	8.4531_n	
	$3\Gamma+4\theta_0+3\Delta_0-\Sigma_0$				7.1976	8.2026_n	
	$4\Gamma+4\theta_0+3\Delta_0-\Sigma_0$				6.978	7.9963_n	
η_0^3	$2\theta_0+2\Delta_0$	5.4181_n	6.292	7.4754_n	8.6636		
	$6\theta_0+6\Delta_0$	5.418_n	6.292	8.6328_n	9.4351		
$\eta_0^2\eta'$	$2\theta_0+\Delta_0$	5.885	6.719_n	8.5059	9.2804_n		
	$2\theta_0+3\Delta_0$	4.974	5.896_n	8.0326_n	8.1975		
	$6\theta_0+5\Delta_0$	5.935	6.780_n	9.2774	0.0330_n		
$\eta_0\eta'^2$	$2\theta_0$	5.744_n	6.535	$[8.3811_n]$	9.1030		
	$2\theta_0+2\Delta_0$	5.44_n	6.327	8.0917	8.6300		
	$6\theta_0+4\Delta_0$	5.919_n	6.744	9.4432_n	0.1464		
η'^3	$2\theta_0+\Delta_0$	5.301	6.149_n	8.2302	9.0152_n		
	$6\theta_0+3\Delta_0$	5.301	6.149_n	9.1294	9.7729_n		
$j^2\eta_0$	$2\theta_0+2\Delta_0$			8.5904	9.3492_n	9.8022	
	$2\theta_0+\Delta_0-\Sigma_0$	4.502_n	5.41	8.1011_n	8.8726		
	$6\theta_0+5\Delta_0-\Sigma_0$	4.502_n	5.41	8.0554_n	8.9263		
$j^2\eta'$	$2\theta_0+\Delta_0$			8.5592_n	9.3245		
	$2\theta_0+2\Delta_0-\Sigma_0$	4.057	5.021_n	6.887	8.1804_n		
	$6\theta_0+4\Delta_0-\Sigma_0$	4.057	5.021_n	8.2718	9.1021_n		
		m'^2	m'^2	m'^2, m'	m'^2, m'	m'	m'

$w-w_0=\Sigma Cw^{s}\eta^{p}\eta'^{q}j^{2t}$ cos Agr., where C represents the respective coefficient.

Turning now to the determination of e and π, let equations (235), (236) be written in the form (244), where

$$S=-\frac{1}{2}z+\left(\frac{1}{4e_0}-\frac{e_0}{3}\right)yz+\frac{1}{6}xz+\ .\ .\ .\ .\ .$$

$$C=-\frac{1}{2}y-\frac{z^2}{4e_0}+\frac{1}{6}xy-\frac{1}{3}e_0y^2+\ .\ .\ .\ .\ .$$

Multiplying the first of these by $\sin\psi$, the second by $\cos\psi$ and adding,

$$S\sin\psi+C\cos\psi=-\frac{1}{2}(y\cos\psi+z\sin\psi)+\frac{1}{6}x(y\cos\psi+z\sin\psi)$$

$$-\frac{1}{3}e_0y(y\cos\psi+z\sin\psi)+\frac{1}{4e_0}z(y\sin\psi-z\cos\psi)+\ .\ .\ .$$

Here, again, the arguments and factors are functions of the elements a, e, π, c, and the expansion in a Taylor's series is necessary.

Let

$$S\sin\psi+C\cos\psi=f(\eta,\ \Gamma_1,\ \theta_1,\ \Delta,\ \Sigma)$$

Then the form of Taylor's series is the same as the expression for $w-w_0$, (p. 148), with the following modification. Within first order quantities,

$$f(\eta,\ \Gamma_1,\ \theta_1,\ \Delta,\ \Sigma)=-\frac{1}{2}(y\cos\psi+z\sin\psi)$$

$$\overline{\frac{\partial f}{\partial\psi}}=\overline{\frac{\partial f}{\partial\psi}}_{\psi=\varepsilon}=\frac{1}{2}(y\sin\varepsilon-z\cos\varepsilon)$$

Hence,

$$S\sin\psi+C\cos\psi=f(\eta_0,\ \Gamma,\ \theta_0,\ \Delta_0,\ \Sigma_0)+\left(\frac{\partial f}{\partial(2\theta_0)}-\frac{\partial f}{\partial\Delta_0}-\frac{\partial f}{\partial\Sigma_0}\right)\frac{1}{4\eta_0}z$$

$$+\left(\frac{\partial f}{\partial\Delta_0}+\frac{\partial f}{\partial\Gamma}+\frac{\partial f}{\partial\Sigma_0}\right)\frac{1}{2}\eta_0z-\frac{\partial f}{\partial\eta}\frac{1}{4}y$$

$$+\left\{(1-\eta_0\cos\varepsilon_0)\frac{\partial f}{\partial\theta_0}+\frac{3}{2}\left(1-\frac{2}{3}\eta_0\cos\varepsilon_0\right)\frac{\partial f}{\partial\Gamma}\right\}\overline{\frac{\partial f}{\partial\psi}}+\ .\ .\ .\ .\ .$$

The order of computation is: calculation of

$$-\frac{1}{2}(y\cos\psi+z\sin\psi)$$

by inspection of the table for W, in which the arguments are to be given the subscript zero, differentiation of the first order terms, calculation of the necessary products of functions of y, z, and the partial derivatives, and the addition of these products to the first calculation. The required function is given in Table LVII.

TABLE LVII.

Logarithmic $S \sin \psi + C \cos \psi$ Unit=1″.

	Cos	w^{-3}	w^{-2}	w^{-1}	w^0	w	w^2
	$\psi-5\Gamma+2\theta_0+2\Delta_0$			8.81	1.082_n	1.5710	1.612_n
	$\psi-4\Gamma+2\theta_0+2\Delta_0$			9.009	1.2314_n	1.5492	0.989_n
	$\psi-3\Gamma+2\theta_0+2\Delta_0$			9.318	0.931	1.604_n	1.916
	$\psi-2\Gamma+2\theta_0+2\Delta_0$			9.207	[1.6478]	2.1070_n	2.2333
	$\psi-\Gamma+2\theta_0+2\Delta_0$			9.711	1.950	2.3426_n	2.3713
	$\psi+2\theta_0+2\Delta_0$		9.196	2.1712_n	2.5678	2.565_n	
	$\psi+\Gamma+2\theta_0+2\Delta_0$			9.230_n	[2.3541_n]	3.1193	3.7107_n
	$\psi+2\Gamma+2\theta_0+2\Delta_0$			9.220_n	[1.9114_n]	2.6867	3.1657_n
	$\psi+3\Gamma+2\theta_0+2\Delta_0$			9.724_n	[1.5372_n]	2.3831	2.8623_n
	$\psi+4\Gamma+2\theta_0+2\Delta_0$			9.494_n	[1.2544_n]	2.1315	2.6333_n
	$\psi+5\Gamma+2\theta_0+2\Delta_0$			9.100_n	1.018_n	1.9034	2.4348_n
η_0	$\psi-5\Gamma+4\theta_0+4\Delta_0$			9.771_n	1.042_n	1.868	2.357_n
	$\psi-4\Gamma+4\theta_0+4\Delta_0$			0.064_n	1.723_n	2.3515	2.6814_n
	$\psi-3\Gamma+4\theta_0+4\Delta_0$			0.3185_n	2.1626_n	2.6961	2.9214_n
	$\psi-2\Gamma+4\theta_0+4\Delta_0$			0.497_n	[2.7787_n]	[3.0649]	3.0993_n
	$\psi-\Gamma+4\theta_0+4\Delta_0$			1.0286_n	3.2379_n	3.1223	3.9385_n
	$\psi+4\theta_0+4\Delta_0$	9.199	9.04_n	[2.6172]	[3.2511_n]	[3.4930]	
	$\psi+\Gamma+4\theta_0+4\Delta_0$			0.7226	3.1702	4.1580_n	4.9365
	$\psi+2\Gamma+4\theta_0+4\Delta_0$			0.669	2.7877	3.7083_n	4.3605
	$\psi+3\Gamma+4\theta_0+4\Delta_0$			0.9435	2.5117	3.4261_n	4.0450
	$\psi+4\Gamma+4\theta_0+4\Delta_0$			0.5122	2.2732	3.2042_n	
η_0	$\psi-5\Gamma$			9.814_n	1.925	2.634_n	2.984
	$\psi-4\Gamma$			0.0434_n	2.0527	2.6896_n	2.9432
	$\psi-3\Gamma$			0.3541_n	2.145	2.675_n	2.744
	$\psi-2\Gamma$		9.140	0.362_n	2.1351	2.3850_n	2.4864_n
	$\psi-\Gamma$			0.4164_n	2.3504_n	3.0929	3.5397_n
	ψ		9.274_n	0.1436_n			
	$\psi+\Gamma$			0.3102_n	2.497	3.1875_n	3.5978
	$\psi+2\Gamma$		9.137_n	9.918	1.9006_n	1.0453	2.8834
	$\psi+3\Gamma$			9.465	0.812_n	2.5218_n	3.3564
	$\psi+4\Gamma$			9.20_n	1.406_n	1.729	
η'	$\psi-5\Gamma+4\theta_0+3\Delta_0$			9.476	1.327	1.889_n	2.2299
	$\psi-4\Gamma+4\theta_0+3\Delta_0$			9.781	1.447	2.1506_n	2.5419
	$\psi-3\Gamma+4\theta_0+3\Delta_0$			9.811	2.1070	2.6309_n	2.8608
	$\psi-2\Gamma+4\theta_0+3\Delta_0$			0.3489	2.5095	2.9557_n	3.0952
	$\psi-\Gamma+4\theta_0+3\Delta_0$			0.9511	3.3599	2.7758	3.9726
	$\psi+4\theta_0+3\Delta_0$	8.76_n	[0.158]	[2.7932_n]	3.3085	3.4526_n	
	$\psi+\Gamma+4\theta_0+3\Delta_0$			9.961_n	3.3609_n	4.3114	5.0691_n
	$\psi+2\Gamma+4\theta_0+3\Delta_0$			0.491_n	2.9943_n	3.8728	4.4922_n
	$\psi+3\Gamma+4\theta_0+3\Delta_0$			1.0464_n	2.7293_n	3.6067	4.1945_n
	$\psi+4\Gamma+4\theta_0+3\Delta_0$			0.678_n	2.4992_n	3.3946	
η'	$\psi-5\Gamma+\Delta_0$			9.848	2.0766_n	2.712	2.9697_n
	$\psi-4\Gamma+\Delta_0$			0.0792	2.1609_n	2.6968	2.7976_n
	$\psi-3\Gamma+\Delta_0$			0.3941	2.157_n	2.491	1.51
	$\psi-2\Gamma+\Delta_0$		9.013_n	0.248	2.0455	2.7898_n	3.2380
	$\psi-\Gamma+\Delta_0$			9.901	2.584	3.2539_n	3.6434
	$\psi+\Delta_0$		9.885_n	0.8518			
	$\psi+\Gamma+\Delta_0$			0.1664	1.836	2.448	3.3029_n
	$\psi+2\Gamma+\Delta_0$		9.009	9.76_n	2.1633	2.6170_n	2.2433
	$\psi+3\Gamma+\Delta_0$			9.38_n	2.1064	2.7194_n	2.9212
	$\psi+4\Gamma+\Delta_0$				1.9892	2.6870_n	
η_0^2	$\psi-5\Gamma+6\theta_0+6\Delta_0$				2.3144	2.9730_n	
	$\psi-4\Gamma+6\theta_0+6\Delta_0$				2.9538	3.3785_n	
	$\psi-3\Gamma+6\theta_0+6\Delta_0$				3.3102	3.5843_n	
	$\psi-2\Gamma+6\theta_0+6\Delta_0$				[3.4970]	[3.8423_n]	
	$\psi-\Gamma+6\theta_0+6\Delta_0$				3.9455	3.7269_n	
	$\psi+6\theta_0+6\Delta_0$	9.95_n	1.1109_n	3.1673_n	[3.9296]	[4.3377_n]	
	$\psi+\Gamma+6\theta_0+6\Delta_0$				3.9144_n	5.0372	
	$\psi+2\Gamma+6\theta_0+6\Delta_0$				3.5594_n	4.5942	
	$\psi+3\Gamma+6\theta_0+6\Delta_0$				3.3121_n	4.3236	

TABLE LVII—Continued.

$S \sin \psi + C \cos \psi$

Logarithmic | Unit=1″.

	Cos	w^{-3}	w^{-2}	w^{-1}	w^0	w	w^2
η_0^2	$\psi-5\Gamma+2\theta_0+2\Delta_0$				2. 1657	2. 7221_n	
	$\psi-4\Gamma+2\theta_0+2\Delta_0$				2. 1255	2. 8004_n	
	$\psi-3\Gamma+2\theta_0+2\Delta_0$				2. 234	3. 1304_n	
	$\psi-2\Gamma+2\theta_0+2\Delta_0$				2. 576	3. 3804_n	
	$\psi-\Gamma+2\theta_0+2\Delta_0$				3. 1995	3. 8325_n	
	$\psi+2\theta_0+2\Delta_0$	0. 344_n	1. 017	2. 689_n	[3. 4822]	3. 9938_n	
	$\psi+\Gamma+2\theta_0+2\Delta_0$				2. 2480	3. 2839_n	
	$\psi+2\Gamma+2\theta_0+2\Delta_0$		9. 45		3. 1612	3. 8424_n	
η_0^2	$\psi-5\Gamma-2\theta_0-2\Delta_0$				2. 700_n	3. 5481	
	$\psi-4\Gamma-2\theta_0-2\Delta_0$				2. 817_n	3. 6251	
	$\psi-3\Gamma-2\theta_0-2\Delta_0$				2. 9247_n	3. 6905	
	$\psi-2\Gamma-2\theta_0-2\Delta_0$		9. 59_n		3. 0241_n	3. 7470	
	$\psi-\Gamma-2\theta_0-2\Delta_0$				3. 1364_n	3. 8346	
	$\psi-2\theta_0-2\Delta_0$	0. 117	0. 95_n	2. 297_n	[2. 7856_n]	3. 6614	
	$\psi+\Gamma-2\theta_0-2\Delta_0$				2. 8942_n	3. 5604	
	$\psi+2\Gamma-2\theta_0-2\Delta_0$				2. 297_n	3. 1129	
$\eta_0\eta'$	$\psi-5\Gamma+6\theta_0+5\Delta_0$				2. 4885_n	3. 1691	
	$\psi-4\Gamma+6\theta_0+5\Delta_0$				2. 976_n	3. 5560	
	$\psi-3\Gamma+6\theta_0+5\Delta_0$				3. 6541_n	3. 8829	
	$\psi-2\Gamma+6\theta_0+5\Delta_0$				[3. 9514_n]	[4. 1632]	
	$\psi-\Gamma+6\theta_0+5\Delta_0$				4. 3903_n	4. 0037_n	
	$\psi+6\theta_0+5\Delta_0$	0. 295	1. 366	3. 6364	[4. 3301_n]	[4. 6662]	
	$\psi+\Gamma+6\theta_0+5\Delta_0$				4. 4005	5. 4966_n	
	$\psi+2\Gamma+6\theta_0+5\Delta_0$				4. 0582	5. 0612_n	
	$\psi+3\Gamma+6\theta_0+5\Delta_0$				3. 8204	4. 8027_n	
$\eta_0\eta'$	$\psi-5\Gamma+2\theta_0+\Delta_0$				2. 426_n	3. 0684	
	$\psi-4\Gamma+2\theta_0+\Delta_0$				2. 399_n	3. 0310	
	$\psi-3\Gamma+2\theta_0+\Delta_0$				2. 410_n	3. 1305	
	$\psi-2\Gamma+2\theta_0+\Delta_0$				2. 701_n	3. 4602	
	$\psi-\Gamma+2\theta_0+\Delta_0$				3. 2842_n	3. 8558	
	$\psi+2\theta_0+\Delta_0$	0. 444	1. 188_n	3. 0569	[3. 7266_n]	4. 1122	
	$\psi+\Gamma+2\theta_0+\Delta_0$				2. 8541	3. 5823_n	
	$\psi+2\Gamma+2\theta_0+\Delta_0$				3. 2191_n	3. 7635	
$\eta_0\eta'$	$\psi-5\Gamma-2\theta_0-\Delta_0$				3. 1551	3. 9530_n	
	$\psi-4\Gamma-2\theta_0-\Delta_0$				3. 2454	3. 9948_n	
	$\psi-3\Gamma-2\theta_0-\Delta_0$				3. 3100	4. 0023_n	
	$\psi-2\Gamma-2\theta_0-\Delta_0$		9. 93		3. 3277	3. 9401_n	
	$\psi-\Gamma-2\theta_0-\Delta_0$				3. 1976	3. 4598_n	
	$\psi-2\theta_0-\Delta_0$	0. 490_n	1. 324	3. 0145_n	3. 7326	4. 2787	
	$\psi+\Gamma-2\theta_0-\Delta_0$				3. 3632	3. 9402_n	
	$\psi+\Gamma-2\theta_0-\Delta_0$				2. 7792	3. 5224_n	
$\eta_0\eta'$	$\psi-5\Gamma+2\theta_0+3\Delta_0$				2. 2738_n	2. 847	
	$\psi-4\Gamma+2\theta_0+3\Delta_0$				2. 116_n	3. 0290	
	$\psi-3\Gamma+2\theta_0+3\Delta_0$				2. 5858_n	3. 3787	
	$\psi-2\Gamma+2\theta_0+3\Delta_0$				2. 809_n	3. 5429	
	$\psi-\Gamma+2\theta_0+3\Delta_0$				2. 650_n	3. 7297	
	$\psi+2\theta_0+3\Delta_0$	9. 98	0. 60_n	2. 873_n	[2. 685]	3. 7980	
	$\psi+\Gamma+2\theta_0+3\Delta_0$				3. 5126_n	4. 2856	
	$\psi+2\Gamma+2\theta_0+3\Delta_0$		9. 46_n		3. 3438_n	4. 1208	
η'^2	$\psi-5\Gamma+6\theta_0+4\Delta_0$				1. 9950	2. 7422_n	
	$\psi-4\Gamma+6\theta_0+4\Delta_0$				2. 6112	3. 1949_n	
	$\psi-3\Gamma+6\theta_0+4\Delta_0$				3. 0556	3. 5583_n	
	$\psi-2\Gamma+6\theta_0+4\Delta_0$				3. 7934	3. 7947_n	
	$\psi-\Gamma+6\theta_0+4\Delta_0$				4. 2260	4. 4064	
	$\psi+6\theta_0+4\Delta_0$	9. 98_n	0. 76_n	3. 5017_n	4. 1098	4. 3552_n	
	$\psi+\Gamma+6\theta_0+4\Delta_0$				4. 2852_n	5. 3521	
	$\psi+2\Gamma+6\theta_0+4\Delta_0$				3. 9567_n	4. 9249	
η'^2	$\psi-5\Gamma+2\theta_0+2\Delta_0$				2. 5018	3. 0963_n	
	$\psi-4\Gamma+2\theta_0+2\Delta_0$				2. 453	3. 0935_n	
	$\psi-3\Gamma+2\theta_0+2\Delta_0$				2. 4799	3. 2779_n	
	$\psi-2\Gamma+2\theta_0+2\Delta_0$				2. 9375	3. 6294_n	
	$\psi-\Gamma+2\theta_0+2\Delta_0$				3. 2833	3. 8982_n	
	$\psi+2\theta_0+2\Delta_0$	0. 025_n	0. 60	2. 634	3. 2781	4. 0439_n	
	$\psi+\Gamma+2\theta_0+2\Delta_0$				3. 5607	4. 2381_n	
	$\psi+2\Gamma+2\theta_0+2\Delta_0$				3. 4629	4. 1704_n	

TABLE LVII—Continued.

Logarithmic $S \sin \psi + C \cos \psi$ Unit=1″.

	Cos	w^{-3}	w^{-2}	w^{-1}	w^0	w	w^2
η'^2	$\psi-5\Gamma-2\theta_0$				3.0090$_n$	3.7477	
	$\psi-4\Gamma-2\theta_0$				3.0676$_n$	3.7445	
	$\psi-3\Gamma-2\theta_0$				3.0764$_n$	3.6664	
	$\psi-2\Gamma-2\theta_0$				2.958$_n$	3.3121	
	$\psi-\Gamma-2\theta_0$				3.1140	4.0201$_n$	
	$\psi-2\theta_0$	0.305	1.127$_n$	2.912	3.5491$_n$	3.9085	
	$\psi+\Gamma-2\theta_0$				3.0396$_n$	3.6320	
	$\psi+2\Gamma-2\theta_0$				2.4706$_n$	3.2330	
j^2	$\psi-5\Gamma+6\theta_0+5J_0-\Sigma_0$				2.006	2.7505$_n$	
	$\psi-4\Gamma+6\theta_0+5J_0-\Sigma_0$				2.335	2.981$_n$	
	$\psi-3\Gamma+6\theta_0+5J_0-\Sigma_0$				2.544	3.1436$_n$	
	$\psi-2\Gamma+6\theta_0+5J_0-\Sigma_0$				2.718	3.2445$_n$	
	$\psi-\Gamma+6\theta_0+5J_0-\Sigma_0$				2.970	2.9116$_n$	
	$\psi+6\theta_0+5J_0-\Sigma_0$	8.6$_n$	9.7	2.114$_n$	2.923	3.4067$_n$	
	$\psi+\Gamma+6\theta_0+5J_0-\Sigma_0$				2.7918$_n$	3.9420	
	$\psi+2\Gamma+6\theta_0+5J_0-\Sigma_0$				2.3824$_n$	3.4488	
j^2	$\psi-5\Gamma+2\theta_0+2J_0$				9.6	2.387	
	$\psi-4\Gamma+2\theta_0+2J_0$				1.916$_n$	2.911	
	$\psi-3\Gamma+2\theta_0+2J_0$				2.5178$_n$	3.3047	
	$\psi-2\Gamma+2\theta_0+2J_0$				2.938$_n$	3.6294	
	$\psi-\Gamma+2\theta_0+2J_0$				3.3406$_n$	3.9330	
	$\psi+2\theta_0+2J_0$		0.5910	3.1266	3.8021$_n$	4.1894	
	$\psi+\Gamma+2\theta_0+2J_0$				3.4070	4.3178$_n$	
	$\psi+2\Gamma+2\theta_0+2J_0$				3.0472	3.9308$_n$	
j^2	$\psi-5\Gamma-2\theta_0-J_0+\Sigma_0$				0.732$_n$	1.085	
	$\psi-4\Gamma-2\theta_0-J_0+\Sigma_0$				0.35	1.895$_n$	
	$\psi-3\Gamma-2\theta_0-J_0+\Sigma_0$				1.463	2.5146$_n$	
	$\psi-2\Gamma-2\theta_0-J_0+\Sigma_0$				2.064	3.0255$_n$	
	$\psi-\Gamma-2\theta_0-J_0+\Sigma_0$				2.6816	3.6280$_n$	
	$\psi-2\theta_0-J_0+\Sigma_0$	9.04	0.11$_n$	2.636	3.3284$_n$	3.7399	
	$\psi+\Gamma-2\theta_0-J_0+\Sigma_0$				3.0572$_n$	3.6430	
	$\psi+2\Gamma-2\theta_0-J_0+\Sigma_0$				2.9121$_n$	3.5491	
η_0^3	$\psi+4\theta_0+4J_0$	0.775	1.65$_n$	3.1052$_n$	3.0342$_n$		
	$\psi-4\theta_0-4J_0$	0.29$_n$	1.10	3.1888	3.6104$_n$		
	$\psi+8\theta_0+8J_0$	0.65	1.54$_n$	3.7520	4.5812$_n$		
$\eta_0^2\eta'$	$\psi+4\theta_0+5J_0$			3.7577	4.3244$_n$		
	$\psi+4\theta_0+3J_0$	1.260$_n$	2.081	3.1240	4.1388		
	$\psi-4\theta_0-3J_0$	1.005	1.77$_n$	3.5356$_n$	3.3560		
	$\psi+8\theta_0+7J_0$	1.228$_n$	2.093	4.3980$_n$	5.1827		
$\eta_0\eta'^2$	$\psi+4\theta_0+4J_0$			4.1155$_n$	4.5547		
	$\psi+4\theta_0+2J_0$	1.106	1.88$_n$	2.831	4.1803$_n$		
	$\psi-4\theta_0-2J_0$	1.146$_n$	1.88	3.0422	4.0180		
	$\psi+8\theta_0+6J_0$	1.321	2.152$_n$	4.5658	5.3010$_n$		
η'^3	$\psi+4\theta_0+3J_0$			3.8375	4.0446$_n$		
	$\psi-4\theta_0-J_0$			3.2197	3.9650$_n$		
	$\psi+8\theta_0+5J_0$			4.2553$_n$	4.9349		
$j^2\eta_0$	$\psi+4\theta_0+3J_0-\Sigma_0$			3.0024	3.8634$_n$		
	$\psi-4\theta_0-3J_0+\Sigma_0$	9.98$_n$	0.8	2.956$_n$	3.8331		
	$\psi+8\theta_0+7J_0-\Sigma_0$			3.0757	3.9759$_n$		
	$\psi+4\theta_0+4J_0$	0.46$_n$	1.32	3.8514$_n$	4.6436		
$j^2\eta'$	$\psi+4\theta_0+4J_0-\Sigma_0$			2.442	1.846$_n$		
	$\psi-4\theta_0-2J_0+\Sigma_0$			3.2486	4.0585$_n$		
	$\psi+8\theta_0+6J_0-\Sigma_0$			3.2818$_n$	4.1441		
	$\psi+4\theta_0+3J_0$	0.27	1.15$_n$	3.9421	4.6972$_n$		

$S \sin \psi + C \cos \psi = \Sigma C_1 w^s \eta^p \eta'^q j^{2t} \cos \text{Arg}$, where C_1 represents the coefficient.

COMPARISON OF TABLES.

Table LVI.—Unless there are errors of calculation, all the discrepancies are due to the accumulation of other discrepancies already discussed. Without going into the details of the construction, it is sufficient to remark that our table is built from practically all of the available auxiliary material. Our table includes many more terms than v. Zeipel's table, but it is wanting in the two arguments 6Γ and 8Γ in the first block of terms. These arguments contain 3ε and 4ε, respectively, and our series were not inclusive of these higher multiples. It would be more consistent to include them, since the argument 7Γ is included.

Table LVII.—Unless there are errors of calculation, all the discrepancies are due to the accumulation of discrepancies already discussed. Our table is built from practically all the available auxiliary material. Large disagreements are to be explained by v. Zeipel's use of the formula following Z 131, equation (244). In this equation the following functions are omitted:

$$-\tfrac{1}{2}(y_2'\cos\psi+z_2'\sin\psi)-\tfrac{1}{2}([y_2]\cos\psi+[z_2]\sin\psi).$$

ERRATA[1] IN H. v. ZEIPEL, ANGENÄHERTE JUPITERSTÖRUNGEN FÜR DIE HECUBA-GRUPPE.

With the exception of § 6, Störungen des Radius-vector, all the developments have been checked.

Page.	Line.[2]	For—	Read—
1	8a	$\frac{d\Omega}{dx}$	$\frac{\partial\Omega}{\partial x}$
1	9a	$\frac{d\Omega}{dy}$	$\frac{\partial\Omega}{\partial y}$
3ff[3]		ν'	$\frac{d\nu}{d\varepsilon}$
5	5b	(15)	(16)
9	2a	$\frac{\delta U}{\delta\psi}$	$\frac{\delta \overline{U}}{\delta\overline{\psi}}$
9	1b	$\omega+(1-\omega)\frac{\overline{W}+\nu^2}{1+\overline{W}}$	$w+(1-w)\frac{\overline{W}+\nu^2}{1+\overline{W}}$
12	2a	$\left(\frac{a}{\Delta}\right)^s$	$\left(\frac{a}{\Delta_0}\right)^s$
12	9a	β^{2i+1}_{n+i}	$\beta^{(2i+1)}_{n+i}$
12	10a	β^q_p	$\beta^{(q)}_p$
12	6b	$(2n+4i+1)\gamma_i^{1\cdot n}(4i+4)\gamma_{i+1}^{1,n}$	$(2n+4i+1)\gamma_i^{1\cdot n}+(4i+4)\gamma_{i+1}^{1\cdot n}$
13	5a	$(2n+4i+3)\gamma_i^{3\cdot n}(4i+4)\gamma_{i+1}^{3,n}$	$(2n+4i+1)\gamma_i^{3\cdot n}+(4i+4)\gamma_{i+1}^{3,n}$
14	2b	$n'g'$	ng'
15	8bff	Σ	$\sum_{n=0}^{n=\infty}$
16	10a	sin	cos
16	6b	$\frac{1}{\iota}a_2\frac{\partial\Omega}{\partial z}$	$\frac{1}{\iota}a^2\frac{\partial\Omega}{\partial z}$
19	6a	$\frac{dF}{da_0}$	$\frac{dF}{d\alpha_0}$
20	5b	$\gamma^{3\cdot n}_{i-1}$	$\overline{\gamma}^{3\cdot n}_{i-1}$
21	4a	$\overline{\vartheta}_i{}^{m,n}$	$\overline{\vartheta}_i{}^{3\cdot n}$
21	4b	Metoden	Methoden
24	9b	$\Sigma P\,{}^i{}_{p\cdot q}$	$-\Sigma P\,{}^i{}_{p\cdot q}$
27	21a	$P_{0\cdot 0}[n-1.-n+1]_{-\sigma}$	$P_{0\cdot 0}[n-1.-n+1]_{-\delta}$
34	21a	$P_{0\cdot 0}(n-1.-n+1)_{-\sigma}$	$P_{0\cdot 0}(n-1.-n+1)_{-\delta}$
42	5b	$P_{0\cdot 2}(n.-n)$	$F_{0\cdot 2}(n.-n)$
44	18b	$H_{1\cdot 1}(n+1.-n+1)$	$H_{1\cdot 1}(n+1.-n-1)$
45	20a	${}_{1\cdot 0}(n.-2-n-1)_{-\sigma}$	${}_{1\cdot 0}(n-2.-n-1)_{-\sigma}$
46	8a	H	G
46	10a	${}_{1\cdot 0}(n-1\cdot-n+2)_{-\delta}$	${}_{0\cdot 1}(n-1\cdot-n+2)_{-\delta}$
46	11a	${}_{1\cdot 0}(n-1\cdot-n)_{+\delta}$	${}_{0\cdot 1}(n-1\cdot-n)_{-\delta}$

[1] Inclusive of those tabulated by v. Zeipel.

[2] The number of the line counting from the top of the page is indicated by a, counting from the bottom of the page by b.

[3] On page 3 and all following pages ν' is defined by $\nu'=\frac{d\nu}{d\epsilon}$. The error consists in the omission of a statement announcing a change of notation. See definition of ν' given on page 2.

Errata in H. v. Zeipel, Angenäherte Jupiterstörungen für die Hecuba-Gruppe—Continued.

Page.	Line.[1]	For—	Read—
49	7b	$\bar{r}\frac{\partial \Omega}{\partial \varepsilon}$	$\bar{r}\frac{\partial \Omega}{\partial \bar{r}}$
50	6b	$\frac{r^2}{a^2}$	$\frac{\bar{r}^2}{a^2}$
50	6b	$3+\eta^2$	$3+14\eta^2$
51	1b	$S_{0\cdot 1}(n.-n+1)$	$S_{0\cdot 1}(n.-n-1)$
53	11b	$\geqq$	$\geqq$
54	5a	Ξ_t	Ξ_i
56	4b	$H_{1\cdot 1}(n-1\cdot n-1)$	$H_{1\cdot 1}(n-1\cdot -n-1)$
61	11b	$2\theta+2\theta$	$2\theta+2\varDelta$
62	17a	$\psi+6\theta+4\theta$	$\psi+6\theta+4\varDelta$
62	5b	$+436$	$+439$
63	9a	$[(1-e\cos\varepsilon)W'_2]$	$[(1-e\cos\varepsilon)\overline{W}_2']$
65	3a	(106)	(106a)
65	5a	(106)	(106a)
68	3a	$w^{-4}\ w^{-3}\ w^{-2}\ w^{-1}\ w$	$w^{-4}\ w^{-3}\ w^{-2}\ w^{-1}\ w\ w^2$
69	6b	$\sin A$	$\eta^p\eta'^q j^{2t}\sin A$
69	5b	$\sin(A-\psi+\varepsilon)$	$\eta^p\eta'^q j^{2t}\sin(A-\psi+\varepsilon)$
69	4b	$\sin(A+\psi-\varepsilon)$	$\eta^p\eta'^q j^{2t}\sin(A+\psi-\varepsilon)$
70	1b	$\overline{W}_4'''$	$\overline{W}_4''$
70	1b	$\cos A$	$\eta^p\eta'^q j^{2t}\cos A$
71	7a	$\cos A$	$\eta^p\eta'^q j^{2t}\cos A$
75	15a	$\underline{A}_{0\cdot 3}$	$\underline{A}_{0\cdot 2}$
75	18a	$\underline{A}_{0\cdot 1}$	$\underline{A}_{0\cdot 0}$
75	2b	$\underline{A}_{1\cdot 0}$	$\underline{A}_{0\cdot 1}$
75	1b	$A_{1\cdot 0}$	$A_{0\cdot 1}$
79	10b	θ	θ_i
81	8b	$1-e\cos\varepsilon)$	$(1-e\cos\varepsilon)$
83	12a	$+3744$	$+3344$
86	4a	(128_2) und (130)	(128_2), (128_3) und (130)
86	6a	$\frac{\partial \overline{W}_1}{\partial \vartheta}$	$\frac{\partial \overline{W}_1}{\partial \vartheta}$
91	9a	$\varepsilon\cos\varepsilon$	$e\cos\varepsilon$
91	11a	W'_2	$\overline{W}_2'$
92	3a	$\eta_1\cos\varepsilon$	$y_1\cos\varepsilon$
92	10b	$-\frac{3}{4}(1-e\cos\varepsilon)(\overline{W}-\frac{1}{3}\Xi)(\overline{W}+\frac{1}{9}\Xi)$	$-\frac{3}{4}[(1-e\cos\varepsilon)(\overline{W}-\frac{1}{3}\Xi)(\overline{W}+\frac{1}{9}\Xi)]$
92	4b	$\frac{\partial W}{\partial \vartheta}$	$\frac{\partial \overline{W}}{\partial \vartheta}$
93	10a	$\sin A$	$\eta^p\eta'^q j^{2t}\sin A$
93	10a	Σ'	Σ
94	19b	$\frac{\partial W}{\partial \vartheta}$	$\frac{\partial \overline{W}}{\partial \vartheta}$
97	15a	(156)	(154)
99	4b	$-\eta w\sin\varepsilon)$	$-\eta w\sin\varepsilon\}$
100	5a	A'	A
100	6a	$-\frac{1}{2}u_2$	$-\frac{1}{2}\bar{u}_2$
115	4b	(191)	(192)
116	7a	(195_1)	(195_{-1})
116	10b	$\frac{\partial}{d\varepsilon}(S_1-[S_1])$	$\frac{\partial}{\partial\varepsilon}(S_1-[S_1])$
119	(1)	$U_{q\cdot q}$	$U_{p\cdot q}$
122	3a	$\nu_0-\pi$	$v_0-\pi$
123	4b	$(1-\xi'^2-\xi'^2)$	$(1-\xi^2-\xi'^2)$
125	3a	$1-e_0\cos\varepsilon_0{}^2$	$1-e_0\cos\varepsilon_0$
128	7a	$\varDelta^0$	$\varDelta_0$
129	5a	μ	p
131	7b	$(\psi+A-\varepsilon)$	$(\psi-A-\varepsilon)$
131	6b	$(\psi+A+\varepsilon)$	$(\psi+A-\varepsilon)$
132	8a	2.9227_n	1.9227_n
132	26a	5.3376	5.0376
134	9a	$(2\vartheta+f_4)$	$(4\vartheta+f_4)$
135	10a	$\frac{w}{2}$	$\frac{w}{4}$
135	11a	$\frac{w}{2}$	$\frac{w}{4}$
140	26a	$[n\delta z]$	$[n\delta z]_1$
141	6a	0″8998	0′8998

[1] The number of the line counting from the top of the page is indicated by a, counting from the bottom of the page by b.

ERRATA IN KARL BOHLIN, SUR LE DÉVELOPPEMENT DES PERTURBATIONS PLANÉTAIRES, § 1–7, AND TABLES I–XX.

Page.	Line.[1]	For—	Read—
3	5a	$\mu^3=(1+m)\alpha^2$	$\mu^2=(1+m)\,a^3$
11	1b	$\frac{\partial \overline{W}}{\partial \zeta}$	$\frac{\partial W}{\partial \zeta}$
14	11a	$1+\nu^2$	$1-\nu^2$
20	3a	$+\frac{1}{2}e^2 \cos 2\varepsilon$	$-\frac{1}{2}e^2 \cos 2\varepsilon$
29	8b	$y'n$	$y'-n$
29	2b	$e^{\sqrt{-1}f'}$	$e^{-\sqrt{-1}f'}$
30	11a	$\frac{a}{r'}$	$\frac{a'}{r'}$
30	11a	$-\frac{e'}{\kappa'}$	$+\frac{e'}{\kappa'}$
30	12a	$2n+m-1$	$2n+m+1$
30	13a	$-\left(\frac{e'}{\kappa'}\right)^3$	$+\left(\frac{e'}{\kappa'}\right)^3$
30	13a	$2n+m-1$	$2n+m+1$
30	13a	$2n+m-2$	$2n+m+2$
30	14a	$2n+m-1$	$2n+m+1$
30	5b	$e^{\sqrt{-1}n(\pi-\pi')}$	$e^{\sqrt{-1}n(\pi-\pi')}$
33	9a	$K_{1.0}^{-2i+s}(n-1.\ n)$	$K_{1.0}^{-2i+s}(n-1.-n)$
35	4a	(73)	(74)
36	1a	$2\Gamma_s{}^{s\cdot n}e^{\sqrt{-1}\,n(\pi-\pi)}$	$2\Gamma_i{}^{s\cdot n}e^{\sqrt{-1}n(\pi-\pi')}$
36	10a	$\overline{K}_{1\cdot 1}(n-1.+n+1)$	$\overline{K}_{1\cdot 1}(n-1.-n+1)$
38	13a	a	α
38	10b	$e^{-\sqrt{-1}\pi-\pi')}$	$e^{-\sqrt{-1}(\pi-\pi')}$
38	3b	$2(\eta')y'^{-1}$	$2(\eta')y^{-1}$
38	2b	$2\eta)y'^{-1}$	$2(\eta)y'^{-1}$
40	3a	$\overline{K}_{1\cdot 0}(n.-1-n)$	$\overline{K}_{1\cdot 0}(n-1.-n)$
41	13a	α	a
45	2a	$\overline{K}_{0\cdot 0}(0.-n)$	$\overline{K}_{0\cdot 0}(n.-n)$
45	9b	à	
46	7a	$-\frac{r(r-1)^2x'^{-'}}{1.1.2}$	$+\frac{r(r-1)^2x'^{-1}}{1.1.2}$
46	7b	$-\frac{(n-s)(n-s+2)}{2}\eta'^2x'^{-2}$	$+\frac{(n-s)(n-s+2)}{2}\eta'^2x'^{-2}$
46	5b	$(n-3)$	$(n-s)$
46	4b	η^4	η'^4
48	14a	$P^1{}_{210}(n-2.-n)$	$P^1{}_{2\cdot 0}(n-2.-n)$
48	7b	$P^1{}_{1\cdot 0}\vert n+1.-n-2\vert$	$P^1{}_{1\cdot 2}\vert n+1.-n-2\vert$
48	7b	$P^1{}_{1\cdot 1}(n+1.-n-2)$	$P^1{}_{1\cdot 2}(n+1.-n-2)$
48	6b	$P^1{}_{1\cdot 1}\vert n-1.-n-2\vert$	$P^1{}_{1\cdot 2}\vert n-1.-n-2\vert$
50	5a	$R^1{}_{1\cdot 0}(n+1.-n-1)_{-\pi}$	$R^1{}_{1\cdot 0}(n+1.-n-1)_{-\pi'}$
50	9b	$R^1{}_{1,0}(n-1.-n+1)_{+\pi'}$	$R^1{}_{1\cdot 0}(n-1.-n+1)_{+\pi'}$
50	3b	$R^1{}_{0\cdot 0}(n.-n+2)_{+\pi'}$	$R^1{}_{0\cdot 1}(n.-n+2)_{+\pi'}$
51	1b	$P^1(n+r.-n+s\,\vert$	$P^1\vert n+r.-n+s\vert$
59	5a	$\Omega^{3\cdot 1}{}_{1\cdot 0}[n+1.-n]$	$\Omega^{3\cdot 1}{}_{1\cdot 0}[n-1.-n]$
59	8a	$\Omega^{3,1}{}_{0\cdot 1}[n.-n+1].$	$\Omega^{3\cdot 1}{}_{0\cdot 1}[n.-n+1]$
60	6a	g'	g'
60	8a	(See footnote [2])	
60	9b	$\frac{-(m\mu)^4}{24}$	$\frac{+(m\mu)^4}{24}$
60	5b	$y^{(-n+s)}$	$y^{(-n+s)\mu}$
61	7b	$P_{0\cdot 1}[n.+n+1]$	$P_{0\cdot 1}[n.-n+1]$
61	5b	$P_{0\cdot 1}[n.+n-1]$	$P_{0\cdot 1}[n.-n-1]$
62	1a	$\frac{n^2\mu^3}{6}$	$\frac{n^3\mu^3}{6}$
62	7a	$P_{2\cdot 1}\ (n+2.-n-1)$	$P_{2\cdot 1}(n+2.-n+1)$
62	7a	$\frac{(n-2)^2\mu^2}{2}$	$\frac{(n-1)^2\mu^2}{2}$
62	8a	$P_{1\cdot 1}[n-1.-n+1]$	$P_{1\cdot 1}[n+1.-n+1]$
63	5a	$P_{0\cdot 0}(n-1.-n+1)_{+\vartheta}$	$P_{0\cdot 0}(n-1.-n+1)_{-\vartheta}$
63	11a	$P_{1\cdot 0}(n+2.-n-1)_{+\sigma}$	$P_{1\cdot 0}(n+2.-n-1)_{+\vartheta}$
63	13a	$P_{0\cdot 0}[n-1.-n-1]_{-\vartheta}$	$P_{0\cdot 0}[n-1.-n+1]_{-\vartheta}$
63	14a	$P_{0\cdot 0}[n-1.-n-1]_{-\vartheta}$	$P_{0\cdot 0}[n-1.-n+1]_{-\vartheta}$
63	5b	$R_{0\cdot 0}[n.-n+1]_{-\pi'}$	$R_{0\cdot 0}[n.-n-1]_{-\pi'}$
63	2b	$R_{0\cdot 0}[n.-u-1]_{-\pi'}$	$R_{0\cdot 0}[n.-n-1]_{-\pi'}$
63	1b	$R_{0\cdot 0}[n.\_n-1]_{-\pi'}$	$R_{0\cdot 0}[n.-n-1]_{-\pi'}$

[1] The number of the line counting from the top of the page is indicated by a, counting from the bottom of the page by b.
[2] The argument ϑ is defined first by eq. (31), p. 20, secondly by eq. (105), p. 60. The first of these definitions is used in § 8.

Errata in Karl Bohlin, Sur le Développement des Perturbations Planétaires, § 1–7, and Tables I–XX—Continued.

Page.	Line.[1]	For—	Read—
64	6a	$R_{2\cdot0}[n+1.-n]_{+\pi'}$	$R_{2\cdot0}[n.-n+1]_{+\pi'}$
64	12a	$R_{0\cdot0}[n.-n-1]_{-\pi'}$	$R_{0\cdot0}[n.-n-1]_{-\pi'}$
64	14a	$(n-n)$	$(n-2)$
64	15a	$R_{1\cdot1}[n+1.-n]_{-\pi'}$	$R_{1\cdot1}[n+1.-n]_{+\pi'}$
66	7a	$F\ (n+r.-n+s$	$F\ (n+r.-n+s)$
66	8a	$G\ (n+r.-n+s$	$G\ (n+r.-n+s)$
70	1a	$+3$ $-3\ P_{1\cdot2}(n+1.-n-2)$ -2	$+3$ $-2\ P_{1\cdot2}(n+1.-n-2)$ -2
71	4a	$F_{1\cdot0}(n.\ n+1)_{+\sigma}$	$F_{1\cdot0}(n.-n+1)_{+\sigma}$
71	9b	$+3$ $-2\ P_{1.0}\,(n.-n+1)_{-\delta}$ -2	$+3$ $-2\ P_{1\cdot0}(n.-n+1)_{-\delta}$ -2
73	2a	$F_{2\cdot0}(n.-n+1)_{-\pi'}$	$F_{2\cdot0}(n.-n-1)_{-\pi'}$
73	18a, ff.	See foot note.[2]	j^2
73	4b	$R_{0\cdot0}(n.-n+1)_{+\pi}$	$R_{0\cdot0}(n.-n+1)_{+\pi'}$
73	3b	$R_{0\cdot0}(n.-n+1)_{+\pi}$	$R_{0\cdot0}(n.-n+1)_{+\pi'}$
74	8a	$R_{0\cdot0}n.-n+1)_{+\pi'}$	$R_{0\cdot0}(n.-n+1)_{+\pi'}$
75	1a	$G_{0\cdot0}(n+1.-n)$	$G_{0\cdot0}(n+1.-n)_{+\sigma+\pi'}$
75	11b	$R_{0\cdot0}(n.-n+1)_{-\pi'}$	$R_{0\cdot0}(n.-n-1)_{-\pi'}$
78	1b	$\gamma_0^{1.n}$	$\gamma_1^{1.n}$
79	1b	$\bar\gamma_i^{m+2.n}$	$\gamma_{i-1}^{m+2.n}$
79	*)	$n=1$	$n=0$
80	9b	$\bar\gamma_{i+1}^{m.n}$	$\bar{\bar\gamma}_{i+1}^{m.n}$
81	12a	*)	
81	13a	(120)	(120)*)
135	7a	$K_{0\cdot3}^{1.i}$	$\bar K_{0.3}^{1.i}$
139	3a	$\Gamma_i^{1.n}$	$2\Gamma_i^{1.n}$
140	2a	$\Gamma_i^{1.n}$	$2\Gamma_i^{1.n}$
154	1a	(86)	(93)
161	1a	$-ar$	ar
169	8b	〃	$\frac{3}{4\alpha^2}2\Gamma_i^{5.n}$
170	3a	$\frac{1}{2\alpha}2\Gamma_i^{3.n}$	$2\gamma_i^{1.n}$
170	4b	〃	$\frac{1}{2\alpha}2\gamma_i^{3.n}$
171 ff.		See foot note.[3]	
185	2a	3. 27886	3. 27887
185	13b	4	3
188	6b	$2.\ 017\ 3_n$	$2.\ 01703_n$
189	14a	3. 27886	3. 27887
197	16a	$0.\ 146128_n$	$1.\ 146128_n$
197	18b	1. 505151	1. 505150
198	15b	$1.\ 662759_n$	$1.\ 662758_n$
198	2b	0. 477121	$0.\ 477121_n$

[1] The number of the line counting from the top of the page is indicated by a, counting from the bottom of the page by b.
[2] The space between lines 18 and 19 should read j^2.
[3] Tables XII, XIII, XIV give the same coefficients in numbers as Tables XVI, XVII, XVIII give in logarithms, respectively. The same factor should therefore occur in the former.

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SECOND REPORT ON RESEARCHES ON THE CHEMICAL AND MINERALOGICAL COMPOSITION OF METEORITES.

BY

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HEAD CURATOR OF GEOLOGY, UNITED STATES NATIONAL MUSEUM.

CONTENTS.

EXPLANATION OF PLATES, MICROSTRUCTURES OF METEORITES.

SECOND REPORT ON RESEARCHES ON THE CHEMICAL AND MINERALOGICAL COMPOSITION OF METEORITES.[1]

By GEORGE P. MERRILL,
Head Curator of Geology, United States National Museum.

The paper here presented contains the detailed results of studies made during the past year under a grant from the J. Lawrence Smith fund. The immediate purpose of the investigation, as noted in my first report, these Memoirs, volume 14, 1916, pages 7–29, was the determination of the presence or absence of sundry reported elements existing in minor quantities, but naturally it was found advisable to extend these boundaries from time to time, as interesting or important features developed in progress of the work. In several instances results deemed of special importance have already received publication elsewhere.[2]

For convenience of reference, the meteorites studied are, in the following pages, considered alphabetically.

Bath, Brown County, S. Dak.—The fall of this stone and the attendant phenomena were briefly described by Foote.[3] Later, Brezina,[4] with even greater brevity, described its lithological features. There is nothing to indicate that he examined the stone in thin sections, and as it has never been subjected to chemical analysis it seemed a fit subject for further investigation.

Macroscopically the stone is gray, but, owing to oxidation, so filled with rust spots as to give it a brownish cast. The crust is rough and dull, a characteristic of stones of this class. The texture is firm, but the chondrules, for a large part at least, break free from it when the stone is fractured. The most unusual feature, when examined with a pocket lens, is the abundance of glittering crystalline facets of nickel-iron. The slipping faces mentioned by Brezina are not evident to the unaided eye in the pieces in the museum collection, but in the thin section are numerous fine black fracture lines, along some of which a differential movement has plainly taken place.

In thin section the stone is seen to be a spherulitic chondrite with crystalline base. (Fig. 1, Pl. I.) The chondrules are extremely variable in detail, but present no unusual features. The essential minerals are olivine and enstatite; more rarely polysynthetically twinned monoclinic forms appear. Fragmental forms are common, particularly among the radiating and cryptocrystalline enstatite types. In one of the latter was observed a single granule of a distinctly red, translucent, but not transparent, mineral, of somewhat rounded outline as though corroded, and completely isotropic. (Fig. 1.) It is believed to be a spinel; possibly osbornite; it is impossible to decide from the single occurrence of so small an object. (See further under Homestead.) The phosphatic mineral I have of late had so frequent occasion to note occurs but rarely.

FIG. 1.

Bjelokrynitschie, Volhynia, Russia.—This stone, which fell on January 1, 1887, has apparently as yet received but brief notice and been subjected to no chemical analyses. Four references to descriptions are cited by Wülfing, two of which are by B. K. Agafonov, the others being even briefer notes in the catalogues of Brezina and Meunier. I have had access to but

[1] Presented April, 1918; read November, 1918.

[2] On the Calcium Phosphate of Meteorites, Amer. Journ. Science, vol. 43, 1917, pp. 322–324, and Tests for Fluorine and Tin in Meteorites, with notes on Maskelynite and the Effect of Dry Heat on Meteoric Stones, Proc. Nat. Acad. Sci., vol. 4, no. 6, 1918, p. 176.

[3] Amer. Journ. Sci., vol. 45, 1893, p. 64.

[4] Wiener Sammlung, 1895, p. 259.

one of the papers cited by Agafonov.[1] In this the stone is described as composed chiefly of chondrules, entire and fragmental, embedded in a ground of crystals and crystal fragments. The mineral composition is given as olivine, bronzite, augite, maskelynite, nickel-iron, and troilite. Brezina describes it[2] as having suffered from oxidation to a depth of 1–3 cm. below the original surface, as of a brecciated structure and with strongly developed slicken-sided surfaces (Harnischflächen). He classes it a brecciated chondrite (Cib), though with occasional black chondrules showing a gradation into the brecciated spherulitic chondrites (Ccb). One fragment badly oxidized he seems inclined to class as a crystalline chondrite (Ck). Meunier in his list[3] states that while the characteristics are not absolutely identical with those of Tadjera, the composition is the same and the differences not sufficiently marked to justify relegating to a distinct type. Neither says anything of the mineral composition other than is to be inferred from the classification.

The stone is represented in the national collection only by a small oxidized mass weighing 8 grams and a thin slice of the fresh, unaltered stone weighing 14 grams. A thin section cut from this last shows the stone to be of a pronounced chondritic type, the entire mass being composed of chondrules and fragments of chondrules closely compressed and with a minimum amount of fragmental interstitial matter. The mineral composition is nickel-iron, iron sulphide, olivine, an orthorhombic and a monoclinic pyroxene, the last named polysynthetically twinned. In two instances interstitial areas of the phosphate provisionally called "francolite" were noted and there are numerous areas of the black irresoluble matter which Meunier regards as fayalite and of secondary origin.

Judging from what has been written and my own observation, the stone is of a somewhat variable character. From the result of study of this one section I feel disposed to class it as a veined spherulitic chondrite (Cca). (See Fig. 2, Pl. I.)

Farmington, Washington County, Kans.—This stone belongs to the group of black chondrites of Brezina, of which but eight representatives are known. The stone was seen to fall and its history is beyond question. It has been described by several writers among whom only Kunz, Weinschenk,[4] and Brezina need here be mentioned. Weinschenk, to whom the microscopic descriptions are doubtless due, refers to the occurrence of "the mineral designated by Tschermak as 'monticellite-like' formed in the usual way. This contains rounded, colorless inclusions with bubbles probably of glass." Brezina[5] says "Auch monticellitartige Chondren kommen vor." I am unable in the five thin sections we have of this stone to find the monticellite-like mineral in chondrules. It occurs rather in irregular cavities, sometimes completely filling them and sometimes merely small, colorless crystalline plates lining their walls. (See Fig. 1, Pl. II.) Naturally there was at once suggested the possibility that these were a phosphate, a possibility made a certainty by treating one of the areas in an uncovered slide with a drop of acid ammonium molybdate, when the mineral was quite dissolved, giving rise to abundant crystals of the phosphomolybdate of ammonium. I have been unable to detect the "asymmetric feldspar," the presence of which was thought to be indicated by the chemical analysis. The structure is, however, very obscure, and it is yet possible that a mineral of this nature may exist and be unrecognizable. Meunier's conclusions relative to the secondary nature of the dark color in the black chondrites are well supported by a comparison of slides of this stone with those from a roasted fragment of Homestead.

Forest City, Winnebago County, Iowa.—The only mineralogical description of this stone that has thus far been given is that of Kunz.[6] This is incomplete and unsatisfactory, made evidently without recourse to thin sections and a microscope. He describes it as a "typical chondrite, apparently of the type of the Parnallite group of Meunier . . . A broken surface shows the interior color to be gray, spotted with brown, black, and white, containing small

[1] Rev. des Sciences Naturelle, St. Petersburg, no. 1, 1891, p. 41.
[2] Die Meteoriten Sammlung, 1895, p. 249.
[3] Revision des Pierres Meteorique, 1894, p. 413.
[4] Min. u. Pet. Mittheil. vol. 12, 1891, pp. 177–182, and Amer. Journ. Sci., vol. 43, 1892, pp. 65–67.
[5] Weiner Sammlung, 1895, p. 253.
[6] Amer. Journ. Sci., vol. 40, 1890, pp. 318–320.

specks of meteoric iron, from 1 to 2 millimeters across. Troilite is also present in small rounded masses of about the same size. On one broken surface was a very thin scum of black substance, evidently graphite, soft enough to mark white paper; a feldspar (anorthite) was likewise observed, and enstatite was also present." Further on, in discussing the analyses by L. G. Eakins he remarks that "it is of course probable that the Cr_2O_3 represents chromite, and possible that the alkalies and alumina with a little lime represent a soda-lime feldspar." Nothing is said as to the presence of olivine, though its presence is to be inferred from the 36.04 per cent soluble in hydrochloric acid.

Under the microscope I find the structure very obscure, confused, and, as is so often the case with meteorites of this class, baffling all efforts at satisfactory descriptions. Few of the constituent minerals are crystallographically well developed, though occasional small forms in the midst of the chondrules present recognizable crystal faces. The recognized constituents, aside from the nickel iron and iron sulphide, are olivine and two pyroxenes, one orthorhombic in crystallization and one monoclinic, the latter polysynthetically twinned. The calcium phosphate is common in the usual interstitial forms. A black carbonaceous matter in veins and coating slicken-sided surfaces is not uncommon. Nothing resembling a feldspar is to be seen in any of the sections examined. (Fig. 2, Pl. II.)

Gargantillo (*Tomatlan*), *Jalisco*, *Mexico*.—This stone was described by Shepard,[1] who seems to have secured 511 grams out of the total known weight of 780 grams. The mineral composition as given by him was as follows:

	Per cent.
Chrysolite	80.00
Chladnite (?)	10.00
Nickeliferous iron	7.00
Troilite, Chromite, Peroxide of iron	3.00
Total	100.00

Specific gravity, 3.47 to 3.48.

He noted as a "striking peculiarity . . . the prevalence everywhere of octahedral crystals of nickeliferous iron," which were "so distinct as to be recognizable with the naked eye, the brilliant equilateral, triangular faces coming into view by every change of position of the specimen." No chemical analysis appears to have been made, nor has it apparently been studied further except by Brezina, who classes it in his catalogue[2] as a "kugelchenchondrit" (Cc) and refers to it as having a very loose and friable ground mass, thick crust, large chondrules, many brown flecks, like Sarbanovas, and the iron abundant with many crystalline faces.

A little more may well be added to this description. The stone is so friable and the abundant chondrules so loosely embedded that it is practically impossible to get a satisfactory section without sacrificing a larger amount of material than is warranted. The microscope shows an indistinct and confused, fine, granular ground of olivine, enstatite, and occasional grains of a monoclinic pyroxene, in addition to the metallic constituents and the sulphide. (See Fig. 1, Pl. III.) The fine powder treated on the slide with a drop of ammonium molybdate yields characteristic globules and crystals of phosphomolybdate of ammonium. No feldspars, even of the maskelynite type, were detected.

A vial of fragments too small for other purposes, found in the Shepard collection, was sacrificed for the purposes of analysis, with the following results:

	Per cent.
Mineral	93.54
Metal	6.46

The metal amounted to 0.41 grains and consisted of:

	Per cent.
Nickel	10.12
Cobalt	1.02
Iron (by difference)	88.86

[1] Amer. Journ. Sci., vol. 30, 1885, pp. 105–108.

[2] Die Meteoritensammlung, etc., 1896, p. 256.

The mineral portion amounted to 5.94 grains and consisted of:

	Per cent.
Silica SiO_2	41.16
Alumina Al_2O_3	3.97
Ferrous oxide FeO	18.48
Manganous oxide MnO	0.39
Chromic oxide Cr_2O_3	0.20
Phosphoric acid P_2O_5	0.30
Sulphuric anhydride SO_3	5.56
Lime CaO	1.92
Magnesia MgO	26.88
Soda Na_2O	1.14
Potash K_2O	0.06
Total	100.06

A recalculation of these figures gives the following, representing the composition of the stone as a whole:

	Per cent.
SiO_2	38.50
Al_2O_3	3.71
Cr_2O_3	0.18
FeO	17.28
MnO	0.36
MgO	25.14
CaO	1.79
Na_2O	1.06
K_2O	0.05
P_2O_5	0.28
SO_3	5.20
Fe	5.74
Ni	0.66
Co	0.06
Total	100.01

These figures fall well within the range of chondritic stones. No barium strontium or other alkaline earths than those mentioned could be detected. No calcium in a water solution, hence no oldhamite. The mineral composition is olivine, monoclinic and orthorhombic pyroxene, calcium phosphate (merrillite of Wherry), chromite, nickel-iron, and troilite.

Hartford (Marion), Linn County, Iowa.—The first descriptions of this stone are by Shepard.[1] His determination of its lithological nature is excusable only in consideration of the times and the means at his command. He wrote: "It appears to contain but a single mineral species of this (i. e., 'earthy') description, and this one which . . . has until now escaped a separate recognition." For this mineral he proposed the name *howardite* and gave the complete mineral composition of the stone as howardite, 83 per cent; nickel-iron, 10.44 per cent; magnetic pyrite, 5 per cent; olivinoid and anorthite, traces. Some twenty and odd years later Rammelsberg[2] reviewed Shepard's work and showed the stone to consist of 10.54 per cent nickel-iron; 6.37 per cent troilite; 41.58 per cent soluble silicate, and 41.24 per cent insoluble, the soluble portion being identified as olivine; the insoluble, which was analyzed, being "almost exactly a bisilicate," but which he does not name.

An examination of thin sections from fragments in the Museum collection shows the essential constituents to be olivine and enstatite, with the usual interstitial calcium phosphate, nickel-iron, and troilite. The structure is not strongly chondritic. (Fig. 2, Pl. III.) No polysynthetically twinned pyroxenes were noted. The phosphatic mineral was evident to the naked eye in two instances as small white spots, perhaps 2 mm. in diameter, on a broken surface of the stone. These were so soft and friable as to fall down to almost dustlike particles when touched with a needle point. It is doubtless this brittle property of the mineral, causing it to break away in the process of grinding the section, that has prevented its earlier detection. It

[1] Amer. Journ. Sci., vol. 4, 1847, p. 288, and vol. 6, 1848, p. 403.

[2] Mon.-Ber. Berlin Akad., 1870, pp. 457–459.

should be stated that a particle tested by the immersion method showed an index of refraction of 1.625. Far more abundant than the phosphate is a limpid, colorless mineral, likewise occurring interstitially, but locally so abundant as to form almost the base in which the other silicates are embedded. The mode of occurrence and appearance are in every way characteristic of the so-called maskelynite, but that in many instances the area between crossed nicols breaks up into granular aggregates which are plainly biaxial and give distinct polarizations in light and dark, rarely yellowish colors in the thicker sections. The dark cloud, as a rule, sweeps over the face of the crystal in a manner indicating conditions of strain, and in no case have I been able to find a satisfactory section showing the emergence of an optic axis, or other indications of its optical properties than the indistinct black brushes sweeping across it as the stage is revolved. It is apparently positive. There are no signs of cleavage, but in a few instances faint lines were observed traversing the section. In these cases I was able to measure extinction angles against these lines, of 8° and 10°. But for its very evident doubly refracting properties the mineral would have been set down at once as maskelynite. As it was, additional tests seemed necessary. Two determinations of its refractive index by the immersion method gave 1.54 and 1.545, which is higher than that of a similar appearing mineral to which I have frequently referred in other publications. All further doubts as to the nature of the substance are, however, in this particular case set at rest by the finding of occasional granules still retaining residual traces of the characteristic twinning bands of a plagioclase feldspar.

Homestead, Iowa.—The Homestead meteoric stone fell on February 12, 1875, and is now represented by 124,492 grams scattered among 62 collections throughout the world. It has been the subject of numerous papers, concerning which a reference to Wülfing's bibliography is here sufficient. The stone is classed by Brezina as a brecciated gray chondrite (Ccb), and by Meunier as a limerickite. Wadsworth, who examined the stone in thin section, states it to consist of "crystals and grains of olivine, enstatite, pyrrhotite, iron, and base," and quotes Lasaulx as stating that it carries plagioclase. Several chemical analyses have been made, none of which show the presence of any unusual constituents. This is little to be wondered at when one considers that in the case of Gumbel but 1.5 grams of material were at his disposal. Much of the interest that is attached to the stone is due to A. W. Wright's work on the gaseous contents of meteorites.

My own attention was first drawn to this stone when studying the occurrence of the calcium phosphate concerning which I have of late written several papers, and which, incidentally, I find here in abundant characteristic forms. I do not find the plagioclase feldspar referred to by Lasaulx, but do find in some of the chondrules a polysynthetically twinned monoclinic pyroxene which seems to have been wholly overlooked by previous observers. The immediate cause of the present note is, however, the occurrence in each of two slides examined of a minute, bright red-brown, scarcely translucent, isotropic mineral embedded in enstatite, as shown in the drawing reproduced here. (Fig. 2.) An attempt at a definite determination of its mineral nature was a partial failure. Finding it insoluble in ordinary acids, one of the slides was sacrificed, painting around the object as closely as possible with vaseline and then covering the exposed portion with a large drop of fluorhydric acid. The silicates were all decomposed badly, but amidst the gelatinous mass of decomposition products I was still able to detect the red granule apparently untouched. In an attempt to remove the granule for further tests and observation, it became hopelessly lost. I can only surmise from its apparent insolubility, subtranslucency, color, isotropic nature, and a suggestion of octahedral form, that it may be a spinel. That it is osbornite does not seem probable. The second slide, from which the accompanying figure was drawn, has been covered and preserved. I may add that eight other small sections, cut from fragments of the stone in the Barker bequest, gave no new occurrences of the mineral. This is probably the same mineral noted by Gumbel[1] but thought to be garnet. The decidedly octahedral termination on the form figured in the present paper seems to warrant its being considered a spinel.

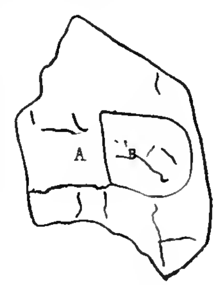

FIG. 2.

[1] Sitz. der Matt-phys. Classe, der K. bayrischen Akad. zu München Dec. 1875, p. 323.

McKinney, Collin County, Tex.—It is remarkable that this interesting stone, which has been known since 1895, should have been allowed so long to remain unstudied, the bibliography consisting only of a brief statement by von Hauer [1] regarding the acquisition by the Vienna museum of upward of 40 kilograms of the material, a description of the stone by Brezina,[2] based evidently only on an examination by the naked eye, aided perhaps by a pocket lens, a brief note by Meunier [3] calling attention to the evidence it afforded of the introduction of the metal and sulphide after consolidation, and lastly an analysis by Whitfield given in my paper on the minor constituents of meteorites[4] the last named made with particular reference to the possible occurrence of barium, strontium, zirconium, or other of the rare elements.

Macroscopically the stone is fine-grained, compact, dull brownish gray, almost black, looking on a broken surface very much like a piece of hard shale, showing here and there a minute fleck of metal, and with chondrules quite inconspicuous except where it is polished. The texture is firm and the chondrules break with the stone. On the polished surface they are of greenish color, suggestive of a serpentinous alteration, which, however, microscopic examination shows not to have taken place.

In the thin section the microscope reveals, in addition to the iron and iron sulphide, three varieties of pyroxene, one occurring in broad plates with wide (25°–30°) extinction angles, a polysynthetically twinned variety and normal enstatite, in addition to olivine and the calcium phosphate, while the whole mass is here and there so impregnated with a coal black compound as to give it the dark color referred to.

The chondrules are varied and interesting. They consist of enstatite in the common radiating and cryptocrystalline forms as well as in good, well developed phenocrysts in a glassy or fibrous base. Sometimes the entire chondrule is composed of small, closely compacted forms with little or no interstitial base. Others are formed wholly of the polysynthetically twinned monoclinic forms. These twinned pyroxenes occur also scattered throughout the groundmass and under such condition with relation to their associated minerals as to suggest a dynamic action, a crowding and crushing, and sometimes even raising the question if the twin structure may not itself be due to this same cause. The occurrence of the twinned forms in the chondrules where there are no signs of strain forbids, however, the universal application of any such theory of origin. Still other chondrules are wholly of olivine. The calcium phosphate occurs in the usual irregular, interstitial, colorless forms with low relief. The groundmass is everywhere so obscured by the black matter that it is impossible to make out a structure for a certainty. It is apparently fragmental, though if we accept Meunier's views, it may have been caused by the reheating to which he ascribes this black color. In this connection Brezina says "Dessen Zugehörigkeit zu den Cs insoferne nicht ganz sichergestellt ist, als die schwarze Farbe nicht mit Bestimmtheit auf einen Kohlegehalt Zurückgeführt ist". (See further under "Effects of dry heat on meteoric stones," Proc. Nat. Acad. Sci., vol. 4, 1918, p. 178.)

This black constituent, which is sufficiently abundant to give the stone a uniform color, is by no means uniformly distributed, but, as shown in the thin section and figures, is injected throughout the ground and along cleavage and fracture lines of the various minerals, being absent in quantity from the chondrules, forming a dense black, opaque ground from which these and the scattered, often fragmental silicates stand out sharply. An attempt was made to determine the possible presence of a hydrocarbon, but the facilities at command did not enable me to arrive at a satisfactory result. One hundred grams of the pulverized stone were digested for 48 hours, first in ether and next in carbon disulphide. Although care was taken to use the purest chemicals obtainable, and the filters were first washed in ether, the slight, colorless extract obtained in the first instance, and the single small drop of a greenish, oil-like matter in the second, were both felt to be perhaps in part due to impurities. Any hydrocarbon, if present at all, is certainly there in very small quantities. The apparent introduction (or perhaps better *production*) of the coloring matter at a late period in the history of the

[1] Ann. Hof-Mus., vol. 10, 1895, p. 34.
[2] Idem, pp. 252, 253.
[3] Revision des Pierres Meteorique, etc., p. 412.
[4] Mem. Nat. Acad. Sci., vol. 14, 1916, p. 19.

stone is beautifully shown in some of the pyroxene sections where the cleavage and fracture lines have become so filled as to form a black network between the threads of which the colorless pyroxenic material stands out sharply and in all its original freshness.

Naturally these observations recalled Meunier's views on the origin of the meteorites of his tadjerite group through a preterrestial heating of aumalites, and the matter seemed of sufficient interest to warrant a partial repetition of his experiments. The results, I have given on page 178 of the Proceedings of the Academy as noted above.

Ness County, Kans.—This is a holocrystalline chondritic stone, of firm texture, the chondrules breaking with the matrix. Thin sections show, where not too badly stained by iron oxides, a granular aggregate of olivine and bronzite with the usual scattering blebs and granules of nickel-iron and iron sulphide. The chondritic structure is very obscure and the chondrules themselves present little variation. (Fig. 2, Pl. IV.) The structure is in places decidedly cataclastic. Aside from the minerals mentioned, I find, rarely, clusters of minute, polysynthetically twinned pyroxenes and numerous limpid, completely colorless interstitial areas, without crystal outlines or determinable cleavage, polarizing only in light and dark colors, often showing conditions of strain, and giving occasionally biaxial interference figures. It is evidently of the nature of the so-called maskelynite. By careful work with a needle point on an uncovered section the edge of one of these areas was sufficiently exposed to permit testing by the immersion method, and found to have an index of refraction of between 1.55 and 1.56, or that of andesine as given by Iddings. In addition, two of the sections show a completely colorless mineral, one of which is isotropic and shows two lines of cleavage cutting at angles of about 56° and 124°, and the other showing extinctions parallel with a single series of cleavage lines and giving a uniaxial interference figure strongly suggestive of the mineral apatite [1] (see Figs. 3 and 4).

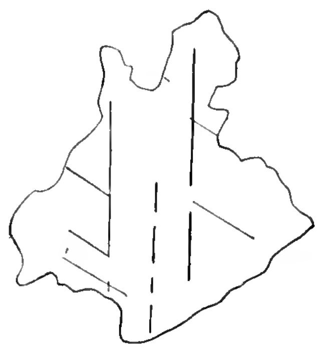

FIG. 3.

FIG. 4.

Although sought for most carefully, this mineral could not be found in any of the six other sections examined, and a more exact determination is impossible. It is perhaps the same mineral referred to by Farrington [2] and which he also failed to determine.

Ochansk, Siberia.—Through an oversight on my part, this stone in my Handbook and Catalogue [3] was stated not to have been analyzed as a whole. Since the issue of that publication, my attention has been called to the paper of Tichomirow and Petrow [4] in which is given the analysis quoted below.

My excuse for taking the matter up once more lies in the somewhat unusually high ratio of nickel to iron [5] (1–3.5) which, so far as I now recall, is equalled only by that of the Middlesborough stone. They also report 0.52 per cent of copper and tin. A quantity of fragments of not over a gram or so each in weight, the residues from the Ward collection, formed abundant opportunity for further investigation, which after sundry qualitative tests by myself, was undertaken in detail by Dr. Whitfield.

As is well known, the stone belongs to the brecciated spherulitic chondrules of Brezina or canellites of Meunier. The texture seems to be somewhat variable. In a sample received from De Kroutschoff in 1887, the texture is firm enough to receive a smooth surface and a rather low-grade polish. The samples in the Ward collection, on the other hand, which are fresh and unoxidized, are quite friable. Otherwise, however, both in structure and mineral com-

[1] A similar mineral described by me in the Mocs meteorite (see Fig. 5, p. 305, Proc. Nat. Acad. Sci., vol. 1, May, 1915) was found to be soluble in acid and to give solutions reacting for phosphorus and calcium.

[2] Meteorite Studies I, Field Columbian Museum Publ. 64, Geol. Ser., vol. 1, 1902, p. 300.

[3] Bull. 94, 1916, U. S. National Museum.

[4] Jour. de russ. phys-chem. Ges. 1888, Part 1, pp. 513–518.

[5] See Prior, on the Genetic Relationship and Classification of Meteorites, Mineralogical Magazine, vol. 18, 1916, no. 83, pp. 29 and 33.

position, the stones seem to be identical and there is apparently no reason for doubting the authenticity of the material now under consideration. A broken surface is light ash gray in color, thickly studded with chondrules, some of which are of a dark color and others very light greenish when broken across. All separate readily from the ground, often in very perfect spherulitic forms. No metal is evident to the unaided eye. In thin sections under the microscope the structure is that of a tufaceous ground carrying the abundant chondrules, entire and fragmental, and scattered crystalline particles with the usual sprinkling of metal and metallic sulphide. It will be recalled that Siemaschko [1] described this last as occurring in pentagododecahedral forms and, therefore, pyrite. The correctness of this has been questioned (see Cohen, p. 208). The recognizable silicates are olivine and enstatite, though as often the case many of the chondrules are densely crypto-crystalline and their mineralogical nature indeterminable other than that they are pyroxenic. The powdered stone treated with a drop of acid ammonium molybdate solution gives rise to abundant reaction for phosphorus, indicative of a lime phosphate which occurs only in minute interstitial granules quite inconspicuous unless specially sought under the microscope.

The results of Dr. Whitfield's work are given below. It should be stated that particular pains were taken, as usual of late, to determine the presence of the rarer elements particularly tin and copper which the previous investigators had reported, and also the presence or absence of nickel and cobalt in the silicate portions.

Several grams of the finely pulverized material boiled for half an hour in distilled water in a platinum vessel yielded no evidences of the presence of oldhamite.

The mineral composition, determined by the usual methods, was found to be—

	Per cent.
Silicate portion (including a small amount of phosphate)	76.274
Troilite	6.100
Metallic portion	16.860
Chromite (calculated)	0.766
Total	100.000

The metallic portion yielded—

	Per cent.
Iron	92.092
Nickel	7.158
Cobalt	0.686
Phosphorus	0.064
Total	100.000

The silicate portion yielded—

	Per cent.
SiO_2	44.438
Al_2O	9.226
Cr_2O_3	[2] 0.550
P_2O_5	0.503
FeO	13.675
MnO	0.376
CaO	1.505
MgO	27.204
NiO	0.678
CoO	0.066
Na_2O	1.186
K_2O	0.222
SO_3	0.371
Total	100.000

[1] Tschermak's Min. u. petro. Mitt, vol. 11, 1890, p. op.

[2] Equals 0.766 chromite.

By recalculation the composition as a whole is found to be—

	Per cent.	
SiO_2	34.235	Silicate portion.
Al_2O	7.107	
Cr_2O_3	0.423	
P_2O_5	0.387	
FeO	10.535	
MnO	0.289	
CaO	1.159	
MgO	20.958	
NiO	0.563	
CoO	0.058	
Na_2O	0.913	
K_2O	0.171	
SO_3	0.285	
Fe	15.526	Metallic portion.
Ni	1.196	
Co	0.115	
P	0.011	
Fe	3.880	Troilite.
S	2.220	
Total	100.031	

The analysis given by Tichomirow and Petrow is as follows:

	Per cent.
SiO_2	36.36
FeO	13.80
MgO	18.54
CaO	3.00
Fe	19.80
Ni	5.55
S	2.30
P	0.05
C	0.08
CuSn	0.52
Mn, Co, Na	Traces.
	100.00

The discordance in these results is altogether too large to be accounted for satisfactorily. That there must have been some error in the percentage of nickel, as suspected, is evident, as Dr. Whitfield's analysis of the metallic portion shows but 7.158 per cent of this constituent, which, when calculated in percentage of the entire stone, amounts to but 1.196 per cent instead of 5.55. The discrepancy in the calcium oxide (1.159 per cent against 3 per cent) is greater than should exist in portions from the same mass, but, singularly enough, the amount of troilite as indicated by the 2.30 per cent of sulphur is about the same and the remaining differences are perhaps not greater than might be expected with the exception of the alkalies, Whitfield reporting 1.084 per cent.

It is to be noted further that Whitfield reports no traces of tin or copper and that the silicate portion freed from all metal by boiling the finely pulverized mineral in mercuric chloride still yields 0.744 nickel and cobalt oxide. It may be recalled that in the table of analyses given in the Memoirs of the Academy [1] there are to be found several instances of this character.

To these, at the time, I made no reference in the text, feeling that in some instances at least they might be due to imperfect separation of the metal from the silicate portion. In analyses since made especial care has been taken to guard against any such possibility and there seems no reasonable doubt but that the silicates–olivines or pyroxenes, or both—in meteorites carry small quantities of these constituents, as is the case in terrestrial rocks. Such being the case, it follows that the statement made by Dr. Prior,[2] together with an explanation by Dr. Wahl [3] to the effect that "the ferromagnesium minerals of chondritic stones contain practically no oxide of nickel," is founded upon faulty analyses and insufficient data.

[1] Vol. 14, 1916, pp. 7-27. [2] Min. Mag. Nov., 1916, p. 39. [3] Min. u. Petr. Mittheil, vol. 26, 1907.

The Ruff's Mountain, South Carolina, meteoric iron, and its included phosphide.—This beautiful iron was first described by Shepard[1] and has since been the subject of numerous other notices, which need reference here only as they bear directly upon the matter in hand. An etched surface shows it to be a medium octahedrite, after Brezina, or a caillite if we follow Meunier. It is chiefly distinguished by the broad fields of plessite and the lack of notable quantities of troilite. The kamacite bands are somewhat swollen and there are occasional rather inconspicuous Reichenbach lamellae. My attention was first drawn to it by the unsatisfactory nature of Shepard's analysis and his supposed discovery of potassium as one of its constituents. A slice weighing a little over 150 grams was therefore submitted to Dr. J. E. Whitfield with the request that he utilize so much as was necessary for an exhaustive analysis. Bulk analysis yields:

	Per cent.
Iron	90. 654
Copper	0. 018
Nickel	8. 550
Phosphorus	0. 233
Cobalt	0. 500
Carbon	0. 025
Sulphur	0. 020
Silicon	None.
Total	100. 000

with no traces of platinum, palladium, iridium, ruthenium, or the allied elements. Shepard, it may be recalled, reported 96 per cent iron; 3.121 per cent nickel, with chromium, cobalt, magnesium, and sulphur in traces. Later Rammelsburg reported a mean of 8.62 per cent nickel, which is substantially the amount given by Whitfield above. There is nothing of especial note in this composition unless it be its freedom from the rare elements.

Ninety grams of the iron yielded 1.4843 grams of material insoluble in hydrochloric acid of one-half ordinary strength. This residue when examined under the microscope was found to consist largely of schreibersite particles, among which were a few of sufficiently perfect crystalline form to permit measurements and determination of crystalline system. The material possessed the well-known physical properties of schreibersite (see Cohen, Meteoritenkunde, pp. 118–131), including the characteristic habit of breaking up readily into cuboidal forms, and which need not be further discussed.

The particles showing well-developed crystal faces were submitted to Dr. Edgar T. Wherry, then assistant curator in charge of the Mineral Department, who reported as follows:[2]

The crystals average about one-half millimeter in diameter and are irregularly distorted, some of the faces being cavernous; the system of crystallization is not evident on superficial examination. The faces yield, however, fairly good reflections, the positions of which can be located in many cases within 5–10 minutes, unquestionable tetragonal symmetry being exhibited by the angular relations. The forms observed are: c (001) a (100), m (110), o (111), and x(362). In addition there are rounded or poorly developed faces of other pyramids and prisms. All of the forms are incomplete, but there is hardly sufficient regularity in the suppression of faces to justify the assignment of the crystals to any particular hemihedral class.

Below are given the angles observed, which compare closely with those measured on artificial crystals by Mallard, Hlawatsch, and Spencer.

TABLE 1.—*Measured and calculated angles of iron phosphide.*

Tetragonal, $c = 0.346 \pm 0.001$.

No.	Letter.	Symbol.	Crystals.	Measurements.	Angles measured.		Angles calculated.	
					ϕ	ρ	ϕ	ρ
0	c	001	1	1	....	0° 00′	....	0° 00′
1	a′	010	2	5	0° 00′—	90° 00′—	0° 00′	90° 00′
2	m	110	2	5	45° 00′±15′	90° 00′—	45° 00′	90° 00′
3	o	111	2	5	45° 00′±60′	26° 05′±15′	45° 00′	26° 05′
4	x	362	1	2	26° 00′±60′	49° 00′±60′	26° 34′	49° 15′

[1] Amer. Journ. Sci., vol. 10, 1850 p. 128.

[2] Amer. Mineralogist, vol. 2, 1917, pp. 80-81; vol. 3, 1198, p. 184.

Several attempts were made at a determination of the chemical composition of this material, but with results so discordant that the matter must be pended awaiting further investigation.

Tennasilm, Estland, Russia.—This stone, which fell on the 28th of June, 1872, was described by G. Baron Schilling some 10 years later.[1] The acquisition of a fragment weighing nearly a kilogram, through Krantz, in Bonn, led me to sacrifice enough for thin sections. An examination of these leads to conclusions relative to its mineral composition somewhat at variance with those of Schilling and is the cause of the present note. It should, however, be stated in advance that Schilling apparently made no use of thin sections, but based his mineralogical determinations wholly upon the results of chemical analysis.

The stone is of a pronounced chondritic type, a veined spherical chondrite (Cca) according to Brezina, or a limerickite if one follows Meunier. Schilling, as a result of analyses which need not be repeated here in their entirety, finds the silicate portion of the stone to consist of 54.45 per cent olivine; 32.27 per cent bronzite, and 13.23 per cent labradorite. Cohen [2] seems to have accepted these results without question and by a further calculation gives the chemical composition of the labradorite as though it had actually been isolated and analyzed, while as a matter of fact, as noted later, labradorite, or other feldspar, is wholly lacking, at least so far as the Museum material is concerned. Meunier [3] apparently accepts this mineralogical determination, placing the stone in his limerickite group, the mineral composition of which is enstatite associated with bronzite and a feldspathic mineral. My observations are based upon a study of four thin sections cut from different portions of the mass mentioned. As described, the stone is of a gray color, plainly chondritic, somewhat soft and friable, the chondrules falling away readily from the matrix when the stone is broken. The metallic constituents are scarcely evident to the unaided eye. Under the microscope the chondritic structure is very pronounced (see Fig. 1, Pl. V). The chondrules are in some cases of beautifully limpid, well developed orthorhombic pyroxenes in a somewhat fibrous base, sometimes of the radiating cryptocrystalline forms, sometimes of polysynthetically twinned monoclinic forms, or again, of olivine. In no case have I been able to find a feldspar, even in the maskelynite condition.

This occurrence offers an interesting illustration of the danger of calculating the mineral composition from chemical analyses, and also the weakness of the quantitative classification when applied to rocks of this type.

Travis County, Tex.—This stone needs a brief reference for the reason that Wülfing in his catalogue raises the question if it does not belong to the Bluff, Fayette County, fall.

Such a suggestion is wholly unwarranted, and it is safe to say would never have been made by one who had seen and compared the two stones. Indeed, if the question of identity were to be raised it might well be with that of McKinney, in Collin County, which it closely resembles. Like the McKinney stone it is black in color, very firm and compact, and presents on a freshly broken surface little to suggest its meteoric nature. It might well be mistaken for a fine-grained basalt. The chondritic structure is very obscure and metallic particles safely identified only with a microscope or pocket lens. Abundant exudations of lawrencite, made conspicuous by globules of iron oxide, serve as a fairly safe criterion of its celestial nature.

Under the microscope the resemblance to the McKinney stone is further augmented. The ground is everywhere impregnated with a black material, carbonaceous [4] in part, which permeates into the borders of the chondrules and cleavage and fracture lines of the enstatites, and the olivines have in many cases the same greenish yellow appearance suggestive of a serpentinous or chloritic alteration. The enstatites of the ground are colorless except where injected with the black matter which gives the dark hue to the stone. These are interspersed in a manner difficult of description, with radiating and polysomatic chondrules of both olivine and pyroxenes often so altered as to break up into scaly and fibrous aggregates when

[1] Arch. Naturk. Liv. Est. u. Kurlands, vol. 9, pt. 2, 1882, pp. 95–114.
[2] Meteoritenkunde, vol. 1, p. 310.
[3] Revision des Pierres Meteorique, etc., pp. 393–406.
[4] Roasted in a closed tube the powdered stone yields moisture and gives a distinct empyreumatic odor.

the stage is revolved between crossed nicols. A monoclinic pyroxene is present in minor quantity, showing indistinct traces of polysynthetic twinning, and there are frequent interstitial, very irregular areas of calcium phosphate. It will be noted from Eakins' analysis that the stone yields 0.41 per cent P_2O_5, an unusually large amount. I find nothing that I can with safety relegate to a feldspar, even of the maskelynite type. The structure is, however, so obscure that it will not do to pronounce too definitely on this point. The general resemblance to the McKinney stone is very close, but in composition, as shown by the two analyses below, it differs radically in the proportional amounts of alumina and ferrous iron, a difference which can be explained by the presence of an aluminous-monoclinic-pyroxene in the stone of McKinney, while magnesian forms prevail in that of Travis County.

	Travis County.	McKinney.
	Per cent.	*Per cent.*
Silica (SiO)	44.75	37.900
Alumina (Sl_2O_3)	2.72	13.290
Chromic oxide (Cr_2O_3)	.52	[1] .110
Ferrous oxide (FeO)	16.04	7.400
Magnesia (MgO)	27.93	26.690
Lime (CaO)	2.23	1.650
Manganous oxide (MnO)	Trace.	.210
Nickel oxide (NiO)	.52	.440
Potash (K_2O)	.13	....
Soda (Na_2O)	1.13	....
Iron (Fe)	1.83	5.070
Nickel (Ni)	.22	.920
Cobalt (Co)	.01	.050
Copper (Cu)	Trace.	.004
Sulphur (S)	1.83	[2] 6.260
Ignition (H_2O)	.84	....
Phosphoric acid (P_2O_5)	.41	.050
Total	101.11	100.044
Less O for S	.92	....
Total	100.19	....

[1] Chromite. [2] FeS.

It is obvious from the above that the Travis County stone is to be classed—following Brezina—as a black chondrite, rather than a Ckb, as is Bluff.

It is greatly to be regretted that so little is known regarding the fall or finding of either of these interesting stones.

Waconda, Kans.—This stone has been the subject of several papers and briefer references, of which only those of Shepard, Smith, Wadsworth, and Brezina, are important. Neither Shepard nor Smith made use of thin sections, a method then practically unknown, and their determinations of mineral composition were surmises based on chemical analyses. Wadsworth based his brief description evidently on a single section, and there is nothing in Brezina's to indicate that he made use of other means than perhaps a pocket lens.

As thus far described, the stone is a brecciated crystalline chondrite, or aumalite of Meunier, consisting of olivine, enstatite and a monoclinic pyroxene with the usual sprinkling of metallic iron and iron sulphide. Smith's analysis, referred to later, showed it to consist of 3.85 per cent troilite, 5.34 per cent nickel-iron, and 90.81 per cent stony matter. In describing the appearance of the stone he mentioned as occurring "only on one part" of his specimen a mineral "in the form of a white, crystalline mass, not exceeding in weight 20 milligrams," which was soluble in hydrochloric acid, the solution reacting for magnesia and silica. This mineral he thought might occupy "the same place among the unisilicates of the meteorites that the enstatite does among the bisilicates."

In looking over a quantity of fragmental material in the Shepard collection my attention was attracted to a small white area, some 2 mm. in diameter, on one of the fragments, and, recalling Smith's work, I undertook its determination. The results are given below, and, as will be apparent, the investigation was much more extended than at first intended.

In the thin section the stone is at once seen to be composed essentially of olivine and pyroxene with nickel-iron and troilite. The chondritic structure is very evident (Fig. 2, Pl. V), the individual chondrules consisting wholly of pyroxenes or of olivine in the customary forms,

embedded in a crystalline ground of the same constituents, and the metallic components. Where not stained by oxidation the silicates are beautifully clear and pellucid. The pyroxene is in part of the normal enstatite type, though many of the larger forms are monoclinic, showing extinction angles as high as 25°. In almost the first section examined attention was attracted to a minute, irregular, colorless area traversed by numerous fracture lines, with only moderate relief, non-pleochroic, and polarized in faint bluish-gray colors. Its appearance at once suggested the phosphatic mineral described by me in a previous paper.[1] Microscopic examination of a considerable number of slides, accompanied in some instances by microchemical tests, showed the mineral to be a calcium phosphate, and occurring not infrequently. In no instance was the mineral found in the crystalline form characteristic of apatite. Nearly altogether it occurs as an interstitial filling, almost isotropic, and, as in the previous cases which I have described, of lower refractive indices than normal apatite. Indeed, in many instances the mode of occurrence and low relief without cleavage or crystal outline causes it to resemble on casual inspection an interstitial glass, for which doubtless it has heretofore been frequently mistaken. In such cases, it is only by treating a slide with a drop of acid and watching the mineral gradually disappear, then testing the solution, that its true nature can be determined.

Further examination showed the presence of this phosphate in the Waconda stone where it could not be recognized even microscopically. It was found that when the surface of an uncovered slide was treated with a dilute solution of hydrochloric acid and allowed to stand for not more than a quarter of an hour, the solution thus obtained would react for phosphorus and calcium, and the slide when examined be found to contain frequent minute, irregular, interstitial pits where the material had been dissolved away.

These determinations naturally suggested the possible phosphatic nature of the white spots before noted. An examination with a pocket lens showed these to be composed of aggregates of minute crystals of a faint yellow-green tint. It being obviously impossible to rely on cutting a thin section including the desired area, recourse was made once more to microchemical tests on minute fragments broken out by a needle point. Reactions for phosphorus and calcium were easily obtained, the mineral being readily soluble in cold nitric acid and less so in hydrochloric acid. Dr. E. S. Larsen kindly determined the indices of refraction by solutions, as follows: $\alpha = 1.627 \pm 0.003$; $\gamma = 1.621 \pm 0.003$. These results are low for normal apatite, agreeing more closely with those obtained by Dr. Wright on material from the Alfianello and Rich Mountain stones, as given in the paper before referred to.

As phosphorus was not determined by J. L. Smith in his analysis of either the metallic or silicate portions of this stone, a second analysis was decided upon. The results as determined by Dr. J. E. Whitfield are given below, Smith's results being also given for purposes of comparison.

Preliminary separations yielded:

	J. E. Whitfield.	J. L. Smith.
	Per cent.	*Per cent.*
Stony matter	87.80	90.81
Nickel-iron	5.93	5.34
Troilite	6.27	3.85
Total	100.00	100.00
Nickel-iron yielded:		
Iron	85.50	86.18
Nickel	13.78	12.02
Cobalt	.71	.91
Copper	(²)	.04
Total	99.99	99.15

[1] On the monticellite-like mineral in meteorites, and on oldhamite as a meteoric constituent, Proc. Nat. Acad. Sci., vol. 1, 1915, p. 302, and On the calcium phosphate of meteoric stones, Amer. Journ. Sci., vol. 43, 1917, p. 322.

[2] Not determined.

Phosphorus not determined in either case.

Smith further determined the stony portion to consist of 69 per cent soluble in aqua regia and 41 per cent insoluble, giving analyses of each, from which the bulk analysis given below was calculated.

	J. E. Whitfield.	J. L. Smith.[1]
	Per cent.	*Per cent.*
SiO_2	35.05	38.14
FeO	16.53	23.44
P_2O	.23	(2)
Al_2O_3	4.94	1.02
CaO	2.25	(3)
MgO	24.98	26.69
NiO	.74	(2)
CoO	.04	(2)
MnO	.29	.47
SO_3	.06	(2)
Na_2O	.76	1.05
K_2O	.17	(3)
H_2O	1.61	(2)
Fe	5.07	4.64
Ni	.81	.65
Co	.04	.05
Fe } Troilite	3.99	3.85
S } Troilite	2.28	
Total	99.84	100.00

[1] Analyses recalculated by Farrington. Smith reported also traces of lithium and copper.
[2] Not determined.
[3] Trace.

Five grams of the finely pulverized stone were boiled in distilled water for an hour, resulting in a solution yielding 0.062 per cent SO_3 and 0.012 per cent CaO. A portion of the SO_3 probably came from the decomposed troilite, rendering any calculations uncertain, while the amount of lime (CaO) is too small to make the results more than suggestive of the presence of a minute quantity of oldhamite. A second 5 grams were boiled for half an hour in acetic acid of 15 per cent normal strength. The solution yielded 0.08 per cent P_2O_5 and 0.122 per cent CaO. Inasmuch as the bulk analysis shows 0.23 per cent P_2O_5, it is evident that a complete solution of the phosphate was not accomplished by the acetic acid. Be this as it may, the relative proportion of acid to base is such as to render it unsafe to draw definite conclusions.[1]

It is difficult to account for all the discrepancies between the two analyses. The difference of some 3 per cent between the amount of stony matter and troilite may perhaps be accounted for on the supposition that Smith worked, as is so often the custom, on very small amounts that did not correctly represent the stone as a whole. (Whitfield had 19 grams of selected material.) The analyses of the metallic portion, it will be noticed, agree fairly well excepting that Whitfield reports no copper. In the bulk analyses, however, we find a difference of 3 per cent (in round numbers) in the total silica, nearly 7 per cent in the ferrous iron, 3.92 per cent in the alumina, and 2.25 per cent in the lime, with minor differences, mainly due to omissions elsewhere. The totals for iron and magnesia do not differ more than might be anticipated from analyses on separate portions, made by even the same analyst. That Smith did not determine the nickel and cobalt in the silicate portion is not strange, it being customary in his day to regard these elements as constituents of the nickel-iron only. The phosphoric acid, amounting to 0.23 per cent, should in this day certainly not be overlooked.

It does not seem in the least probable that the phosphate to which I have referred above as evident to the unaided eye can be the white mineral mentioned by Smith. Nevertheless, a most careful examination of all the material in the Museum and Shepard collections reveals nothing that is even suggestive of his doubtful unisilicate.

[1] Since the above was written, Dr. E. T. Wherry (Amer. Mineralogist, vol. 2, No. 9, 1917) has complimented me by suggesting that the problematic phosphate be given the name *merrillite*. Had I been consulted in the matter I should have suggested a postponement until a more definite statement of its composition could be given. Incidentally, it may be stated, I had considered the use of Shepard's name, apatoid (Amer. Journ. Sci., vol. 2, 1846), but abandoned it because of his definite statement that his mineral contained no phosphorus.

Weston, Conn.—Notwithstanding that this is the oldest known of American falls, it is deserving of more detailed study than it has yet received either from a mineralogical or chemical standpoint. The work of Shepard (in 1809, 1846–1848) would naturally at this date be considered faulty. He described the stone as composed principally of howardite and olivinoid, with scattered grains of magnetic pyrites and nickel-iron. Little advance over this seems to have been made by subsequent workers, excepting Meunier, who, in classifying the stone as a limerickite, recognized its chondritic character and mineral composition. Brezina classified it as a spherical chondrite, brecciated, apparently without regard to its composition or microscopic structure. The breccia-like structure is very evident, and is produced by angular pieces of a light gray color embedded in the prevailing dark-gray material. The chondritic structure is equally pronounced in both, and so far as can be determined by the unaided eye or a pocket lens there are no essential differences between the two kinds of fragments other than that of color. The mineral composition I find to be chiefly a pyroxene with a low angle of extinction, about 10°, which therefore relegates it to the clino-enstatite of Wahl, a polysynthetically twinned pyroxene, olivine, "merrillite," nickel-iron, and iron sulphide. No feldspars, even in the form of maskelynite, were observed.

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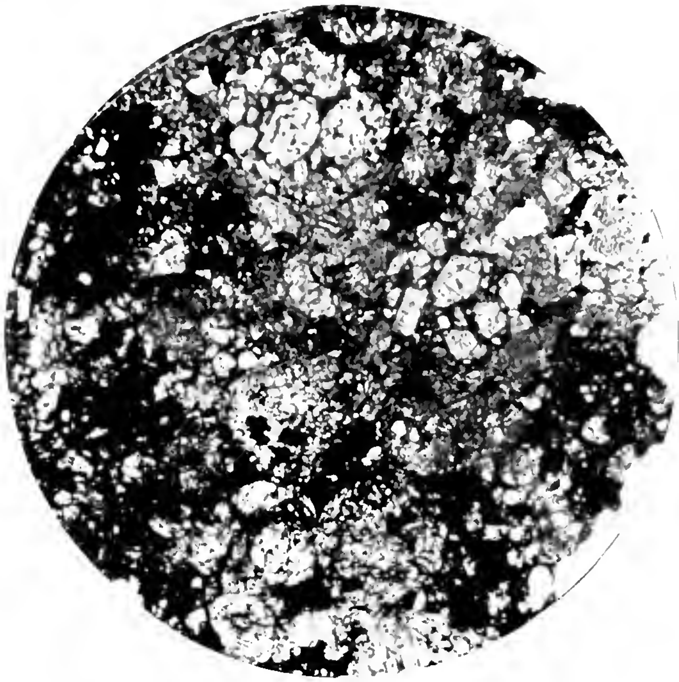

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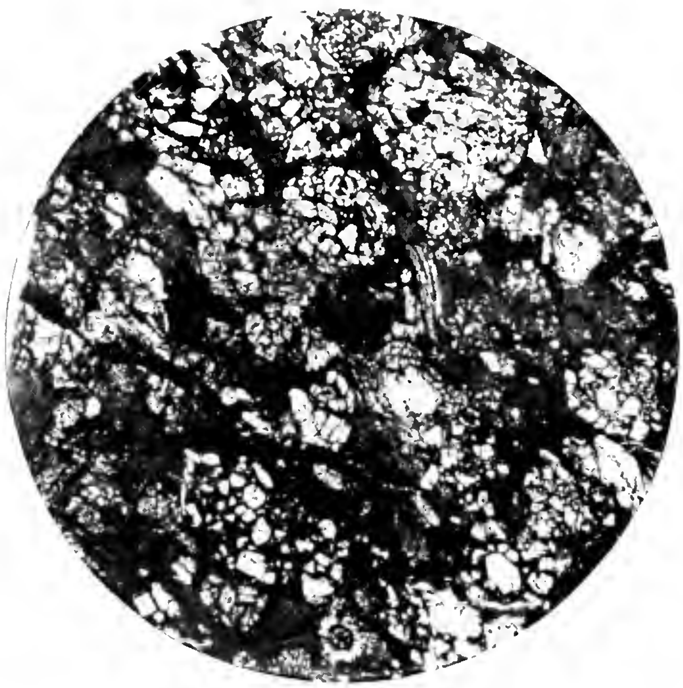

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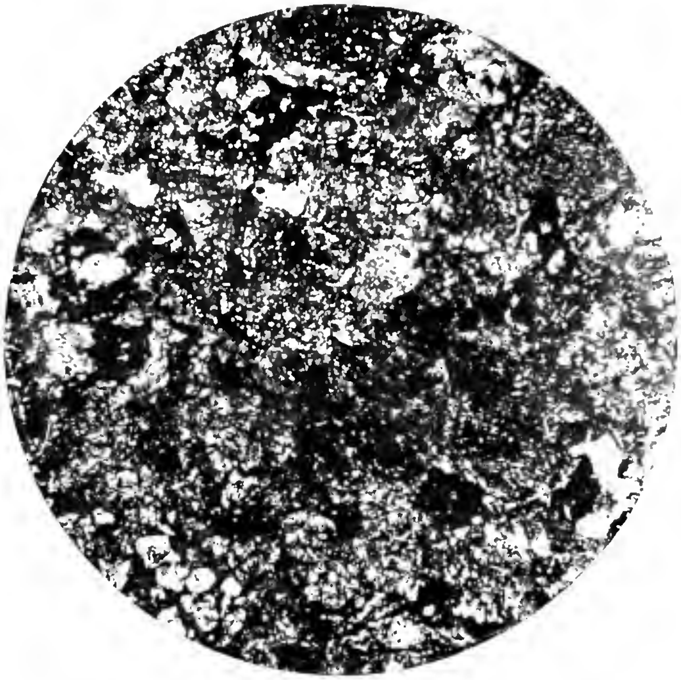

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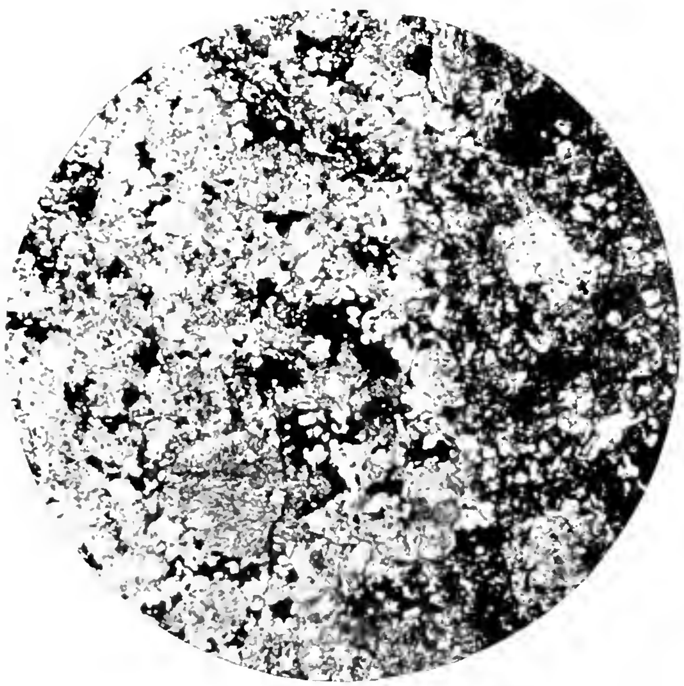

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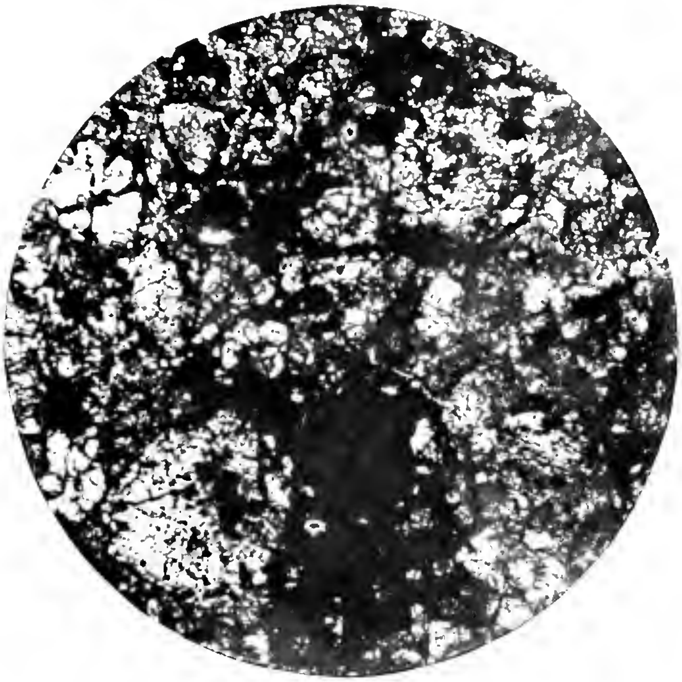

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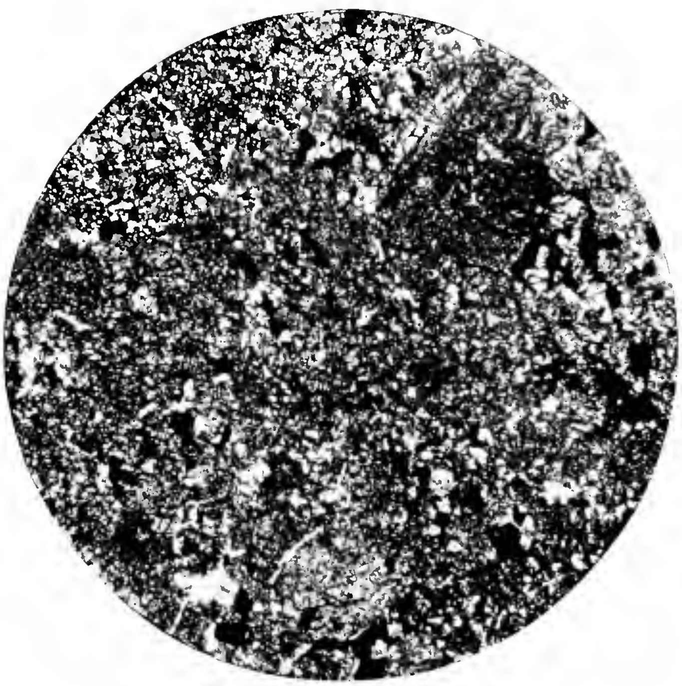

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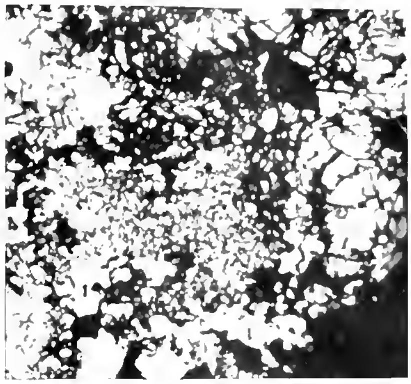

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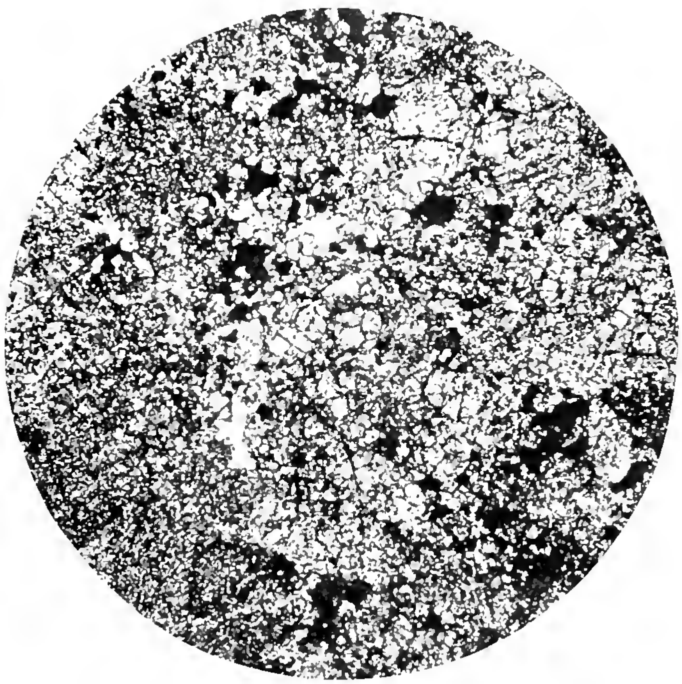

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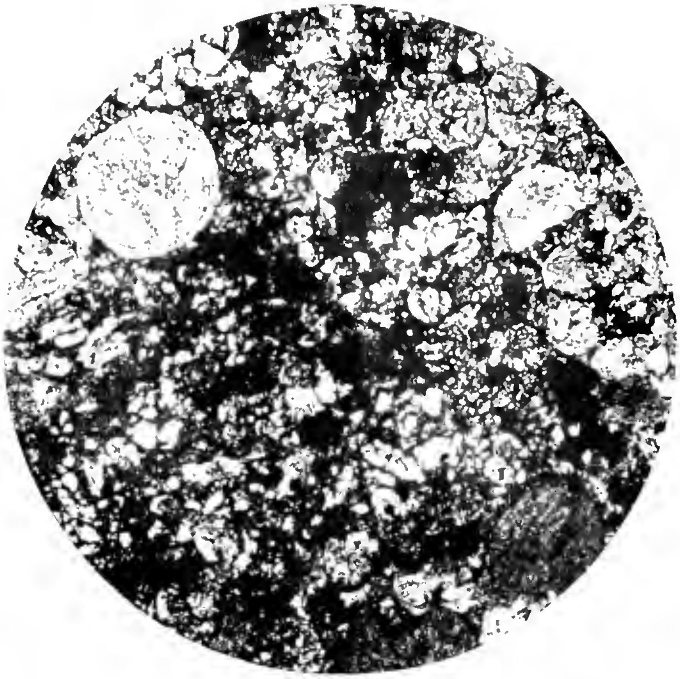

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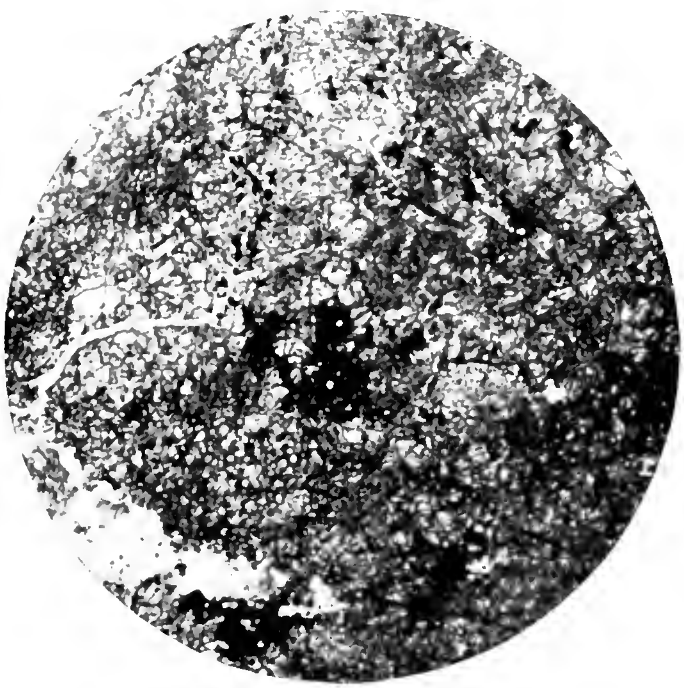

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TABLES OF THE EXPONENTIAL FUNCTION AND OF THE CIRCULAR SINE AND COSINE TO RADIAN ARGUMENT.

BY

C. E. VAN ORSTRAND.

TABLES OF THE EXPONENTIAL FUNCTION AND OF THE CIRCULAR SINE AND COSINE TO RADIAN ARGUMENT.[1]

By C. E. Van Orstrand.

The tables accompanying this paper have been prepared with the expectation of meeting a twofold requirement. The first was to obtain a few high place values at sufficiently small intervals of argument for general use in the evaluation of integrals and other functions; the other object was to obtain a basis for subsequent interpolation to small intervals of argument for use in the construction of complete 10-place tables which are applicable in the various fields of pure and applied mathematics. The need of tables meeting these and other requirements has been emphasized by various authors.

The most important tables of extended values of the exponential function in which the exponents are integers or fractions have been constructed by Schulze, Bretschneider, Newman, Gram, Glaisher, and Burgess. Bretschneider included a few high place values of the circular sine and cosine to radian argument, but with the exception of these and a few values computed by Gudermann, there appears to be no extended values of these important functions.

Schulze[2] gives values of the ascending exponential at intervals of unity between the limits 1 and 24, inclusive, to 28 or 29 significant figures, and for the special arguments 25, 30, and 60 his values include 32 or 33 figures. In so far as I have been able to ascertain, Schulze gives no information regarding methods of computation or accuracy of results. Glaisher[3] verified the first 15 figures of Schulze's value of e^{16} by direct substitution in the series; the first 13 powers of e were verified to 22 places of decimals; and the values of e^{14}, e^{15}, ... e^{25} to 15 places of decimals by means of the relation

$$\frac{e^n-1}{e-1}=e^{n-1}+e^{n-2}\ \dots\ +e+1.$$

Bretschneider[4] evaluated e, e^{-1}, sin 1 and cos 1 each to 105 places of decimals; also values of the same functions at intervals of unity between the limits 1 and 10, inclusive, to 20 places of decimals. He corrected the erroneous value of e given in Callet's tables and Vega's Thesaurus and the slightly erroneous values of sin 1 and cos 1 given to the twenty-fifth decimal by Gudermann.[5]

Bretschneider obtained his values by direct substitution in the exponential series in connection with the evaluation of the three transcendants,

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[1] Published by permission of the Director of the U. S. Geological Survey.

[2] I. S. Schulze, Sammlung logarithmischer Trigonometrischer-Tafeln (Berlin, 1778).

[3] J. W. L. Glaisher, Tables of the exponential function. Camb. Phil. Trans., vol. 13 (1883), pp. 243-272. (In Salomon's Tafeln (1827) the values of e^n, e^{-n} $e^{0.0n}$, ... $e^{0.000000n}$ where n has the values 1, 2, ... 9 are given to 12 places.

[4] C. A. Bretschneider. Berechnung der Grundzahl der natürlichen Logarithmen, so wie mehrerer anderer mit ihr zusammenhängender Zahlwerthe. Grunert's Archiv der Math. und Phys., Bd. 3 (1843), pp. 27-34.

[5] C. Gudermann. Potenzial oder cykisch-hyperbolische Functionen. Jour. für die reine und angewandte Math., Bd. VI (1830), pp. 1-39.

$$\mathrm{Si}\,x=\int_0^x \frac{\sin x}{x}\,dx=x-\frac{1}{3}\frac{x^3}{3!}+\frac{1}{5}\frac{x^5}{5!}-\frac{1}{7}\frac{x^7}{7!}+\cdots$$

$$=\frac{\pi}{2}-\cos x\left[\frac{1}{x}-\frac{2!}{x^3}+\frac{4!}{x^5}-\frac{6!}{x^7}+\cdots\right]$$

$$-\sin x\left[\frac{1}{x^2}-\frac{3!}{x^4}+\frac{5!}{x^6}-\frac{7!}{x^8}+\cdots\right]$$

$$\mathrm{Ci}\,x=\int_\infty^x \frac{\cos x}{x}\,dx=\gamma+\frac{1}{4}\log_e(x^4)-\frac{1}{2}\frac{x^2}{2!}+\frac{1}{4}\frac{x^4}{4!}-\cdots$$

$$=\sin x\left[\frac{1}{x}-\frac{2!}{x^3}+\frac{4!}{x^5}-\frac{6!}{x^7}+\cdots\right]$$

$$-\cos x\left[\frac{1}{x^2}-\frac{3!}{x^4}+\frac{5!}{x^6}-\frac{7!}{x^8}+\cdots\right]$$

$$\mathrm{Ei}\,x=\int_{-\infty}^x \frac{e^x}{x}\,dx=\gamma+\frac{1}{4}\log_e(x^4)+x+\frac{1}{2}\frac{x^2}{2!}+\frac{1}{3}\frac{x^3}{3!}+\cdots$$

$$=e^x\left[\frac{1}{x}+\frac{1}{x^2}+\frac{2!}{x^3}+\frac{3!}{x^4}+\frac{4!}{x^5}+\cdots\right],$$

known, respectively, as the sine integral, the cosine integral and the exponential integral. The quantity γ is the Eulerian constant 0.5772156 .... .

Newman's[1] contribution to the subject consists of the following:

18-place values of e^{-x} from $x=0.0$ to $x=37.0$ at intervals of 0.1.
12-place values of e^{-x} from $x=0.000$ to $x=15.349$ at intervals of 0.001.
14-place values of e^{-x} from $x=15.350$ to $x=17.298$ at intervals of 0.002.
14-place values of e^{-x} from $x=17.300$ to $x=27.635$ at intervals of 0.005.
16-place values of e^x from $x=0.1$ to $x=3.0$ at intervals of 0.1.
12-place values of e^x from $x=0.001$ to $x=2.000$ at intervals of 0.001.

The 18-place table is hardly the equivalent of a 16-place table, as the original computation included only 18 decimals.

All of Newman's computations are based on formulas of the type

$$M\pm N=e^{-x\pm h}=e^{-x}\left[1\pm\frac{h}{1!}+\frac{h^2}{2!}\pm\frac{h^3}{3!}+\cdots\right]$$

wherein h assumes the constant values 1, 0.1, 0.01, . . . dependent upon the interval of interpolation. Having given e^{-x} and e^{-x+h} the value of e^{-x-h} is computed from the formula by putting

$$M=\Sigma\frac{h^m}{m!}e^{-x} \text{ and } N=\Sigma\frac{h^n}{n!}e^{-x},$$

m being an even and n an odd integer. The values of the separate terms in these expressions may be computed by successive divisions. Then the appropriate summations give

$$M+N=e^{-x+h}$$

a known quantity, and

$$M-N=e^{-x-h}$$

[1] F. W. Newman. Tables of the descending exponential function to 12 or 14 places of decimals. Trans. Camb. Phil. Soc., vol. XIII (1883), pp. 146–241; table of the exponential function e^x to 12 places of decimals. Trans. Camb. Phil. Soc., vol. XIV (1889), pp. 237–249.

the quantity to be determined. The equation for $M \pm N$ provides a check on the values of M and N, but the sum or difference which is the quantity sought is not verified by this method until another interpolation is made.

Gram[1] gives values of the ascending exponential to 24 places of decimals at intervals of unity between the limits $x=0$ and $x=20$, inclusive; also values of the same function from $x=5.00$ to $x=20.00$, inclusive, at intervals of 0.02, the number of tabular decimals ranging from 4 to 15; and from $x=0.1$ to $x=15.0$, the values are given to one decimal only at intervals of 0.1. Some of the values were obtained by either repeated multiplication or logarithmic computation, and the remainder were borrowed from Schulze, Bretschneider, and Oppermann.

Glaisher[2] gives 10-place logarithmic values and 9 significant figures of the natural values of both the ascending and descending function for the following ranges of argument:

From $x=0.001$ to $x=0.100$ at intervals of 0.001.
From $x=0.01$ to $x=2.00$ at intervals of 0.01.
From $x=0.1$ to $x=10.0$ at intervals of 0.1.
From $x=1$ to $x=500$ at intervals of unity.

Since the natural values were computed from the logarithmic values, the maximum tabular error is one unit in the ninth significant figure with the exception of values of e^{-x} contained in Newman's tables. The remaining values of Glaisher's tables were checked either by differences or by duplicate computation. Glaisher gives also the reciprocals of the factorials from 1 to 50, inclusive, to 28 significant figures, and verifies his values by forming the summations for e and e^{-1}. A further verification is obtained by evaluating e^{-10} to 32 decimal places by means of the formula

$$e^{-x}=\frac{y+h}{10^n}.$$

The quantity $y \times 10^{-n}$ is here an approximate value of e^{-x}. The equation gives

$$\log_e(y+h)=n\,\log_e 10-x$$

a known quantity. Since $\log_e y$ is also known, we may evaluate the expressions

$$[\log_e(y+h)-\log_e y] \text{ and } y[\log_e(y+h)-\log_e y].$$

The expansion of the first by Taylor's series gives

$$h=y[\log_e(y+h)-\log_e y]+\frac{1}{2}\frac{h^2}{y}-\frac{1}{3}\frac{h^3}{y^2}+\ .\ .\ .\ ,$$

from which an approximate value of $h\,y^{-1}$ may be computed by neglecting terms in h beginning with the square. Finally the substitution of

$$\frac{1}{2}\,y\,\frac{h^2}{y^2} \text{ and} -\frac{1}{3}\,h\,\frac{h^2}{y^2}$$

in the preceding equation gives a corrected value of h.

Burgess[3] gives 30-place values of e^{-x} for $x=0.5,\ 1,\ 2,\ .\ .\ .\ 10$; and 14 values of e^{-x^2} at irregular intervals between the limits 1.0 and 3.0, ranging in extent from 23 to 27 decimals. These values were used in his evaluation of the probability integral, but no information seems to have been given with regard to either method or accuracy of computation.

In his "Rectification of the Circle" (1853), Shanks evaluates the Naperian base by direct substitution in the series to 137 places of decimals. His second computation[4] was carried to

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[1] J. P. Gram, Undersogelser angaaende Maengden af Primtal under en given Graense. Copenhagen Academy, 6 vol. II (1884), pp. 183–308.

[2] J. W. L. Glaisher. Tables of the exponential function. Trans. Camb. Phil. Soc., vol. 13 (1883), pp. 244–272.

[3] James Burgess. On the definite integral $\frac{2}{\sqrt{\pi}}\int_0^t e^{-t^2}dt$, with extended tables of values. Trans. Roy. Soc. Edinburgh, vol. 39 (1900), pp. 257–321.

[4] William Shanks. On the extension of the value of the base of Napier's logarithms; of the Naperian logarithms of 2, 3, 5, and 10, and of the Modulus of Brigg's, or the Common system of logarithms; all to 205 places of decimals. Proc. Roy. Soc. Vol., VI (1850–1854), p. 397.

the two hundred and fifth decimal by a method given in J. R. Young's Elementary essay on the computation of logarithms (pp. 13–14). Glaisher [1] verified the first result, using the continued fraction

$$\frac{e-1}{2}=\frac{1}{1}+\frac{1}{6}+\frac{1}{10}+\cdots\frac{1}{4n+2+\ldots},$$

but the second result was shown by Boorman [2] to be incorrect after the one hundred and eighty-seventh decimal.

Boorman's formula is readily deduced. Since we have identically

$$\frac{1}{m}+\frac{1}{mn}=\frac{1}{n}+\frac{n+1-m}{mn},$$

we obtain

$$\frac{1}{m}+\frac{1}{mn}=\frac{1}{n}+\frac{2}{mn},$$

by the substitution

$$m=n-1$$

in the numerator of the right-hand member. The series for the Naperian base may thus be transformed into the series

$$e=\left(1+\frac{1}{1}\right)+\frac{1}{1}\left(\frac{1}{n}+\frac{2}{mn}\right)+\frac{1}{1}\cdot\frac{1}{mn}\left(\frac{1}{n_1}+\frac{2}{m_1n_1}\right)+$$
$$\frac{1}{1}\cdot\frac{1}{mn}\cdot\frac{1}{m_1n_1}\left(\frac{1}{n_2}+\frac{2}{m_2n_2}\right)+\ldots,$$

wherein $m=2$, $n=3$; $m_1=4$, $n_1=5$; $m_2=6$, $n_2=7$; . . . . .

Tichánek [3] and Minks verified Boorman's value of e as far as the two hundred and twenty-third decimal, making use of Euler's continued fraction

$$F=\frac{1}{2.1}+\frac{1}{2.3}+\frac{1}{2.5}+\cdots,$$

in connection with the relations

$$e=\frac{1+F}{1-F}$$

$$\frac{e-1}{2}=\frac{1}{1}-\frac{1}{1.7}+\frac{1}{7.71}-\frac{1}{71.1001}+\frac{1}{1001.18089}-\cdots$$

Gauss [4] gives values of $e^{n\pi}$ ranging from 15 to 57 decimals for 13 values of n at irregular intervals between the limits 1/2 and -16. He used the formula

$$e^{n\pi}=Ne^{\log a+10\log b-\log c-\log b\,\ldots\,-n\pi}$$

The quantity N is an approximate value of $e^{n\pi}$ multiplied by $a\times10^b$, and the quantities $c, d, \ldots$ are the factors of N so selected that their natural logarithms may be taken from Wolfram's [5] tables.

The present contribution consists of the following tables:

Table I: Values of the reciprocal of $n!$ to 108 places of decimals at intervals of unity from 1 to 74.

Table II: Values of e^x to 42 significant figures at intervals of unity from 0 to 100.

Table III: Values of e^x to 33 significant figures at intervals of 0.1 from 0.0 to 50.0.

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[1] J. W. L. Glaisher. On the calculation of e from a continued fraction. Brit. Assoc. Rep. (1871), pp. 16–18.

[2] J. Marcus Boorman. Computation of the Naperian base. Math. Mag. vol. I (1882–1884), pp. 204–205; see also L'Intermédiaire des mathématiciens, vol. 7 (1900), p. 53; G. Peano, Formulaire de mathematiques. Tome 11, No. 3, p. 125.

[3] F. J. Studnička. Ueber die Berechnung die transcendenten Zahl e. Jahr. über die Fort. der Math. Bd. 23 (1891), p. 440: Vorträge über monoperiodische Functionen. Jahr. über die Fort. der Math. Bd. 25 (1893–1894), p. 736.

[4] Lemniscatische Functionen. Werke 3, pp. 413–432.

[5] Logarithmorum Naturalium. (48 decimals). See Vega's Thesaurus, pp. 641–684.

Table IV: Values of e^x to 62 places of decimals at decimal intervals from 1×10^{-10} to 9×10^{-1}.

Table V: Values of e^{-x} ranging from 52 to 62 places of decimals at intervals of unity from 0 to 100.

Table VI: Values of e^{-x} ranging from 33 to 48 places of decimals at intervals of 0.1 from 0.0 to 50.0.

Table VII: Values of e^{-x} to 62 places of decimals at decimal intervals from 1×10^{-10} to 9×10^{-1}.

Table VIII: Values of $e^{\pm(n\pi/360)}$ to 23 places of decimals or significant figures at intervals of unity from $n=0$ to $n=360$.

Table IX: Values of $e^{\pm n\pi}$ to 25 places of decimals or significant figures for various values of n.

Table X: Values of sin x and cos x to 23 places of decimals at intervals of unity from 0 to 100.

Table XI: Values of sin x and cos x to 23 places of decimals at intervals of 0.1 from 0.0 to 10.0.

Table XII: Values of sin x and cos x to 23 places of decimals at intervals of 0.001 from 0.000 to 1.600.

Table XIII: Values of sin x and cos x to 25 places of decimals at decimal intervals from 1×10^{-10} to 9×10^{-4}.

Table XIV: Miscellaneous values of e^x, e^{-x}, sin x and cos x to a great number of decimals, including Boorman's value of e.

The tabular error of the preceding tables may in some cases slightly exceed 5 units in the next succeeding tabular digit as two digits only were dropped from most of the values.

In my preliminary computations, Glaisher's [1] table of the reciprocals of the factorials was used. It contains all of the recurring decimals from $n=1$ to $n=12$, inclusive, and from $n=13$ to $n=50$, inclusive, the values are given to 28 significant figures. This table frequently failed to give the requisite number of decimals in the vicinity of $n=13$ and upwards, so it was afterwards extended roughly to 110 decimals, and the range of the argument extended from 50 to 74, the results being verified by forming the summations for e and e^{-1}, and then computing the product of these two quantities, in addition to making a direct comparison with the well known value of e. A further check on the value of e^{-1} consisted in reciprocating e, written in the form $(a+b)^{-1}$.

With the table of reciprocal factorials as a basis, it was easy to compute the value of $e^{0.1}$ from the series: and afterwards by repeated multiplications by this factor, in accordance with the formula,

$$e^{x+\Delta x} = e^x \cdot e^{\Delta x} \text{ - - - - - - - - - } (2),$$

the values of $e^{0.2}$, $e^{0.3}$, ... e were obtained and verified by comparing the last computed value with the well known value of e. Similarly the value of e^{10} was computed from e and the value of e^{100} was computed from e^{10}. Values of the descending exponential for the same intervals of argument were determined in the same manner, and the evaluation of both functions was verified at frequent intervals by means of the product relation, $e^x e^{-x}$. Another check consisted in substituting values of x and Δx in (2). Subsequent interpolations to one-tenth the previous interval of interpolation provided a further complete check on the entire computation. The maximum difference between any value and the corresponding value obtained by 10 interpolations did not exceed 15 units in the last decimal or significant figure. Practically all of the computations were made with a 10-groove computing machine of the millionaire type.

[1] J. W. L. Glaisher, Tables of the exponential function. Trans. Camb. Phil. Soc., vol. 13 (1883), pp. 244-272.

[2] C. F. Degen, Tabularum Enneas (Copenhagen 1824) gives 18-place values of $\log_{10}(n!)$ from $n=1$ to $n=1200$. De Morgan reprinted the same to 6 places in his article on "Probabilities" in Encyclopedia Metropolitana. J. W. L. Glaisher gives 20-place values of $n \times n!$ and 10-place values of $-\log(n \times n!)$ to $n=71$ in Phil. Trans. Roy. Soc., vol. 160, 1870, p. 370. Shortrede, Tables (1849, Vol. I) contains 5-place values of $\log(n!)$ to $n=1000$ and 8-place values for arguments ending in 0.

The values of $e^{\pm n\pi/360}$ contained in Tables VIII and IX were computed in the manner just described from the values, $e^{\pm\pi/360}$. The latter function was evaluated for the first 10 decimals of the exponent by successive multiplication of the appropriate factors taken from Tables IV and VII. The values for the remaining decimals of the exponents were obtained by substitution in the exponential series. The product of the two factors thus obtained is the required result. Checks were applied in the usual manner, also by comparison with the values given by Gauss.[1]

Values of the exponential function previously obtained provided an excellent check on the fundamental values needed in the computation of sin x and cos x. These values were computed at intervals of 0.1 from 0.0 to 1.6, inclusive, by direct substitution in the series and verified by means of the relation

$$e^x = \sin x + \cos x + 2\left[\frac{x^2}{2!} + \frac{x^6}{6!} + \dots\right] + 2\left[\frac{x^3}{3!} + \frac{x^7}{7!} + \dots\right] \dots (3).$$

Interpolations were made by means of the formulas

$$\sin(x+\Delta x) = \sin x + \frac{\Delta x}{1!}\cos x - \frac{(\Delta x)^2}{2!}\sin x + \dots$$

$$\cos(x+\Delta x) = \cos x - \frac{\Delta x}{1!}\sin x - \frac{(\Delta x)^2}{2!}\cos x + \dots$$

in which Δx assumes the values 0.1, 0.01, ... according to the interval of interpolation. It will be noted that the two equations together contain terms of the form

$$(\Delta x)^n \sin x/n! \text{ and } (\Delta x)^n \cos x/n!$$

wherein n assumes successive values of the natural numbers beginning with unity. There are thus two series of terms,

$$\frac{1}{2!}\sin x,\ \frac{1}{3!}\sin x,\ \frac{1}{4!}\sin x,\ \dots$$

$$\frac{1}{2!}\cos x,\ \frac{1}{3!}\cos x,\ \frac{1}{4!}\cos x,\ \dots$$

which may be evaluated by dividing the sine or cosine, as the case may be, first by 2, this quotient by 3, the last by 4, and so on, thus avoiding the use of large factors. The computation of both functions is made at the same time, and a complete check is obtained on the tenth interpolation. The maximum difference between the interpolated values is 10 units, and as there were two interpolations, the maximum error of interpolation is of the order of magnitude of 20 units in the twenty-fifth decimal. Table X was computed with the assistance of a computing machine by substitution in the trigonometric expansions for sin $(x+\Delta x)$ and cos $(x+\Delta x)$, and verified by assigning various values to x and Δx; also by forming the sum of the squares of the sine and cosine for several values of the argument. The values of sin x and cos x contained in Tables XIII and XIV were computed by substitution in the respective series and verified by means of equation (3).

Writers on interpolation emphasize the importance of interpolation by differences while not much attention is given to interpolation by means of derivatives. This procedure does not seem justifiable as the time lost in rotabulating and differencing the quantities is sometimes much greater than the loss of time due to the possible increased labor and difficulty of computation by the derivative formula. Furthermore the check provided by the derivative formula is much more reliable than that of the difference formula when both the interval of interpolation and the interpolated values are large. Neither method provides an absolute check, for experience proves that positive and negative errors of equal or approximately equal magnitudes very frequently escape detection. The same is true of the various methods of mechanical quadratures which could be used for the same purpose.

[1] Loc. cit.

A comparison of the present values with those mentioned in the first part of this paper shows some interesting results. The values given by Schulze are generally incorrect in the last or the next to the last decimal. Newman's 18-place table of the descending exponential is correct to 16 decimals when the last two decimals are taken into account. His values for $x = -3.5$, -26.1, -26.4, and -26.9 contain misprints. The values of $e^{-0.5}$ by Burgess is in error by approximately one unit in the thirtieth decimal. Glaisher's value of e^{-10} computed from formula (1) is correct. His table of the reciprocals of the factorials contain errors slightly in excess of 5 units in the next succeeding decimal for $n=20$, 27, 41, and 50. All of Bretschneider's values are correct and the values of e^x given by Gram to 24 decimals are correct. The value of $e^{\pi/2}$ given by Gauss is incorrect in the twenty-third and following decimals.

The present paper was completed before the 1916 report[1] of the British Association for the Advancement of Science was received. The report of the committee on the calculation of mathematical tables (pp. 59–126) contains the following tables of sin x and cos x to radian argument:

Table I: Values of sin x and cos x to 11 places of decimals at intervals of 0.001 from 0.000 to 1.600.

Table II: Values of $x - \sin x$ and $1 - \cos x$ to 11 places of decimals at intervals of 0.00001 from 0.00001 to 0.00100.

Table III: Values of sin x and cos x to 15 places of decimals at intervals of 0.1 from 0.1 to 10.0.

In one value only, does the tabular error of Tables I and III exceed 10 units in the next succeeding decimal; the value of sin 9.1 should read 52 instead of 53 in the last two tabular digits.

Nearly all of the numerical computations were made by A. G. Seiler, piece work computer, and R. Weinstein and A. T. Harris, aids in the physical laboratory of the Geological Survey. I am indebted to F. A. Wolff, of the United States Bureau of Standards, Washington, D. C., for valuable suggestions in regard to the contents of Table IX, and to E. B. Escott, who kindly called my attention to the omission of several important references which had been overlooked in my preliminary publications in the Journal of the Washington Academy of Sciences (1912–13). The values given by Gram and Bretschneider were especially useful as a partial check on certain values which I had previously carried to a slightly greater number of decimals. No errors were discovered in my computations.

[1] The same report, pp. 123–126, contains the following:
10 place values of the logarithmic gamma function at intervals of 0.005 from 0.005 to 1.000.
10 place values of the integral of the logarithmic gamma function at intervals of 0.01 from 0.01 to 1.00.
13 place values of the logarithmic derivate of the gamma function at intervals of unity from 1 to 101, and from 0.5 to 100.5.

Table I.—*Values of the reciprocal of n! to 108 places of decimals at intervals of unity from 1 to 74.*

n	$\frac{1}{n!}$
1	1.
2	0. 5
3	. 1$\dot{6}$
4	. 041$\dot{6}$
5	0. 008$\dot{3}$
6	. 0013$\dot{8}$
7	. 0001$\dot{9}$ 8412$\dot{6}$
8	00002 48$\dot{0}$15 87$\dot{3}$
9	. (5) 27$\dot{5}$57 31922 39858 9065$\dot{2}$
10	0. (5) 027$\dot{5}$5 73192 23985 89065 $\dot{2}$
11	. (5) 002$\dot{5}$0 52108 38544 17187 $\dot{7}$
12	. (5) 00020 $\dot{8}$7675 69878 68098 97921 00903 21201 43231 25434 23654 53476 564$\dot{5}$
13	. (5) 00001 $\dot{6}$0590 43836 82161 45993 92377 17015 49479 32725 71050 34882 812$\dot{6}$
14	. (10) 11470 74559 77297 24713 85169 79786 82105 66623 26503 59634 48661 86136 02740 58686 75709 94555 12153 92485 23375 508
15	0. (10) 00764 71637 31819 81647 59011 31985 78807 04441 55100 23975 63244 12409 06849 37245 78380 66303 67476 92832 34891 701
16	. (10) 00047 79477 33238 73852 97438 20749 11175 44027 59693 76498 47702 75775 56678 08577 86148 79143 97967 30802 02180 731
17	. (10) 00002 81145 72543 45520 76319 89455 83010 32001 62334 92735 20453 10339 73922 24033 99185 22302 58703 95929 53069 455
18	. (15) 15619 20696 85862 26462 21636 43500 57333 42351 94040 84469 61685 54106 79112 99954 73461 25483 55329 41837 192
19	. (15) 00822 06352 46624 32971 69559 81236 87228 07492 20738 99182 61141 34426 67321 73681 82813 75025 45017 33780 905
20	0. (15) 00041 10317 62331 21648 58477 99061 84361 40374 61036 94959 13057 06721 33366 08684 09140 68751 27250 86689 045
21	. (15) 00001 95729 41063 39126 12308 47574 37350 54303 55287 47379 00621 76510 53969 81365 90911 46131 01297 66032 812
22	. (20) 08896 79139 24505 73286 74889 74425 02468 34331 24880 86391 89841 38816 80971 17768 70278 68240 80274 219
23	. (20) 00386 81701 70630 68403 77169 11931 52281 23231 79342 64625 73471 36470 29607 44250 81316 46445 25229 314
24	. (20) 00016 11737 57109 61183 49048 71330 48011 71801 32472 61026 07227 97352 92900 31010 45054 85268 55217 888
25	0. (25) 64469 50284 38447 33961 94853 21920 46872 05298 90441 04289 11894 11716 01240 41802 19410 74208 716
26	. (25) 02479 59626 32247 97460 07494 35458 47956 61742 26555 42472 65842 08142 92355 40069 31515 79777 258
27	. (25) 00091 83689 86379 55461 48425 71683 64739 13397 86168 71943 43179 33634 92309 45928 49315 39991 750
28	. (25) 00003 27988 92370 69837 91015 20417 27312 11192 78077 45426 55113 54772 67582 48068 87475 54999 705
29	. (30) 11309 96288 64477 16931 55876 45769 38316 99244 05014 70865 98440 43709 74071 34050 88103 438
30	0. (30) 00376 99876 28815 90564 38529 21525 64610 56641 46833 82362 19948 01456 99135 71135 02936 781
31	. (30) 00012 16125 04155 35179 49629 97468 56922 92149 72478 51043 94191 87143 77391 47455 96868 928
32	. (35) 38003 90754 85474 35925 93670 89278 84129 67889 95345 12318 49598 24293 48357 99902 154
33	. (35) 01151 63356 20771 95028 05868 81493 29822 11148 18040 76130 86351 46190 71162 36360 671
34	. (35) 00033 87157 53552 11618 47231 43573 33230 06210 24060 02239 14304 45476 19740 06951 784
35	0. (40) 96775 92958 63189 09920 89816 38092 28748 86401 71492 54694 41299 31992 57341 480
36	. (40) 02688 22026 62866 36386 69161 56613 67465 24622 26985 90408 17813 86999 79370 597

The numbers in the parentheses represent the number of zeros between the first tabular figure and the decimal point.

TABLE I.—*Values of the reciprocal of n! to 108 places of decimals at intervals of unity from 1 to 74*—Continued.

n	$\frac{1}{n!}$										
37	0. (40)	00072	65460	17915	30713	15382	74503	07228	79043	84513	13254
		27508	48297	29172	178						
38	. (40)	00001	91196	32050	40281	92510	07223	76506	02080	10118	76664
		58618	64428	87609	794						
39	. (45)	04902	46975	65135	43397	69415	99397	59027	69490	22478	57913
		29857	15066	918							
40	0. (45)	00122	56174	39128	38584	94235	39984	93975	69237	25561	96447
		83246	42876	673							
41	. (45)	00002	98931	08271	42404	51078	91219	14487	21200	90867	36498
		72762	10801	870							
42	. (50)	07117	40673	12914	39311	40267	12249	69552	40258	74678	54113
		38352	425								
43	. (50)	00165	52108	67742	19518	86982	95633	71384	93959	50573	91956
		12519	824								
44	. (50)	00003	76184	28812	32261	79249	61264	40258	74862	71603	95271
		73011	814								
45	0. (55)	08359	65084	71828	03983	32472	54227	97219	17146	75450	48289
		151									
46	. (55)	00181	73154	01561	47912	68097	22917	99939	54720	58161	96701
		938									
47	. (55)	00003	86662	85139	60593	88682	91976	97871	05419	58684	29717
		063									
48	. (60)	08055	47607	07512	37264	22749	52038	98029	57472	58952	439
49	. (60)	00164	39747	08316	57903	35158	15347	73429	17499	44060	254
50	0. (60)	00003	28794	94166	33158	06703	16306	95468	58349	98881	205
51	. (65)	06446	95964	04571	72680	45417	78342	52124	50958	455	
52	. (65)	00123	97999	30857	14859	23950	34198	89463	93287	663	
53	. (65)	00002	33924	51525	60657	72150	00645	26216	30062	031	
54	. (70)	04331	93546	77049	21706	48160	09744	74630	778		
55	0. (70)	00078	76246	30491	80394	66330	18358	99538	741		
56	. (70)	00001	40647	25544	49649	90470	18184	98206	049		
57	. (75)	02467	49570	95607	89306	49441	84179	053			
58	. (75)	00042	54302	94751	86022	52576	58347	915			
59	. (80)	72106	82961	89593	60213	16243	185				
60	0. (80)	01201	78049	36493	22670	21937	386				
61	. (80)	00019	70131	95680	21683	11835	039				
62	. (85)	31776	32188	39059	40513	468					
63	. (85)	00504	38606	16493	00643	071					
64	. (85)	00007	88103	22132	70322	548					
65	0. (90)	12124	66494	34928	039						
66	. (90)	00183	70704	45983	758						
67	. (90)	00002	74189	61880	355						
68	. (95)	04032	20027	652							
69	. (95)	00058	43768	517							
70	0. (100)	83482	407								
71	. (100)	01175	809								
72	. (100)	00016	331								
73	. (105)	224									
74	. (105)	003									

The numbers in the parentheses represent the number of zeros between the first tabular figure and the decimal point.

Table II.—*Values of e^x to 42 significant figures at intervals of unity from 0 to 100.*

x	e^x
0	1. 00000 00000 00000 00000 00000 00000 00000 00000 0
1	2. 71828 18284 59045 23536 02874 71352 66249 77572 5
2	7. 38905 60989 30650 22723 04274 60575 00781 31803 2
3	20. 08553 69231 87667 74092 85296 54581 71789 69879
4	54. 59815 00331 44239 07811 02612 02860 87840 27907
5	148. 41315 91025 76603 42111 55800 40552 27962 3488
6	403. 42879 34927 35122 60838 71805 43388 27960 5900
7	1096. 63315 84284 58599 26372 02382 88121 43244 222
8	2980. 95798 70417 28274 74359 20994 52888 67375 597
9	8103. 08392 75753 84007 70999 66894 32759 96501 148
10	22026. 46579 48067 16516 95790 06452 84244 36635 35
11	59874. 14171 51978 18455 32648 57922 57781 61426 11
12	1 62754. 79141 90039 20808 00520 48984 86783 17020 9
13	4 42413. 39200 89205 03326 10277 59490 88281 78439 1
14	12 02604. 28416 47767 77749 23677 07678 59449 41249
15	32 69017. 37247 21106 39301 85504 60917 21315 50574
16	88 86110. 52050 78726 36763 02374 07814 50350 80272
17	241 54952. 75357 52982 14775 43518 03858 23879 8676
18	656 59969. 13733 05111 38786 50325 90600 33569 2164
19	1784 82300. 96318 72608 44910 03378 87227 03883 620
20	4851 65195. 40979 02779 69106 83054 15405 58684 639
21	13188 15734. 48321 46972 09998 88374 53027 85091 44
22	35849 12846. 13159 15616 81159 94597 84206 89222 69
23	97448 03446. 24890 26000 34632 68482 29752 77649 39
24	2 64891 22129. 84347 22941 39162 15281 18823 40870 2
25	7 20048 99337. 38587 25241 61351 46612 61579 15223 5
26	19 57296 09428 83876 42697 76397 87609 53427 92036
27	53 20482 40601. 79861 66837 47304 34117 74416 59256
28	144 62570 64291. 47517 36770 47422 99692 88569 0206
29	393 13342 97144. 04207 43886 20580 84352 76857 9694
30	1068 64745 81524. 46214 69904 68650 74140 16500 245
31	2904 88496 65247. 42523 10856 82111 67982 56667 647
32	7896 29601 82680. 69516 09780 22635 10822 42199 562
33	21464 35797 85916. 06462 42977 61531 26088 03692 26
34	58346 17425 27454. 88140 29027 34610 39101 90036 59
35	1 58601 34523 13430. 72812 96446 25774 66012 51762 0
36	4 31123 15471 15195. 22711 34222 92856 92539 07888 6
37	11 71914 23728 02611. 30877 29397 91190 19452 16754
38	31 85593 17571 13756. 22032 86717 01298 64599 95422
39	86 59340 04239 93746. 95360 69327 19264 93424 97019
40	235 38526 68370 19985. 40789 99107 49034 80450 8872
41	639 84349 35300 54949. 22266 34035 15570 81887 9337
42	1739 27494 15205 01047. 39468 13036 11235 22614 798
43	4727 83946 82293 46561. 47445 75627 44280 37081 975
44	12851 60011 43593 08275. 80929 96321 43099 25780 11
45	34934 27105 74850 95348. 03479 72334 06099 53341 17
46	94961 19420 60244 88745. 13364 91171 18323 10181 72
47	2 58131 28861 90067 39623. 28580 02152 73380 43163 7
48	7 01673 59120 97631 73865. 47159 98861 17405 45593 8
49	19 07346 57249 50996 90525. 09984 09538 48447 38819
50	51 84705 52858 70724 64087. 45332 29334 85384 82747
51	140 93490 82426 93879 64492. 14331 23701 68788 6848
52	383 10080 00716 57684 93035. 69548 78619 93898 7056
53	1041 37594 33029 08779 71834. 72933 49379 64398 047
54	2830 75330 32746 93900 44206. 35480 14074 54085 033
55	7694 78526 51420 17138 18274. 55901 29393 99207 077
56	20916 59496 01299 61539 07071. 15721 46737 78152 97
57	56857 19999 33593 22226 40348. 82063 32533 03372 16
58	1 54553 89355 90103 93035 30766. 91117 46200 68363 7
59	4 20121 04037 90514 25495 65934. 30719 16176 84111 1

TABLE II.—*Values of e^x to 42 significant figures at intervals of unity from 0 to 100*—Continued.

x	e^x										
60				11	42007	38981	56842	83662	95718. 31447	65630	19805
61				31	04297	93570	19199	08707	34214. 11071	00372	06295
62				84	38356	66874	14544	89073	32948. 03731	17960	08069
63				229	37831	59469	60987	90993	52840. 26861	36004	6328
64				623	51490	80811	61688	29092	38708. 92846	97448	3139
65				1694	88924	44103	33714	14178	36114. 37197	49489	262
66				4607	18663	43312	91542	67731	84428. 06008	68933	490
67				12523	63170	84221	37805	13521	96074. 43657	67534	89
68				34042	76049	93174	05213	76907	18700. 43505	95373	88
69				92537	81725	58778	76002	42397	91668. 73458	73476	60
70			2	51543	86709	19167	00626	57811	74252. 11296	14074	1
71			6	83767	12297	62743	86675	58928	26677. 71095	59458	4
72			18	58671	74528	41279	80340	37018	12545. 41194	69464	
73			50	52393	63027	61041	94557	03833	21857. 64648	53672	
74			137	33829	79540	17618	77841	88529	80853. 89315	7998	
75			373	32419	96799	00164	02549	08317	26470. 01434	2778	
76			1014	80038	81138	88727	83246	17841	31716. 97577	666	
77			2758	51345	45231	70206	28646	98199	02661. 94334	152	
78			7498	41699	69901	20434	67563	05912	24060. 45470	466	
79			20382	81066	51266	87668	32313	75371	72632. 37469	74	
80			55406	22384	39351	00525	71173	39583	16612. 92485	67	
81		I	50609	73145	85030	54835	25941	30167	67498. 18994	0	
82		4	09399	69621	27454	69666	09142	29327	82904. 32005	4	
83		11	12863	75479	17594	12087	07147	81839	40805. 73408		
84		30	25077	32220	11423	38266	56639	64434	28742. 46903		
85		82	23012	71462	29135	10304	32801	64077	74695. 48629		
86		223	52466	03734	71504	74430	65732	33271	47398. 7754		
87		607	60302	25056	87214	95223	28938	13027	60752. 6138		
88		1651	63625	49940	01855	52832	97962	64858	76706. 963		
89		4489	61281	91743	45246	28424	55796	45316	27776. 598		
90		12204	03294	31784	08020	02710	03513	63697	53970. 75		
91		33174	00098	33574	26257	55516	10785	25919	09603. 01		
92		90176	28405	03429	89314	00995	98217	09052	59128. 75		
93	2	45124	55429	20085	78555	27729	43110	91534	23487. 6		
94	6	66317	62164	10895	83424	48140	50240	87326	26873. 9		
95	18	11239	08288	90232	82193	79875	80988	15925	04790.		
96	49	23458	28601	20583	99754	86205	91133	04494	83780.		
97	133	83347	19204	26950	04617	36408	70611	50290	7672 .		
98	363	79709	47608	80457	92877	43826	76018	57298	9310 .		
99	988	90303	19346	94677	05600	30967	13803	71014	0508 .		
100	2688	11714	18161	35448	41262	55515	80013	58736	111 .		

Table III.—*Values of e^x to 33 significant figures at intervals of 0.1 from 0.0 to 50.0.*

x.	e^x						
0.0	1.00000	00000	00000	00000	00000	00000	00
.1	1.10517	09180	75647	62481	17078	26490	25
.2	1.22140	27581	60169	83392	10719	94639	67
.3	1.34985	88075	76003	10398	37443	13328	01
.4	1.49182	46976	41270	31782	48529	52837	22
0.5	1.64872	12707	00128	14684	86507	87814	16
.6	1.82211	88003	90508	97487	53676	68162	86
.7	2.01375	27074	70476	52162	45493	88583	07
.8	2.22554	09284	92467	60457	95375	31395	08
.9	2.45960	31111	56949	66380	01265	63602	47
1.0	2.71828	18284	59045	23536	02874	71352	66
.1	3.00416	60239	46433	11205	84079	53588	67
.2	3.32011	69227	36547	48953	07674	29601	64
.3	3.66929	66676	19244	22045	74899	16011	49
.4	4.05519	99668	44674	58722	41088	95228	62
1.5	4.48168	90703	38064	82260	20554	60119	28
.6	4.95303	24243	95114	80365	42863	56423	96
.7	5.47394	73917	27199	76079	08626	63009	10
.8	6.04964	74644	12946	08373	10239	53027	72
.9	6.68589	44422	79269	41607	25307	27692	86
2.0	7.38905	60989	30650	22723	04274	60575	01
.1	8.16616	99125	67650	07344	97274	10478	63
.2	9.02501	34994	34120	92647	17771	66888	66
.3	9.97418	24548	14720	73995	76151	56908	86
.4	11.02317	63806	41601	65223	79397	69667	8
2.5	12.18249	39607	03473	43807	01759	51168	0
.6	13.46373	80350	01690	39775	08253	32584	1
.7	14.87973	17248	72834	11186	89930	19468	4
.8	16.44464	67710	97049	87149	80160	10925	0
.9	18.17414	53694	43060	94267	62565	74128	1
3.0	20.08553	69231	87667	74092	85296	54581	7
.1	22.19795	12814	41633	40482	79743	81257	2
.2	24.53253	01971	09348	64356	02637	27964	2
.3	27.11263	89206	57887	42681	83721	10231	2
.4	29.96410	00473	97013	34816	27530	33730	2
3.5	33.11545	19586	92313	75065	32493	50388	6
.6	36.59823	44436	77987	75259	47658	99183	7
.7	40.44730	43600	67390	52889	41892	39039	1
.8	44.70118	44933	00823	03755	78287	29065	3
.9	49.40244	91055	30173	87976	14865	41220	3
4.0	54.59815	00331	44239	07811	02612	02860	9
.1	60.34028	75973	61969	49748	72197	08124	4
.2	66.68633	10409	25141	64502	17346	53992	0
.3	73.69979	36995	95796	91176	19511	70652	5
.4	81.45086	86649	68117	44440	08117	26181	1
4.5	90.01713	13005	21813	55011	54567	45574	4
.6	99.48431	56419	33808	73545	40534	87566	7
.7	109.94717	24521	23498	87972	87004	55366	
.8	121.51041	75187	34880	75704	81162	9788[illegible]	
.9	134.28977	96849	35484	84005	86277	74352	
5.0	148.41315	91025	76603	42111	55800	40552	
.1	164.02190	72999	01743	94514	82613	02021	
.2	181.27224	18751	51179	36998	41338	23353	
.3	200.33680	99747	91684	83525	66156	38620	
.4	221.40641	62041	87087	02509	46801	14279	
5.5	244.69193	22642	20387	91518	89495	11839	
.6	270.42640	74261	52628	15292	10465	31487	
.7	298.86740	09670	60232	67202	80305	55296	
.8	330.29955	99096	48654	12024	52287	64816	
.9	365.03746	78653	28777	31505	32150	83072	

TABLE III.—*Values of e^x to 33 significant figures at intervals of 0.1 from 0.0 to 50.0*—Continued.

x	e^x
6.0	403.42879 34927 35122 60838 71805 43388
.1	445.85777 00825 16931 79233 21972 16812
.2	492.74904 10932 56254 57006 20910 66389
.3	544.57191 01259 29033 05938 86677 33165
.4	601.84503 78720 82056 60929 82761 16979
6.5	665.14163 30443 61840 69396 14942 42634
.6	735.09518 92419 72894 90710 17107 60161
.7	812.40582 51675 43113 47226 72512 95340
.8	897.84729 16504 17697 57784 39706 81908
.9	992.27471 56050 25876 97253 10085 94319
7.0	1096.63315 84284 58599 26372 02382 8812
.1	1211.96707 44925 76721 19815 40043 4583
.2	1339.43076 43944 17829 68735 15152 9872
.3	1480.29992 75845 45222 83730 58693 3122
.4	1635.98442 99959 26540 06633 38342 5709
7.5	1808.04241 44560 63206 90380 14827 7881
.6	1998.19589 51041 17959 25232 48348 4882
.7	2208.34799 18872 08523 98030 94345 1393
.8	2440.60197 76244 99077 24871 55411 2634
.9	2697.28232 82685 08847 21116 61148 7690
8.0	2980.95798 70417 28274 74359 20994 5289
.1	3294.46807 52838 41333 08812 83565 2825
.2	3640.95030 73323 54721 56857 18339 5742
.3	4023.87239 38223 09841 54472 32070 1925
.4	4447.06674 76998 56085 59847 50173 2566
8.5	4914.76884 02991 34375 43137 36763 4783
.6	5431.65959 13629 80321 56806 91897 0967
.7	6002.91221 72610 21980 07565 92099 0448
.8	6634.24400 62778 85158 52737 29275 5448
.9	7331.97353 91559 92905 24450 31452 0296
9.0	8103.08392 75753 84007 70999 66894 3276
.1	8955.29270 34825 11710 77437 86428 2849
.2	9897.12905 87439 15886 85434 02479 7437
.3	10938.01920 81651 83753 33850 61222 010
.4	12088.38073 02169 84397 55833 57238 533
9.5	13359.72682 96618 72275 90175 59729 146
.6	14764.78156 55772 72615 55426 11148 697
.7	16317.60719 80154 32232 76797 34500 972
.8	18033.74492 78285 11245 99526 53348 081
.9	19930.37043 82302 89490 56032 14677 875
10.0	22026.46579 48067 16516 95790 06452 842
.1	24343.00942 44083 88345 98557 99428 153
.2	26903.18607 42975 60998 95889 84543 248
.3	29732.61885 28914 13820 76842 75016 320
.4	32859.62567 44433 12762 26957 08978 804
10.5	36315.50267 42466 37738 91202 69013 166
.6	40134.83743 08757 93109 47683 09703 197
.7	44355.85513 02978 66938 62836 34286 021
.8	49020.80113 63817 18305 10499 68773 316
.9	54176.36379 66987 33990 00463 83753 492
11.0	59874.14171 51978 18455 32648 57922 578
.1	66171.16016 83766 04182 26482 33834 845
.2	73130.44183 34154 97311 60903 28180 212
.3	80821.63754 03135 52465 42612 50238 593
.4	89321.72336 08055 55699 37363 40540 407
11.5	98715.77101 07604 97428 11026 81147 200
.6	1 09097.79927 65075 80429 18173 80085 19
.7	1 20571.71498 64506 07884 32987 03867 70
.8	1 33252.35294 55309 39735 38206 60578 27
.9	1 47266.62524 05526 56665 65566 98194 62

Table III.—*Values of ϵ^x to 33 significant figures at intervals of 0.1 from 0.0 to 50.0*—Continued.

x	ϵ^x						
12.0	1	62754.79141	90039	20808	00520	48984	87
.1	1	79871.86225	37510	99202	55498	70958	42
.2	1	98789.15114	29545	30399	15171	71329	96
.3	2	19695.98867	21377	34715	78951	40139	70
.4	2	42801.61749	83235	41021	99665	43832	72
12.5	2	68337.28652	08744	56956	47967	37871	50
.6	2	96558.56529	82029	28131	06698	18068	83
.7	3	27747.90187	38118	24915	27613	20512	30
.8	3	62217.44961	12478	85014	64554	45272	23
.9	4	00312.19132	98824	57935	63962	48069	30
13.0	4	42413.39200	89205	03326	10277	59490	88
.1	4	88942.41461	54600	59140	29689	39772	44
.2	5	40364.93724	66919	42887	77702	08966	51
.3	5	97195.61379	28162	51018	72789	72271	12
.4	6	60003.22476	61566	27675	08247	66901	27
13.5	7	29416.36984	77013	31861	08259	40363	04
.6	8	06129.75912	39902	17000	49212	27173	34
.7	8	90911.16597	91609	45513	21710	16782	56
.8	9	84609.11122	90349	84647	14285	05695	62
.9	10	88161.35540	26400	42869	04190	82607	5
14.0	12	02604.28416	47767	77749	23677	07678	6
.1	13	29083.28081	20933	72415	65547	31032	3
.2	14	68864.18965	40950	11264	71279	19631	2
.3	16	23345.98500	84583	73176	94920	55661	4
.4	17	94074.77260	62144	46062	26766	69215	2
14.5	19	82759.26353	75687	67141	76278	73256	4
.6	21	91287.87560	68098	30730	21834	00372	8
.7	24	21747.63325	24135	50747	88825	38372	5
.8	26	76445.05518	90966	65944	60323	31294	3
.9	29	57929.23882	23613	37256	83192	42565	6
15.0	32	69017.37247	21106	39301	85504	60917	2
.1	36	12822.93074	02438	44330	52318	26886	2
.2	39	92786.83521	09471	82558	38605	78417	8
.3	44	12711.89235	04420	61860	72912	49413	9
.4	48	76800.85327	22664	04847	12229	15576	7
15.5	53	89698.47628	30123	67815	21079	20761	8
.6	59	56538.01318	46158	94525	78083	82516	5
.7	65	82992.58458	37360	04428	51377	35395	1
.8	72	75331.95838	95879	21060	75789	28904	5
.9	80	40485.29975	85202	66729	31241	77682	7
16.0	88	86110.52050	78726	36763	02374	07814	5
.1	98	20670.92207	13565	82889	22079	08745	3
.2	108	53519.89906	44180	45529	12596	65383	
.3	119	94994.55120	13332	33724	02003	53020	
.4	132	56519.14046	35683	00166	44194	17466	
16.5	146	50719.42895	35169	10097	65773	23551	
.6	161	91549.04176	52861	89444	22585	06037	
.7	178	94429.11955	46139	05552	62473	92240	
.8	197	76402.65849	77754	61390	92622	74676	
.9	218	56305.08232	56648	96058	58443	63455	
17.0	241	54952.75357	52982	14775	43518	03858	
.1	266	95351.31074	27049	13394	12187	57749	
.2	295	02925.91644	54583	71110	68906	11219	
.3	326	05775.72099	58447	95506	06223	13988	
.4	360	34955.08814	16391	55271	54298	50110	
17.5	398	24784.39757	62250	21870	67634	98518	
.6	440	13193.53483	40439	30710	38742	44398	
.7	486	42101.50633	36988	59843	73758	07283	
.8	537	57835.97888	36562	28073	19655	81474	
.9	594	11596.94254	29315	75595	99097	66876	

TABLE III.—*Values of e^x to 33 significant figures at intervals of 0.1 from 0.0 to 50.0*—Continued.

x	e^x						
18.0		656	59969.13733	05111	38786	50325	90600
.1		725	65488.37232	22497	75110	99891	90593
.2		801	97267.40504	71134	14452	49662	99965
.3		886	31687.64519	41289	61081	78952	16349
.4		979	53163.60543	32304	45541	27301	00064
18.5		1082	54987.75023	07572	48748	04460	1217
.6		1196	40264.19819	05133	97759	51385	3688
.7		1322	22940.62272	72454	49131	49731	9827
.8		1461	28948.67868	13129	20356	77145	8982
.9		1614	97464.36864	74215	13410	14609	5743
19.0		1784	82300.96318	72608	44910	03378	8723
.1		1972	53448.41573	97114	12668	18600	1486
.2		2179	98774.67921	04573	69563	01720	4717
.3		2409	25905.95158	92662	02664	76985	1347
.4		2662	64304.66872	50454	28992	02822	0165
19.5		2942	67566.04150	88065	66680	80045	3345
.6		3252	15956.12198	05562	88545	56147	9971
.7		3594	19216.80017	87860	03058	99120	6792
.8		3972	19665.80508	38215	53744	05532	4200
.9		4389	95622.73550	64203	80154	45375	0896
20.0		4851	65195.40979	02779	69106	83054	1541
.1		5361	90464.42938	89023	64651	69867	1124
.2		5925	82107.83683	56144	86124	27127	3255
.3		6549	04512 15323	80392	40495	98782	8846
.4		7237	81420.94827	82113	22801	67333	7645
20.5		7999	02177.47550	54067	04598	83728	3990
.6		8840	28623.85131	39326	49420	01192	2695
.7		9770	02725.82690	79801	12264	26784	8714
.8		10797	54999.46453	41371	25566	62697	510
.9		11933	13824.05498	96018	57459	21390	201
21.0		13188	15734.48321	46972	09998	88374	530
.1		14575	16796.05142	39203	84629	61823	210
.2		16108	05175.60282	86330	43100	62026	135
.3		17802	15034.76198	29093	45688	56781	729
.4		19674	41884.33997	16024	55721	30300	926
21.5		21743	59553.57648	85454	85310	20243	562
.6		24030	38944.05268	31647	45191	75991	160
.7		26557	68755.97023	86819	92208	74387	373
.8		29350	78394.23224	92632	94732	06188	947
.9		32437	63283.57765	25326	80093	80230	715
22.0		35849	12846.13159	15616	81159	94597	842
.1		39619	41421.38043	39369	91055	46827	187
.2		43786	22438.02895	04595	53310	41691	167
.3		48391	26179.74308	56773	45193	39107	005
.4		53480	61522.75056	74038	45957	13828	733
22.5		59105	22063.02329	06142	72278	94443	044
.6		65321	37094.69782	08990	20985	63179	874
.7		72191	27949.94318	43117	28947	94442	179
.8		79783	70264.14427	69362	61695	05249	058
.9		88174	62789.57177	77864	45437	53886	624
23.0		97448	03446.24890	26000	34632	68482	298
.1	1	07696	73371.15763	45779	38536	25530	80
.2	1	19023	29806.97713	79397	34848	01024	86
.3	1	31541	08760.01606	93214	92804	01732	96
.4	1	45375	38454.77387	81109	95934	22322	54
23.5	1	60664	64720.62247	86090	61991	59775	50
.6	1	77561	89565.52034	81110	48593	83852	42
.7	1	96236	24323.65135	78359	05185	09236	20
.8	2	16874	58909.74138	08217	35308	44175	78
.9	2	39683	48874.00676	57400	68251	23095	68

TABLE III.—*Values of e^x to 33 significant figures at intervals of 0.1 from 0.0 to 50.0.*—Continued.

x	e^x						
24.0	2	64891	22129.84347	22941	39162	15281	19
.1	2	92750	07423.25706	46440	91249	77678	54
.2	3	23538	86830.63240	94606	10181	71913	49
.3	3	57565	74811.92562	51762	98574	72157	75
.4	3	95171	26612.13642	04797	38119	69976	30
24.5	4	36731	79097.64641	45304	17828	87281	52
.6	4	82663	27438.62807	18527	03687	39155	11
.7	5	33425	41407.48840	78591	38870	16973	92
.8	5	89526	25459.80221	23469	39592	04239	87
.9	6	51527	27202.37940	92103	46415	77990	24
25.0	7	20048	99337.38587	25241	61351	46612	62
.1	7	95777	20706.64333	60674	37733	94535	75
.2	8	79469	82651.72848	99796	19698	43300	30
.3	9	71964	47559.19382	99044	89837	30276	11
.4	10	74186	87182.68578	47334	69615	59214	4
25.5	11	87160	09132.16965	09652	01023	04023	3
.6	13	12014	80802.87690	06069	24450	49061	0
.7	14	50000	60991.79992	16792	65970	81555	6
.8	16	02498	50527.33242	01261	84906	43941	9
.9	17	71034	74428.77727	54108	41351	39504	1
26.0	19	57296	09428.83876	42697	76397	87609	5
.1	21	63146	72147.05767	28406	29286	74083	0
.2	23	90646	84809.99645	04520	70140	43285	0
.3	26	42073	37190.92910	83050	91670	72539	2
.4	29	19942	65405.62132	14147	79370	61947	9
26.5	32	27035	70371.15483	07849	19455	52377	3
.6	35	66426	01133.37854	36755	39770	00888	2
.7	39	41510	30919.46297	12378	08766	28766	3
.8	43	56042	56701.72586	52960	40096	29323	2
.9	48	14171	56296.70645	41109	23997	81144	6
27.0	53	20482	40601.79861	66837	47304	34117	7
.1	58	80042	42526.42283	53382	81240	46422	3
.2	64	98451	88545.30248	85133	02409	44687	4
.3	71	81900	03631.65428	08266	99454	22658	5
.4	79	37227	05666.34806	33381	19699	12653	4
27.5	87	71992	51318.76492	83096	93392	27847	5
.6	96	94551	01915.23018	62951	00965	77092	2
.7	107	14135	85016.77547	89905	47059	90553	
.8	118	40951	35391.71069	44133	88465	01118	
.9	130	86275	07869.76518	22787	42513	40126	
28.0	144	62570	64291.47517	36770	47422	99693	
.1	159	83612	47516.40054	90476	75454	54890	
.2	176	64623	67334.23784	68124	17444	60541	
.3	195	22428	36252.86153	64001	24396	78170	
.4	215	75620	07648.18119	79284	16722	73105	
28.5	238	44747	84797.67787	68074	52711	03802	
.6	263	52521	87043.08195	72729	35918	88625	
.7	291	24040	78915.26116	09982	34926	73407	
.8	321	87042	89702.04007	25720	34279	28499	
.9	355	72183	74864.02890	79644	36208	11659	
29.0	393	13342	97144.04207	43886	20580	84353	
.1	434	47963	34436.96185	95024	47682	30942	
.2	480	17425	53781.40567	43880	77859	92347	
.3	530	67462	26525.50089	96447	48663	05030	
.4	586	48615	99163.66652	71853	60117	55890	
29.5	648	16744	77934.32021	79214	42218	51631	
.6	716	33581	33446.16669	80045	98003	56392	
.7	791	67350	84845.35758	16856	01647	30249	
.8	874	93453	81880.23393	20218	01897	03542	
.9	966	95220	68253.50589	75038	08871	22088	

TABLE III.—*Values of e^x to 33 significant figures at intervals of 0.1 from 0.0 to 50.0.*—Continued

x	e^x
30.0	1068 64745 81524. 46214 69904 68650 7414
.1	1181 03809 24255. 46209 01487 22090 7004
.2	1305 24895 28882. 52476 97252 90255 6973
.3	1442 52318 35807. 87724 46546 87919 4322
.4	1594 23467 11433. 85149 18434 29655 0257
30.5	1761 90179 51355. 63141 21609 84760 9319
.6	1947 20262 44891. 01937 17824 87753 5604
.7	2151 99171 21859. 31322 47658 79141 9410
.8	2378 31865 62477. 10567 98775 07647 3359
.9	2628 44861 28017. 22881 21183 58524 8783
31.0	2904 88496 65247. 42523 10856 82111 6798
.1	3210 39438 53582. 96612 04636 67150 6906
.2	3548 03451 02513. 33135 61277 07371 6175
.3	3921 18455 70585. 46635 65946 87032 8266
.4	4333 57913 68684. 45663 26741 42381 5711
31.5	4789 34563 32463. 72707 54403 58901 8061
.6	5293 04551 04764. 87666 92400 05063 9297
.7	5849 71996 62294. 84813 15789 57425 7372
.8	6464 94038 55632. 86150 88365 43609 9270
.9	7144 86410 12173. 08287 18592 40090 3405
32.0	7896 29601 82680. 69516 09780 22635 1082
.1	8726 75671 99064. 03195 78574 92376 1833
.2	9644 55773 59617. 86912 13519 98498 2802
.3	10658 88472 74864. 77539 68973 56901 625
.4	11779 88941 99387. 29527 90142 63008 392
32.5	13018 79120 50632. 93871 26745 74701 261
.6	14387 98942 83349. 67368 39501 94593 527
.7	15901 18748 57756. 68324 09867 00264 957
.8	17573 52997 21476. 94368 79223 01141 844
.9	19421 75425 31483. 77619 09187 05800 909
33.0	21464 35797 85916. 06462 42977 61531 261
.1	23721 78421 31044. 37760 36693 34787 219
.2	26216 62603 71890. 35741 51580 61740 413
.3	28973 85266 63661. 34260 27596 09521 264
.4	32021 05935 14764. 11460 39540 04425 132
33.5	35388 74356 12259. 87392 92482 12571 573
.6	39110 61021 11037. 88087 97251 60053 907
.7	43223 90899 35043. 72041 86857 86658 642
.8	47769 80718 51694. 68936 96145 39196 174
.9	52793 80166 31304. 10421 90549 88835 133
34.0	58346 17425 27454. 88140 29027 34610 391
.1	64482 49496 51084. 44647 43293 22486 580
.2	71264 17816 03972. 25326 59900 26513 914
.3	78759 09720 34327. 18570 84270 17314 623
.4	87042 26376 31269. 08969 92798 21172 377
34.5	96196 57855 44776. 41048 71247 85963 930
.6	1 06313 66103 67882. 10250 93915 64891 44
.7	1 17494 76637 20104. 34037 07747 33623 41
.8	1 29851 79882 04385. 00894 18446 81468 97
.9	1 43508 43171 61583. 15354 06356 61404 02
35.0	1 58601 34523 13430. 72812 96446 25774 66
.1	1 75281 59431 73561. 61212 33756 44114 37
.2	1 93716 12051 34757. 28303 31243 64910 33
.3	2 14089 42275 39307. 66424 27337 78435 43
.4	2 36605 40389 52471. 09566 20224 56329 49
35.5	2 61489 41144 45696. 60738 41656 44430 36
.6	2 88990 49291 32558. 10962 41749 67431 50
.7	3 19383 88836 80768. 63352 22189 34573 53
.8	3 52973 78512 63176. 61523 70828 13371 84
.9	3 90096 36216 46888. 64411 37401 49460 41

TABLE III.—*Values of e^x to 33 significant figures at intervals of 0.1 from 0.0 to 50.0*—Continued.

x	e^x
36.0	4 31123 15471 15195. 22711 34222 92856 93
.1	4 76464 77269 61994. 98745 07283 67350 68
.2	5 26575 01027 13635. 63500 14974 59058 17
.3	5 81955 38753 72964. 47397 44109 37106 06
.4	6 43160 16992 36632. 16821 94211 47485 19
36.5	7 10801 91546 42244. 06486 15833 68420 94
.6	7 85557 60548 35257. 60090 79367 46411 88
.7	8 68175 42005 35355. 65163 79156 01753 73
.8	9 59482 22603 12769. 18144 67304 04477 19
.9	10 60391 85262 02523. 59786 47544 17130 2
37.0	11 71914 23728 02611. 30877 29397 91190 2
.1	12 95165 53352 09485. 45824 65306 72850 0
.2	14 31379 28174 12826. 72968 64500 92801 3
.3	15 81918 75491 64744. 53050 66757 27100 8
.4	17 48290 60269 21254. 81056 07962 69122 2
37.5	19 32159 93044 02836. 20844 22759 20919 7
.6	21 35366 96419 36677. 02922 94101 03254 3
.7	23 59945 46824 63243. 03968 93611 44775 8
.8	26 08143 09975 02543. 50501 04825 61149 7
.9	28 82443 90402 36540. 01941 88916 28577 7
38.0	31 85593 17571 13756. 22032 86717 01298 6
.1	35 20624 93461 64588. 56603 83582 49016 4
.2	38 90892 29119 00887. 26618 70087 82885 6
.3	43 00101 00558 80104. 30725 35475 49330 3
.4	47 52346 57616 37170. 45041 48507 32909 5
38.5	52 52155 22859 25158. 15729 58254 60641 7
.6	58 04529 21585 94036. 20184 79777 28230 9
.7	64 14996 88248 82561. 06182 08507 97636 0
.8	70 89667 99407 19634. 02423 10740 15188 6
.9	78 35294 88586 00468. 95973 47814 57630 8
39.0	86 59340 04239 93746. 95360 69327 19264 9
.1	95 70050 78458 77343. 61342 17376 38971 4
.2	105 76541 81163 33982. 46846 15693 64994
.3	116 88886 42402 83560. 86698 34176 84070
.4	129 18217 34052 53920. 49038 85620 11921
39.5	142 76838 11812 91985. 91758 33666 53263
.6	157 78346 29023 02477. 43715 59460 30821
.7	174 37769 45528 92517. 50602 13224 60008
.8	192 71715 67809 35081. 54052 84561 81318
.9	212 98539 70885 14544. 13549 39399 97850
40.0	235 38526 68370 19985. 40789 99107 49035
.1	260 14095 14517 50670. 00948 59692 83355
.2	287 50021 41450 03765. 77508 14124 91637
.3	317 73687 56135 79105. 26803 06064 59178
.4	351 15355 45283 47072. 98994 83315 42259
40.5	388 08469 62436 20324. 02317 21875 72699
.6	428 89992 00386 70710. 64074 96477 64288
.7	474 00771 83917 09565. 01140 65056 82629
.8	523 85954 53099 08701. 59178 65130 82966
.9	578 95433 46328 43124. 57201 85917 65939
41.0	639 84349 35300 54949. 22266 34035 15571
.1	707 13642 11693 40529. 35962 83009 93650
.2	781 50660 77884 47896. 96629 84417 00348
.3	863 69837 52117 44030. 73926 44719 24967
.4	954 53432 62732 08325. 47163 27987 89991
41.5	1054 92357 77020 81418. 45053 91711 6615
.6	1165 87085 88686 56114. 75075 60769 2286
.7	1288 48656 74535 16481. 07600 60987 0147
.8	1423 99788 26807 42680. 14854 81712 4332
.9	1573 76104 73400 54746. 30325 05193 6633

TABLE III.—*Values of e^x to 33 significant figures at intervals of 0.1 from 0.0 to 50.0*—Continued.

x	e^x
42.0	1739 27494 15205 01047. 39468 13036 1124
.1	1922 19608 39061 80476. 58968 93786 0494
.2	2124 35521 07720 08071. 35617 33172 1242
.3	2347 77559 86076 86074. 87538 89370 8665
.4	2594 69331 37488 59588. 85140 73151 4199
42.5	2867 57959 16805 71559. 56393 30187 8547
.6	3169 16556 99926 08018. 31712 57371 5547
.7	3502 46962 25224 63704. 82268 16568 0900
.8	3870 82756 82552 18196. 19954 73844 1926
.9	4277 92605 73211 46063. 77834 40083 1416
43.0	4727 83946 82293 46561. 47445 75627 4428
.1	5225 07068 56173 08600. 14746 75513 6950
.2	5774 59616 66338 34529. 15450 27874 7849
.3	6381 91574 69948 30360. 69842 21728 8812
.4	7053 10768 51877 09178. 99594 13708 6927
43.5	7794 88949 57253 06399. 59362 37456 5717
.6	8614 68518 02889 58825. 49151 03780 4263
.7	9520 69952 96326 24602. 82213 30778 8026
.8	10522 00023 98864 74241. 01297 41703 5134
.9	11628 60866 51075 19279. 13929 36648 7541
44.0	12851 60011 43593 08275. 80929 96321 4310
.1	14203 21469 71275 74732. 70231 52164 8956
.2	15696 97982 64500 13186. 98266 35129 2425
.3	17347 84560 58126 80995. 55797 31144 1650
.4	19172 33445 48105 90107. 58191 10264 4550
44.5	21188 70647 10763 90948. 92010 98100 2595
.6	23417 14218 34749 10750. 51432 14409 6533
.7	25879 94452 56189 42730. 17190 82453 0870
.8	28601 76205 11251 17788. 91534 78007 8747
.9	31609 83562 46231 64724. 23096 09490 4714
45.0	34934 27105 74850 95348. 03479 72334 061
.1	38608 34041 69043 28227. 13228 24629 719
.2	42668 81502 39272 88395. 41377 10669 376
.3	47156 33347 31937 07791. 30039 79777 515
.4	52115 80835 76508 83053. 61114 98817 745
45.5	57596 87576 88795 35865. 26851 87821 549
.6	63654 39207 17816 19332. 12347 80923 907
.7	70348 98292 55181 17701. 87293 63572 117
.8	77747 65004 54829 17232. 44579 79926 261
.9	85924 44177 89905 22451. 73279 26071 196
46.0	94961 19420 60244 88745. 13364 91171 183
.1	1 04948 35018 22319 54147. 90908 66367 30
.2	1 15985 86452 14218 49473. 77180 46271 79
.3	1 28184 20437 69374 71210. 96429 62509 89
.4	1 41665 45483 40564 33686. 31382 49717 92
46.5	1 56564 54077 85583 41656. 97621 59025 54
.6	1 73030 57727 03314 92805. 23175 74257 30
.7	1 91228 36193 70115 42258. 16763 34724 90
.8	2 11340 02432 40292 75711. 22343 53462 67
.9	2 33566 84870 83171 34964. 40927 95531 96
47.0	2 58131 28861 90067 39623. 28580 02152 73
.1	2 85279 19322 71176 49551. 26703 03268 27
.2	3 15282 26788 66936 88624. 40776 84037 26
.3	3 48440 79345 33095 38552. 71985 37705 33
.4	3 85086 63159 58012 11272. 71553 28309 79
47.5	4 25586 54617 93903 18634. 74249 65238 53
.6	4 70345 87396 17208 02496. 15995 73329 63
.7	5 19812 58133 93678 24359. 56886 42328 44
.8	5 74481 74774 61013 95112. 69333 79245 43
.9	6 34900 52057 42614 89472. 44726 63512 18

Table III.—*Values of ϵ^x to 33 significant figures at intervals of 0.1 from 0.0 to 50.0*—Continued.

x	ϵ^x						
48.0	7	01673	59120	97631	73865. 47159	98861	17
.1	7	75469	24698	67306	37991. 61667	18811	74
.2	8	57026	05963	17562	39656. 99046	28407	19
.3	9	47160	27713	79827	71146. 82358	95213	41
.4	10	46773	99304	93692	57138. 00626	58991	2
48.5	11	56864	17491	60830	07521. 04037	92992	4
.6	12	78532	64228	08340	57448. 38688	41765	1
.7	14	12997	09405	91929	46708. 68045	97513	8
.8	15	61603	29567	96204	89284. 58072	41039	8
.9	17	25838	54795	62031	90250. 54761	43153	1
49.0	19	07346	57249	50996	90525. 09984	09538	5
.1	21	07943	96261	28491	13381. 14667	72236	0
.2	23	29638	36441	28610	87238. 76943	67133	6
.3	25	74648	56998	24118	27728. 21369	31403	7
.4	28	45426	72380	96153	72548. 47735	30218	9
49.5	31	44682	86466	96548	51738. 26948	84493	5
.6	34	75412	04860	37200	08737. 07381	37971	6
.7	38	40924	32444	65405	26989. 78298	37803	3
.8	42	44877	86190	76698	38363. 39647	78017	9
.9	46	91315	56376	34916	34374. 10241	66244	6
50.0	51	84705	52858	70724	64087. 45332	29334	9

TABLE IV.—*Values of e^x to 62 places of decimals at decimal intervals from 1×10^{-10} to 9×10^{-1}.*

x	e^x												
1×10^{-10}	1.00000	00001	00000	00000	50000	00000	16666	66666	70833	33333	34166	66666	67
2	1.00000	00002	00000	00002	00000	00001	33333	33334	00000	00000	26666	66666	76
3	1.00000	00003	00000	00004	50000	00004	50000	00003	37500	00002	02500	00001	01
4	1.00000	00004	00000	00008	00000	00010	66666	66677	33333	33341	86666	66672	36
5	1.00000	00005	00000	00012	50000	00020	83333	33359	37500	00026	04166	66688	37
6	1.00000	00006	00000	00018	00000	00036	00000	00054	00000	00064	80000	00064	80
7	1.00000	00007	00000	00024	50000	00057	16666	66766	70833	33473	39166	66830	07
8	1.00000	00008	00000	00032	00000	00085	33333	33504	00000	00273	06666	67030	76
9	1.00000	00009	00000	00040	50000	00121	50000	00273	37500	00492	07500	00738	11
1×10^{-9}	1.00000	00010	00000	00050	00000	00166	66666	67083	33333	34166	66666	68055	56
2	1.00000	00020	00000	00200	00000	01333	33333	40000	00000	26666	66667	55555	56
3	1.00000	00030	00000	00450	00000	04500	00000	33750	00002	02500	00010	12500	00
4	1.00000	00040	00000	00800	00000	10666	66667	73333	33341	86666	66723	55555	56
5	1.00000	00050	00000	01250	00000	20833	33335	93750	00026	04166	66883	68055	57
6	1.00000	00060	00000	01800	00000	36000	00005	40000	00064	80000	00648	00000	06
7	1.00000	00070	00000	02450	00000	57166	66676	67083	33473	39166	68300	68055	72
8	1.00000	00080	00000	03200	00000	85333	33350	40000	00273	06666	70307	55555	98
9	1.00000	00090	00000	04050	00001	21500	00027	33750	00492	07500	07381	12500	95
1×10^{-8}	1.00000	00100	00000	05000	00001	66666	66708	33333	34166	66666	80555	55557	54
2	1.00000	00200	00000	20000	00013	33333	34000	00000	26666	66675	55555	55809	52
3	1.00000	00300	00000	45000	00045	00000	03375	00002	02500	00101	25000	04339	29
4	1.00000	00400	00000	80000	00106	66666	77333	33341	86666	67235	55555	88063	49
5	1.00000	00500	00001	25000	00208	33333	59375	00026	04166	68836	80557	10565	48
6	1.00000	00600	00001	80000	00360	00000	54000	00064	80000	06480	00005	55428	58
7	1.00000	00700	00002	45000	00571	66667	66708	33473	39166	83006	80571	89569	46
8	1.00000	00800	00003	20000	00853	33335	04000	00273	06667	03075	55597	16571	47
9	1.00000	00900	00004	05000	01215	00002	73375	00492	07500	73811	25094	90017	96
1×10^{-7}	1.00000	01000	00005	00000	01666	66670	83333	34166	66668	05555	55753	96825	64
2	1.00000	02000	00020	00000	13333	33400	00000	26666	66755	55555	80952	38158	73
3	1.00000	03000	00045	00000	45000	00337	50002	02500	01012	50004	33928	58770	09
4	1.00000	04000	00080	00001	06666	67733	33341	86666	72355	55588	06349	36888	89
5	1.00000	05000	00125	00002	08333	35937	50026	04166	88368	05710	56548	58785	97
6	1.00000	06000	00180	00003	60000	05400	00064	80000	64800	00555	42861	30857	17
7	1.00000	07000	00245	00005	71666	76670	83473	39168	30068	07189	56958	74206	71
8	1.00000	08000	00320	00008	53333	50400	00273	06670	30755	59716	57184	46730	53
9	1.00000	09000	00405	00012	15000	27337	50492	07507	38112	59490	01892	47699	73
1×10^{-6}	1.00000	10000	00500	00016	66667	08333	34166	66680	55555	75396	82787	69844	03
2	1.00000	20000	02000	00133	33340	00000	26666	67555	55580	95238	73015	88712	52
3	1.00000	30000	04500	00450	00033	75002	02500	10125	00433	92873	41518	39955	37
4	1.00000	40000	08000	01066	66773	33341	86667	23555	58806	35083	17467	54144	91
5	1.00000	50000	12500	02083	33593	75026	04168	83680	71056	55730	71730	41021	76
6	1.00000	60000	18000	03600	00540	00064	80006	48000	55542	89880	00277	71445	23
7	1.00000	70000	24500	05716	67667	08473	39183	00682	18957	08742	07709	26023	06
8	1.00000	80000	32000	08533	35040	00273	06703	07559	71657	55895	91000	27103	65
9	1.00000	90000	40500	12150	02733	75492	07573	81134	49002	85334	23622	70826	94
1×10^{-5}	1.00001	00000	50000	16666	70833	34166	66805	55575	39685	01984	40255	75947	97
2	1.00002	00002	00001	33334	00000	26666	75555	58095	24444	44585	53820	10587	14
3	1.00003	00004	50004	50003	37502	02501	01250	43393	01986	66138	40912	95086	65
4	1.00004	00008	00010	66677	33341	86672	35558	80636	54603	89700	46532	73294	25
5	1.00005	00012	50020	83359	37526	04188	36821	05664	45007	86247	45526	60354	36
6	1.00006	00018	00036	00054	00064	80064	80055	54327	37170	62873	80580	51744	80
7	1.00007	00024	50057	16766	70973	39330	06968	95837	42177	17672	36470	63988	79
8	1.00008	00032	00085	33504	00273	07030	75971	66130	39100	02976	67262	81645	47
9	1.00009	00040	50121	50273	37992	08238	12199	01246	20911	38661	76502	68546	00
1×10^{-4}	1.00010	00050	00166	67083	34166	68055	57539	70734	15454	17217	83810	34635	39
2	1.00020	00200	01333	40000	26667	55558	09530	15887	12550	26506	33402	81249	62
3	1.00030	00450	04500	33752	02510	12543	39448	44292	42698	70509	44536	57909	05
4	1.00040	00800	10667	73341	86723	55880	65117	53256	02089	23421	81273	13638	99

TABLE IV.—*Values of e^x to 62 places of decimals at decimal intervals from 1×10^{-10} to 9×10^{-1}*—Continued.

x	e^x
5×10^{-4}	1.00050 01250 20835 93776 04383 69605 75164 84877 16767 77200 56676 54052 53
6	1.00060 01800 36005 40064 80648 05554 70231 34873 80662 32142 77422 18779 52
7	1.00070 02450 57176 67223 40800 84397 12431 78087 29972 04988 70829 20697 54
8	1.00080 03200 85350 40273 10307 97169 87567 14849 70787 02383 19576 98136 83
9	1.00090 04051 21527 34242 14882 07410 85590 92409 60364 43887 07063 86170 96
1×10^{-3}	1.00100 05001 66708 34166 80557 53993 05831 15630 76200 58070 14602 28514 67
2	1.00200 20013 34000 26675 55809 58731 56994 71412 62360 35588 16507 84254 33
3	1.00300 45045 03377 02601 29340 91348 90020 53318 72719 56193 06400 58163 87
4	1.00400 80106 77341 87235 88079 75325 86225 67866 79584 56844 65158 24520 10
5	1.00501 25208 59401 06338 35662 41124 06858 07348 75538 59395 63607 58053 70
6	1.00601 80360 54064 86485 55845 42073 81480 76633 97023 13120 72381 28209 31
7	1.00702 45572 66848 55523 16000 31941 33738 72606 26958 32484 35735 15364 69
8	1.00803 20855 04273 43117 20736 14608 63184 74612 48193 62989 84773 19670 85
9	1.00904 06217 73867 81406 25704 81311 87427 40577 45096 80755 75335 36793 79
1×10^{-2}	1.01005 01670 84168 05754 21654 56902 86003 38073 62201 52429 25151 64404 03
2	1.02020 13400 26755 81016 01439 20483 15143 53035 08991 19392 55772 74241 06
3	1.03045 45339 53516 85561 24399 53831 19813 29050 25142 98822 33256 69945 48
4	1.04081 07741 92388 22675 70447 57916 85474 40829 77050 31231 20352 33957 19
5	1.05127 10963 76024 03969 75176 36335 64522 01748 21296 05506 25287 83938 48
6	1.06183 65465 45359 62222 46848 77168 37232 84282 60420 33007 90597 72946 22
7	1.07250 81812 54216 47905 31039 49889 11460 55749 58973 09301 36313 68582 00
8	1.08328 70676 74958 55443 59877 58674 88850 01987 13572 83659 39689 77149 14
9	1.09417 42837 05210 35787 28976 23544 88601 18465 19908 74708 51134 95537 27
1×10^{-1}	1.10517 09180 75647 62481 17078 26490 24666 82245 47194 73751 87187 92863 29
2	1.22140 27581 60169 83392 10719 94639 67417 03075 80941 52050 36412 73425 10
3	1.34985 88075 76003 10398 37443 13328 00733 03782 99697 35936 58030 49917 99
4	1.49182 46976 41270 31782 48529 52837 22228 06432 82773 93742 52815 95633 15
5	1.64872 12707 00128 14684 86507 87814 16357 16537 76100 71014 80115 75079 31
6	1.82211 88003 90508 97487 53676 68162 86451 33822 38808 54643 53863 20547 48
7	2.01375 27074 70476 52162 45493 88583 06527 00175 42394 14586 73115 68989 30
8	2.22554 09284 92467 60457 95375 31395 07675 70536 34135 04848 45961 18583 96
9	2.45960 31111 56949 66380 01265 63602 47069 54217 72306 44008 30207 48545 74

TABLE V.—*Values of e^{-x} ranging from 52 to 62 places of decimals at intervals of unity from 0 to 100.*

x	e^{-x}										
0	1. 00000	00000	00000	00000	00000	00000	00000	00000	00000	00000	00
1	0. 36787	94411	71442	32159	55237	70161	46086	74458	11131	03176	78
2	. 13533	52832	36612	69189	39994	94972	48440	34076	31545	90957	59
3	. 04978	70683	67863	94297	93424	15650	06177	66316	99592	18842	32
4	. 01831	56388	88734	18029	37180	21273	24124	22119	12067	55347	56
5	0. 00673	79469	99085	46709	66360	48423	14842	42488	49585	02735	51
6	. 00247	87521	76666	35842	30451	67430	81666	78915	06479	58553	39
7	. 00091	18819	65554	51620	80031	36084	40928	26264	73724	52743	61
8	. 00033	54626	27902	51183	88213	89125	78086	10193	10900	13372	03
9	. 00012	34098	04086	67954	94976	36690	73003	38260	72152	83228	89
10	0. 00004	53999	29762	48485	15355	91515	56055	06102	37918	08886	66
11	. 00001	67017	00790	24565	93126	35517	36058	08790	77938	04695	93
12	. 00000	61442	12353	32820	97586	82308	17880	55323	11223	98931	49
13	. 00000	22603	29406	98105	43257	85277	29053	86894	69353	14242	27
14	. 00000	08315	28719	10356	78840	63985	14256	52622	94607	65836	50
15	0. 00000	03059	02320	50182	57883	71479	49770	22896	39370	82078	08
16	. 00000	01125	35174	71925	91145	13775	17906	01271	91637	94080	07
17	. 00000	00413	99377	18785	16665	96510	27718	95528	06229	36694	37
18	. 00000	00152	29979	74471	26284	36136	62923	35174	31862	17484	33
19	. 00000	00056	02796	43753	72675	40012	98281	62064	63079	78387	37
20	0. 00000	00020	61153	62243	85578	27965	94038	01558	20976	37580	73
21	. 00000	00007	58256	04279	11906	72794	17432	41268	12644	29803	62
22	. 00000	00002	78946	80928	68924	80771	89130	30644	29320	76931	73
23	. 00000	00001	02618	79631	70189	03039	27527	84061	24977	59833	84
24	. 00000	00000	37751	34544	27909	77516	44969	54752	34067	79168	61
25	0. 00000	00000	13887	94386	49640	20594	66176	37460	86856	91039	98
26	. 00000	00000	05109	08902	80633	24719	87440	01934	79215	76659	41
27	. 00000	00000	01879	52881	65390	83294	75827	04184	22192	62122	87
28	. 00000	00000	00691	44001	06940	20300	94125	84658	74140	92711	82
29	. 00000	00000	00254	36656	47376	92291	03033	85614	85768	16666	03
30	0. 00000	00000	00093	57622	96884	01746	04915	83222	33787	06744	96
31	. 00000	00000	00034	42477	10846	99764	58392	38933	28515	57284	62
32	. 00000	00000	00012	66416	55490	94175	72312	09041	55965	09638	21
33	. 00000	00000	00004	65888	61451	03397	36418	42455	43610	16841	14
34	. 00000	00000	00001	71390	84315	42012	96630	27203	42576	04924	12
35	0. 00000	00000	00000	63051	16760	14698	93856	39021	19224	65427	61
36	. 00000	00000	00000	23195	22830	24356	93883	12263	60973	80800	41
37	. 00000	00000	00000	08533	04762	57440	65794	27804	98229	41244	17
38	. 00000	00000	00000	03139	13279	20480	29628	70896	46522	31919	65
39	. 00000	00000	00000	01154	82241	73015	78598	62624	42063	32386	87
40	0. 00000	00000	00000	00424	83542	55291	58899	53292	34782	85865	80
41	. 00000	00000	00000	00156	28821	89334	98876	80908	82995	10583	41
42	. 00000	00000	00000	00057	49522	26429	35598	06664	38088	05734	23
43	. 00000	00000	00000	00021	15131	03759	10804	86631	40100	70226	51
44	. 00000	00000	00000	00007	78113	22411	33796	51571	33167	29279	90
45	. 00000	00000	00000	00002	86251	85805	49393	64447	01216	29183	94
46	. 00000	00000	00000	00001	05306	17357	55381	23787	63324	44942	81
47	. 00000	00000	00000	00000	38739	97628	68718	71129	31477	49726	91
48	. 00000	00000	00000	00000	14251	64082	74093	51062	85321	02803	41
49	. 00000	00000	00000	00000	05242	88566	33634	63937	17180	53028	32
50	0. 00000	00000	00000	00000	01928	74984	79639	17783	01734	28165	27

TABLE V.—*Values of e^{-x} ranging from 52 to 62 places of decimals at intervals of unity from 0 to 100*—Continued.

x	$e^{-x}\times10^{20}$								
50	0.01928	74984	79639	17783	01734	28165	27012	57475	28
51	.00709	54741	62284	70413	89832	69387	80807	34876	89
52	.00261	02790	69667	70480	47026	95315	33186	48093	16
53	.00096	02680	05450	86760	30230	76967	00074	90907	63
54	.00035	32628	57220	08070	29735	39281	01772	08837	39
55	0.00012	99581	42500	75030	73600	71340	60714	85530	28
56	.00004	78089	28838	85469	08127	71770	42317	96289	39
57	.00001	75879	22024	24311	64895	58751	28803	43631	78
58	.00000	64702	34925	64546	03261	54039	55292	64893	76
59	.00000	23802	66408	69440	06058	94324	58880	24963	31
60	0.00000	08756	51076	26965	20338	48873	28007	39166	04
61	.00000	03221	34028	59925	16089	00124	77758	48943	75
62	.00000	01185	06486	42339	81006	28503	07390	97280	99
63	.00000	00435	96100	00063	08097	36231	24815	88845	96
64	.00000	00160	38108	90548	63785	29760	87034	14233	54
65	0.00000	00059	00090	54159	70613	91401	26029	55584	23
66	.00000	00021	70522	01130	36394	11986	56925	95727	06
67	.00000	00007	98490	42456	86978	80839	26942	66474	24
68	.00000	00002	93748	21117	10802	94660	88806	42392	87
69	.00000	00001	08063	92777	07278	49453	66496	16247	34
70	0.00000	00000	39754	49735	90864	68077	89099	75379	48
71	.00000	00000	14624	86227	25123	09468	26378	73083	16
72	.00000	00000	05380	18616	00211	38413	81818	72704	54
73	.00000	00000	01979	25987	79469	04553	74919	15336	02
74	.00000	00000	00728	12901	78321	64383	42969	73716	88
75	0.00000	00000	00267	86369	61808	07794	43444	15201	08
76	.00000	00000	00098	54154	68611	12580	28938	09797	36
77	.00000	00000	00036	25140	91914	35592	24240	83319	73
78	.00000	00000	00013	33614	81550	22613	41453	01407	91
79	.00000	00000	00004	90609	47306	49280	56613	53873	52
80	0.00000	00000	00001	80485	13878	45415	17231	21283	57
81	.00000	00000	00000	66396	77199	58073	44007	02255	27
82	.00000	00000	00000	24426	00737	74052	76794	40802	61
83	.00000	00000	00000	08985	82594	40493	80669	66884	82
84	.00000	00000	00000	03305	70062	67607	34298	45509	64
85	0.00000	00000	00000	01216	09929	92528	25564	41682	63
86	.00000	00000	00000	00447	37793	06181	12073	46276	56
87	.00000	00000	00000	00164	58114	31082	27365	11660	34
88	.00000	00000	00000	00060	54601	89540	11858	84531	86
89	.00000	00000	00000	00022	27363	56179	57437	39222	91
90	0.00000	00000	00000	00008	19401	26239	90515	43036	11
91	.00000	00000	00000	00003	01440	87850	65374	55326	31
92	.00000	00000	00000	00001	10893	90193	12136	37945	96
93	.00000	00000	00000	00000	40795	58667	17756	01577	01
94	.00000	00000	00000	00000	15007	85762	70739	48875	45
95	0.00000	00000	00000	00000	05521	08227	70285	32731	72
96	.00000	00000	00000	00000	02031	09266	27348	10925	69
97	.00000	00000	00000	00000	00747	19723	37342	99016	06
98	.00000	00000	00000	00000	00274	87850	07910	21493	00
99	.00000	00000	00000	00000	00101	12214	92610	44852	99
100	0.00000	00000	00000	00000	00037	20075	97602	08359	63

TABLE VI.—*Values of e^{-x} ranging from 33 to 48 places of decimals at intervals of 0.1 from 0.0 to 50.0.*

x	e^{-x}						
0.0	1.00000	00000	00000	00000	00000	00000	000
.1	0.90483	74180	35959	57316	42490	59446	437
.2	.81873	07530	77981	85866	99355	08619	039
.3	.74081	82206	81717	86606	68737	79317	817
.4	.67032	00460	35639	30074	44329	25147	826
0.5	0.60653	06597	12633	42360	37995	34991	180
.6	.54881	16360	94026	43262	84589	17232	568
.7	.49658	53037	91409	51470	48000	93397	529
.8	.44932	89641	17221	59143	01023	85015	563
.9	.40656	96597	40599	11188	34542	39645	626
1.0	0.36787	94411	71442	32159	55237	70161	461
.1	.33287	10836	98079	55328	88469	06431	316
.2	.30119	42119	12202	09664	49776	07083	222
.3	.27253	17930	34012	60312	23331	67563	350
.4	.24659	69639	41606	47693	98612	39833	768
1.5	0.22313	01601	48429	82893	32804	70764	013
.6	.20189	65179	94655	40848	51792	67643	350
.7	.18268	35240	52734	65022	39008	37758	940
.8	.16529	88882	21586	53829	68047	20432	214
.9	.14956	86192	22635	05264	10120	69103	735
2.0	0.13533	52832	36612	69189	39994	94972	484
.1	.12245	64282	52981	91021	86473	76072	626
.2	.11080	31583	62333	88333	41444	25849	939
.3	.10025	88437	22803	73372	99406	93797	987
.4	.09071	79532	89412	50337	51722	20079	691
2.5	0.08208	49986	23898	79516	95286	74467	160
.6	.07427	35782	14333	88042	82105	70169	975
.7	.06720	55127	39749	76512	65517	00855	966
.8	.06081	00626	25217	96499	56213	88183	941
.9	.05502	32200	56407	22902	99465	30834	175
3.0	0.04978	70683	67863	94297	93424	15650	062
.1	.04504	92023	93557	80606	83350	92178	335
.2	.04076	22039	78366	21516	60792	62144	425
.3	.03688	31674	01240	00544	56037	04741	515
.4	.03337	32699	60326	07948	24001	31470	948
3.5	0.03019	73834	22318	50073	97862	92363	620
.6	.02732	37224	47292	56080	15630	62435	553
.7	.02472	35264	70339	39120	27573	82983	403
.8	.02237	07718	56165	59577	85833	22540	823
.9	.02024	19114	45804	38847	20275	43743	654
4.0	0.01831	56388	88734	18029	37180	21273	241
.1	.01657	26754	01761	24754	19836	98083	451
.2	.01499	55768	20477	70621	19843	60228	729
.3	.01356	85590	12200	93175	72305	74525	767
.4	.01227	73399	03068	44117	89393	86236	542
4.5	0.01110	89965	38242	30649	61431	34286	931
.6	.01005	18357	44633	58164	21330	94331	550
.7	.00909	52771	01695	81709	20540	74291	388
.8	.00822	97470	49020	02884	13620	26766	074
.9	.00744	65830	70924	34051	82360	46420	128
5.0	0.00673	79469	99085	46709	66360	48423	148
.1	.00609	67465	65515	63610	71345	64785	425
.2	.00551	65644	20760	77241	79937	54667	303
.3	.00499	15939	06910	21621	22867	25942	075
.4	.00451	65809	42612	66798	16490	18705	780
5.5	0.00408	67714	38464	06699	34647	02684	721
.6	.00369	78637	16482	93082	06926	36441	249
.7	.00334	59654	57471	27276	57324	36020	607
.8	.00302	75547	45375	81474	81920	44595	488
.9	.00273	94448	18768	36923	27755	20842	145

TABLE VI.—*Values of e^{-x} ranging from 33 to 48 places of decimals at intervals of 0.1 from 0.0 to 50.0*—Continued.

x	e^{-x}						
6.0	0.00247	87521	76666	35842	30451	67430	817
.1	.00224	28677	19485	80247	32236	16521	454
.2	.00202	94306	36295	73436	33862	53459	782
.3	.00183	63047	77028	90682	52279	36299	895
.4	.00166	15572	73173	93449	90832	54173	641
6.5	0.00150	34391	92977	57244	73829	03332	168
.6	.00136	03680	37547	89341	68557	63685	588
.7	.00123	09119	02673	48118	46234	76276	674
.8	.00111	37751	47844	80307	87892	19640	468
.9	.00100	77854	29048	51076	14475	35575	166
7.0	0.00091	18819	65554	51620	80031	36084	409
.1	.00082	51049	23265	90427	01462	25456	749
.2	.00074	65858	08376	67936	80906	47515	335
.3	.00067	55387	75193	84423	78367	24317	781
.4	.00061	12527	61129	57255	56702	29776	745
7.5	0.00055	30843	70147	83358	31020	00088	530
.6	.00050	04514	33440	61069	55020	35820	856
.7	.00045	28271	82886	79705	79972	07264	598
.8	.00040	97349	78979	78670	84619	67840	934
.9	.00037	07435	40459	08837	44300	21422	977
8.0	0.00033	54626	27902	51183	88213	89125	781
.1	.00030	35391	38078	86666	08655	09532	209
.2	.00027	46535	69972	14232	76277	89393	761
.3	.00024	85168	27107	95202	08034	74470	637
.4	.00022	48673	24178	84827	27986	33560	122
8.5	0.00020	34683	69010	64417	43689	33430	487
.6	.00018	41057	93667	57912	49547	76189	858
.7	.00016	65858	10987	63341	14921	30507	125
.8	.00015	07330	75095	47660	06434	06463	915
.9	.00013	63889	26482	01144	78477	65082	136
9.0	0.00012	34098	04086	67954	94976	36690	730
.1	.00011	16658	08490	11473	56400	85376	178
.2	.00010	10394	01837	09335	07306	72733	712
.3	.00009	14242	31478	17333	78629	43248	947
.4	.00008	27240	65556	63226	27291	71823	338
9.5	0.00007	48518	29887	70059	14711	89319	355
.6	.00006	77287	36490	85387	29971	88458	992
.7	.00006	12834	95053	22209	55132	40931	438
.8	.00005	54515	99432	17698	18088	77544	465
.9	.00005	01746	82056	17530	21858	33726	590
10.0	0.00004	53999	29762	48485	15355	91515	561
.1	.00004	10795	55225	30070	84235	23804	485
.2	.00003	71703	18684	12670	45551	91140	292
.3	.00003	36330	95185	71899	37288	28297	935
.4	.00003	04324	83008	40363	65062	22204	232
10.5	0.00002	75364	49349	74715	78574	11097	102
.6	.00002	49160	09731	50319	62336	14140	627
.7	.00002	25449	37913	21219	44136	71874	562
.8	.00002	03995	03411	17193	64237	54489	295
.9	.00001	84582	33995	78056	47431	58551	471
11.0	0.00001	67017	00790	24565	93126	35517	361
.1	.00001	51123	23819	85502	79896	87011	424
.2	.00001	36741	96065	68095	33745	88026	747
.3	.00001	23729	24261	78823	05171	26757	815
.4	.00001	11954	84842	59094	36391	53266	479
11.5	0.00001	01300	93598	63071	07289	41355	749
.6	.00000	91660	87736	24761	44734	00846	298
.7	.00000	82938	19160	75736	50913	52973	571
.8	.00000	75045	57915	07686	33506	65463	934
.9	.00000	67904	04807	37947	30051	67517	816

TABLE VI.—*Values of e^{-x} ranging from 33 to 48 places of decimals at intervals of 0.1 from 0.0 to 50.0*—Continued.

x	$e^{-x}\times10^5$						
12.0	0.61442	12353	32820	97586	82308	179	
.1	.55595	13241	65014	42782	64691	467	
.2	.50304	55607	11144	43312	43226	772	
.3	.45517	44463	08323	47633	86288	946	
.4	.41185	88707	53570	92504	35497	759	
12.5	0.37266	53172	07867	09929	24851	476	
.6	.33720	15234	13918	32115	08762	687	
.7	.30511	25558	03642	02182	53527	096	
.8	.27607	72572	03720	07929	85038	819	
.9	.24980	50325	86663	59687	66158	817	
13.0	0.22603	29406	98105	43257	85277	291	
.1	.20452	30624	52348	88517	46920	038	
.2	.18506	01197	58190	67532	46783	110	
.3	.16744	93209	43426	71792	86875	852	
.4	.15151	44112	14324	96163	05769	581	
13.5	0.13709	59086	38408	43645	02599	613	
.6	.12404	95079	95671	29562	42723	616	
.7	.11224	46365	23434	33766	81407	425	
.8	.10156	31471	00249	09180	75395	895	
.9	.09189	81357	89795	74280	69218	720	
14.0	0.08315	28719	10356	78840	63985	143	
.1	.07523	98299	21642	10566	16958	717	
.2	.06807	98134	39763	37737	59450	189	
.3	.06160	11626	13205	31394	82053	038	
.4	.05573	90369	26945	98049	07658	331	
14.5	0.05043	47662	56788	80758	92222	233	
.6	.04563	52636	79039	92229	75634	380	
.7	.04129	24941	58732	68861	37841	627	
.8	.03736	29937	98852	62858	23297	360	
.9	.03380	74348	39047	38111	80934	638	
15.0	0.03059	02320	50182	57883	71479	49770	229
.1	.02767	91865	85408	06275	11062	90303	601
.2	.02504	51637	23276	19971	12378	74692	052
.3	.02266	18012	77657	11644	48865	71008	072
.4	.02050	52457	56119	27503	43940	88106	805
15.5	0.01855	39136	26159	78240	71710	86473	493
.6	.01678	82752	99956	62558	32695	60836	212
.7	.01519	06596	75689	62781	87087	19027	273
.8	.01374	50772	79213	96984	28825	38479	717
.9	.01243	70602	36028	70065	42338	53941	588
16.0	0.01125	35174	71925	91145	13775	17906	013
.1	.01018	26036	93120	00088	98479	40768	981
.2	.00921	36008	34566	12805	18330	14487	091
.3	.00833	68107	89962	77760	99947	21871	855
.4	.00754	34583	49844	24816	62940	02631	941
16.5	0.00682	56033	76334	86975	53833	89689	872
.6	.00617	60613	35580	37163	68288	57083	203
.7	.00558	83313	92518	26353	28123	48887	751
.8	.00505	65313	48335	52410	44488	14372	545
.9	.00457	53387	69445	80493	73240	17735	568
17.0	0.00413	99377	18785	16665	96510	27718	955
.1	.00374	59705	56295	25069	03124	10250	558
.2	.00338	94943	26196	92178	25752	20721	809
.3	.00306	69412	94563	55723	45060	05384	994
.4	.00277	50832	42240	75246	48379	49380	576

TABLE VI.—*Values of e^{-x} ranging from 33 to 48 places of decimals at intervals of 0.1 from 0.0 to 50.0*—Continued.

x	$e^{-x}\times10^5$						
17.5	0.00251	09991	55743	98180	35473	43740	193
.6	.00227	20459	92773	85882	20171	05083	212
.7	.00205	58322	29760	44687	85201	69381	462
.8	.00186	01939	26691	55236	15658	41796	203
.9	.00168	31730	69673	75730	08575	95356	349
18.0	0.00152	29979	74471	26284	36136	62923	352
.1	.00137	80655	54894	57374	37067	27046	561
.2	.00124	69252	78575	09801	76125	51553	128
.3	.00112	82646	49549	66131	01387	32549	077
.4	.00102	08960	72359	76231	78545	44852	002
18.5	0.00092	37449	66197	05948	97883	17038	460
.6	.00083	58390	10137	46206	26295	28090	620
.7	.00075	62984	11826	51341	18612	28066	415
.8	.00068	43271	02221	79922	76049	65053	096
.9	.00061	92047	68266	40298	69210	94237	034
19.0	0.00056	02796	43753	72675	40012	98281	621
.1	.00050	69619	86232	22936	08100	57441	536
.2	.00045	87181	74664	75209	98545	73639	897
.3	.00041	50653	68769	82261	54061	18520	217
.4	.00037	55666	76593	82970	51383	03756	200
19.5	0.00033	98267	81949	50712	25140	73787	681
.6	.00030	74879	87958	66105	71369	28807	703
.7	.00027	82266	37101	58709	84770	56340	891
.8	.00025	17498	71943	82798	50011	88884	873
.9	.00022	77927	04120	53677	29238	72891	527
20.0	0.00020	61153	62243	85578	27965	94038	016
.1	.00018	65008	92190	27697	33189	22598	761
.2	.00016	87529	85750	85307	37206	62805	645
.3	.00015	26940	15912	66097	18605	61466	414
.4	.00013	81632	59107	95387	89246	35435	639
20.5	0.00012	50152	86638	67426	28937	55311	923
.6	.00011	31185	09177	16341	53263	71739	791
.7	.00010	23538	59775	94154	25949	62818	845
.8	.00009	26136	02205	67760	50298	19435	554
.9	.00008	38002	52694	79477	46801	25352	979
21.0	0.00007	58256	04279	11906	72794	17432	413
.1	.00006	86098	43996	93450	45164	74732	749
.2	.00006	20807	54094	03619	76867	91816	079
.3	.00005	61729	89244	17303	97323	90528	550
.4	.00005	08274	22551	05926	15332	48072	717
21.5	0.00004	59905	53786	52316	77907	05925	361
.6	.00004	16139	73942	24154	70246	73925	373
.7	.00003	76538	80736	11354	32827	62725	700
.8	.00003	40706	40224	29893	53380	54451	820
.9	.00003	08283	90131	38675	51913	32631	879
22.0	0.00002	78946	80928	68924	80771	89130	306
.1	.00002	52401	51068	45210	21742	73861	281
.2	.00002	28382	33123	61576	64470	64849	105
.3	.00002	06648	87892	07581	80510	04029	230
.4	.00001	86983	63804	26844	64135	94383	017
22.5	0.00001	69189	79226	15130	36130	19439	206
.6	.00001	53089	25478	79478	29098	85778	291
.7	.00001	38520	88603	13758	75438	72331	863
.8	.00001	25338	88086	06835	66073	58263	954
.9	.00001	13411	30933	74976	68297	68776	038
23.0	0.00001	02618	79631	70189	03039	27527	841
.1	.00000	92853	32670	14494	21797	43920	867
.2	.00000	84017	16438	85889	17671	85250	385
.3	.00000	76021	87409	60735	66299	49779	464
.4	.00000	68787	43627	13460	03812	13048	694

TABLE VI.—*Values of e^{-x} ranging from 33 to 48 places of decimals at intervals of 0.1 from 0.0 to 50.0*—Continued.

x	$e^{-x}\times10^{10}$						
23.5	0.62241	44622	90778	32321	36689	302	
.6	.56318	38950	07427	98158	22576	375	
.7	.50958	98614	37956	07815	32879	224	
.8	.46109	59744	80822	57885	97649	739	
.9	.41721	68910	16002	20720	16549	798	
24.0	0.37751	34544	27909	77516	44969	548	
.1	.34158	82993	78385	77078	17927	744	
.2	.30908	18748	40832	95528	56856	901	
.3	.27966	88455	92692	90452	20914	954	
.4	.25305	48361	51189	69970	95413	063	
24.5	0.22897	34845	64555	28940	85224	694	
.6	.20718	37765	72088	85115	75223	970	
.7	.18746	76334	52428	00712	00483	363	
.8	.16962	77294	18406	63987	44513	620	
.9	.15348	55167	14253	44644	47610	507	
25.0	0.13887	94386	49640	20594	66176	37460	869
.1	.12566	33126	86023	89591	28573	24497	477
.2	.11370	48673	92667	30575	06855	57781	111
.3	.10288	44186	29702	25556	66047	29002	133
.4	.09309	36717	09030	56681	48387	57235	814
25.5	0.08423	46375	44686	47405	87646	52628	816
.6	.07621	86519	45129	01041	86087	42569	880
.7	.06896	54882	32212	00165	51703	34317	985
.8	.06240	25543	05624	06152	61704	36655	883
.9	.05646	41661	16749	62792	72023	10934	402
26.0	0.05109	08902	80633	24719	87440	01934	792
.1	.04622	89492	46686	68940	73403	07967	410
.2	.04182	96830	74887	40238	10483	40957	393
.3	.03784	90624	30743	59536	01635	97079	765
.4	.03424	72479	24915	87477	54037	63906	063
26.5	0.03098	81913	87218	25441	64178	60818	385
.6	.02803	92750	84414	72566	40350	61367	907
.7	.02537	09852	70981	83277	48943	37439	574
.8	.02295	66168	05623	56169	40441	23976	976
.9	.02077	20058	77241	34158	48280	41113	052
27.0	0.01879	52881	65390	83294	75827	04184	222
.1	.01700	66800	14814	06878	50865	79529	609
.2	.01538	82804	33968	11670	38458	16312	667
.3	.01392	38919	35884	98620	87677	77862	518
.4	.01259	88584	28277	88967	69065	98718	199
27.5	0.01139	99185	30443	55345	31786	95696	403
.6	.01031	50728	48906	83550	57811	58485	712
.7	.00933	34638	83457	69077	84065	71852	384
.8	.00844	52673	61639	73721	35688	85617	335
.9	.00764	15939	14129	46027	61067	13778	082
28.0	0.00691	44001	06940	20300	94125	84658	741
.1	.00625	64079	40031	33604	79651	20550	275
.2	.00566	10320	06637	63070	77960	29403	337
.3	.00512	23135	84304	92092	59075	13264	538
.4	.00463	48609	97992	98618	53973	63255	792
28.5	0.00419	37956	58379	54442	52680	72672	186
.6	.00379	47032	35298	56414	35892	66116	403
.7	.00343	35894	77640	25514	74190	24136	674
.8	.00310	68402	37543	44761	24890	39628	848
.9	.00281	11852	98789	04044	93344	23703	026

Table VI.—*Values of e^{-x} ranging from 33 to 48 places of decimals at intervals of 0.1 from 0.0 to 50.0*—Continued.

x	$e^{-x}\times10^{10}$						
29.0	0.00254	36656	47376	92291	03033	85614	858
.1	.00230	16038	56719	30252	98961	98444	442
.2	.00208	25772	91055	50034	41929	59577	002
.3	.00188	43938	58898	98201	66055	82710	392
.4	.00170	50700	73848	97320	95731	68146	459
29.5	0.00154	28112	03191	88783	29721	02046	747
.6	.00139	59933	05613	09997	76718	93894	576
.7	.00126	31469	78246	44161	41824	90288	582
.8	.00114	29426	50396	43462	40460	89725	866
.9	.00103	41772	76747	88631	14871	35865	680
30.0	0.00093	57622	96884	01746	04915	83222	338
.1	.00084	67127	40607	93341	71972	67749	165
.2	.00076	61373	70029	83365	22776	85364	170
.3	.00069	32297	59758	65523	77079	03801	763
.4	.00062	72602	25925	71015	47054	72075	602
30.5	0.00056	75685	23263	27224	61872	78872	381
.6	.00051	35572	37148	02171	53091	68192	004
.7	.00046	46858	04474	69695	17315	32710	431
.8	.00042	04651	03518	84753	93216	26484	901
.9	.00038	04525	58642	21646	74684	33532	907
31.0	0.00034	42477	10846	99764	58392	38933	285
.1	.00031	14882	09847	58694	36580	83265	878
.2	.00028	18461	87547	13372	67019	06804	197
.3	.00025	50249	76623	42730	31718	36825	785
.4	.00023	07561	41382	62290	86183	09083	925
31.5	0.00020	87967	91164	59335	50509	88967	622
.6	.00018	89271	49411	56410	78926	37742	140
.7	.00017	09483	54070	45362	63790	25908	984
.8	.00015	46804	67314	60627	92270	38923	531
.9	.00013	99606	74665	54398	29638	60755	089
32.0	0.00012	66416	55490	94175	72312	09041	560
.1	.00011	45901	08570	22324	18777	38161	942
.2	.00010	36854	17971	14108	12572	72255	979
.3	.00009	38184	45884	98657	78521	57413	808
.4	.00008	48904	40338	71765	13373	59415	985
32.5	0.00007	68120	46852	02094	90674	25977	989
.6	.00006	95024	14147	63979	20547	50203	795
.7	.00006	28883	84964	61633	73835	11081	070
.8	.00005	69037	63875	83490	75590	29624	051
9	.00005	14886	54781	93836	54103	95285	602
33.0	0.00004	65888	61451	03397	36418	42455	436
.1	.00004	21553	45104	58862	97123	52302	914
.2	.00003	81437	33620	85080	38776	35563	724
.3	.00003	45138	77443	74206	47237	60762	601
.4	.00003	12294	47752	60511	44037	56739	759
33.5	0.00002	82575	72871	15611	21020	28754	875
.6	.00002	55685	09276	69987	34084	22688	697
.7	.00002	31353	43916	95759	36958	40314	990
.8	.00002	09337	24855	19385	25874	99128	963
.9	.00001	89416	17547	84879	72752	75933	317
34.0	0.00001	71390	84315	42012	96630	27203	426
.1	.00001	55080	84799	46536	18659	27328	468
.2	.00001	40322	95408	63094	99211	85099	450
.3	.00001	26969	45946	66347	89949	81449	683
.4	.00001	14886	71787	32112	48054	47810	798
34.5	0.00001	03953	80116	70221	94395	13367	453
.6	.00000	94061	28904	29918	83443	95415	969
.7	.00000	85110	17391	47948	70615	64947	459
.8	.00000	77010	87001	36544	68253	41936	893
.9	.00000	69682	31678	38580	11813	28543	327
35.0	0.00000	63051	16760	14698	93856	39021	192

TABLE VI.—*Values of e^{-x} ranging from 33 to 48 places of decimals at intervals of 0.1 from 0.0 to 50.0*—Continued.

x	$e^{-x}\times10^{15}$						
35.0	0.63051	16760	14698	93856	39021	19224	654
.1	.57051	05569	66665	64836	20656	86873	686
.2	.51621	92993	27974	97344	74467	39592	088
.3	.46709	45379	44257	03580	57879	67927	598
.4	.42264	46156	92181	08441	86703	53694	809
35.5	0.38242	46628	09713	53519	42886	25672	271
.6	.34603	21444	90013	64814	69714	18553	590
.7	.31310	28321	77790	04439	10367	50706	083
.8	.28330	71582	47497	90491	45219	11627	172
.9	.25634	69175	79771	01372	82219	04476	870
36.0	0.23195	22830	24356	93883	12263	60973	808
.1	.20987	91048	79305	26873	44597	66680	239
.2	.18990	64673	58688	94402	30391	82482	057
.3	.17183	44775	93166	33948	23021	57303	245
.4	.15548	22650	34958	57953	08297	21321	002
36.5	0.14068	61712	44614	67672	48913	72822	964
.6	.12729	81119	42342	00516	07241	99744	214
.7	.11518	40949	30761	29059	26337	74188	051
.8	.10422	28790	55958	90584	32069	53886	118
.9	.09430	47607	85267	94412	20739	28211	370
37.0	0.08533	04762	57440	65794	27804	98229	412
.1	.07721	02078	16561	35572	52863	52358	518
.2	.06986	26850	86757	24087	00541	33072	436
.3	.06321	43715	90960	76015	74919	93941	739
.4	.05719	87287	73130	64818	04795	09826	246
37.5	0.05175	55500	58018	68534	85109	07057	388
.6	.04683	03582	83528	48493	54683	16855	962
.7	.04237	38604	74966	82593	34655	87997	980
.8	.03834	14545	04384	98234	92936	93413	804
.9	.03469	27826	97490	91944	53846	38105	277
38.0	0.03139	13279	20480	29628	70896	46522	319
.1	.02840	40481	04287	51936	53646	27988	899
.2	.02570	10455	48452	71119	81485	05469	214
.3	.02325	52676	94886	54372	34296	62932	824
.4	.02104	22363	76776	20152	57806	72554	547
38.5	0.01903	98028	32864	52319	09651	51045	524
.6	.01722	79260	35202	88389	09152	96528	151
.7	.01558	84721	11807	46343	27407	39887	817
.8	.01410	50328	56773	42732	90339	44269	328
.9	.01276	27615	11435	24275	64109	91551	699
39.0	0.01154	82241	73015	78598	62624	42063	324
.1	.01044	92653	43612	05827	71170	02291	972
.2	.00945	48862	73886	56872	67683	46735	659
.3	.00855	51348	83887	15734	63340	96583	931
.4	.00774	10061	59285	82425	54038	35334	841
39.5	0.00700	43520	26168	64522	06111	20117	684
.6	.00633	77998	02373	37888	31289	73849	824
.7	.00573	46784	09208	34299	62399	53206	969
.8	.00518	89516	05054	64108	86055	09785	357
.9	.00469	51575	72631	18967	64182	28650	614
40.0	0.00424	83542	55291	58899	53292	34782	859
.1	.00384	40698	95260	12322	93217	25582	228
.2	.00347	82582	78776	93145	38283	85687	124
.3	.00314	72582	40230	71953	77048	96788	377
.4	.00284	77570	19982	76205	37354	14757	803
40.5	0.00257	67571	09154	98094	81244	03947	486
.6	.00233	15462	49553	59620	87332	35047	951
.7	.00210	96702	88477	50105	40714	22490	401
.8	.00190	89086	16733	16004	79492	13432	437
.9	.00172	72519	44031	42767	18344	04012	470

TABLE VI.—*Values of e^{-x} ranging from 33 to 48 places of decimals at intervals of 0.1 from 0.0 to 50.0*—Continued.

x	$e^{-x} \times 10^{15}$						
41.0	0.00156	28821	89334	98876	80908	82995	106
.1	.00141	41542	84892	25895	04128	48638	316
.2	.00127	95797	11846	40038	13520	71489	987
.3	.00115	78116	02638	29407	39244	53079	409
.4	.00104	76312	61103	31040	88099	91446	531
41.5	0.00094	79359	65350	47559	45429	56113	551
.6	.00085	77279	31351	14917	47393	06788	324
.7	.00077	61043	26781	09860	09063	86820	825
.8	.00070	22482	35171	14588	95695	22686	530
.9	.00063	54204	79932	56898	16115	01639	407
42.0	0.00057	49522	26429	35598	06664	38088	057
.1	.00052	02382	88056	36486	16013	12077	002
.2	.00047	07310	69328	36896	66194	65095	023
.3	.00042	59350	85360	38765	82354	83665	338
.4	.00038	54020	02888	41921	20220	99004	186
42.5	0.00034	87261	53199	44467	34281	84859	880
.6	.00031	55404	72062	59800	09588	74554	002
.7	.00028	55128	26026	96901	02510	06254	767
.8	.00025	83426	88318	39275	69583	84336	693
.9	.00023	37581	31066	48315	69181	60051	823
43.0	0.00021	15131	03759	10804	86631	40100	702
.1	.00019	13849	70686	16334	16651	97089	136
.2	.00017	31722	82726	55584	82545	15506	796
.3	.00015	66927	61177	68999	51381	41226	635
.4	.00014	17814	73448	94625	92151	05608	824
43.5	0.00012	82891	82360	87848	92767	77284	128
.6	.00011	60808	52529	36166	06882	27545	051
.7	.00010	50342	98886	08059	21548	25900	989
.8	.00009	50389	63809	29842	77008	33343	029
.9	.00008	59948	10626	01859	42848	36603	861
44.0	0.00007	78113	22411	33796	51571	33167	293
.1	.00007	04065	96064	63864	02594	11819	481
.2	.00006	37065	22595	82837	94977	89144	024
.3	.00005	76440	45417	65886	78362	28242	312
.4	.00005	21584	89220	86203	68464	15609	422
44.5	0.00004	71949	52715	26123	41636	05846	918
.6	.00004	27037	59159	20617	46712	38215	253
.7	.00003	86399	59178	04557	49710	54130	889
.8	.00003	49628	80895	67763	67822	60571	334
.9	.00003	16357	22876	74373	05000	03529	236
45.0	0.00002	86251	85805	49393	64447	01216	292
.1	.00002	59011	39215	04273	31305	92431	185
.2	.00002	34363	19931	52920	73161	89041	858
.3	.00002	12060	59215	10958	48008	45775	273
.4	.00001	91880	35866	91742	41361	20466	612
45.5	0.00001	73620	52831	00294	72541	72775	788
.6	.00001	57098	35055	40862	91529	13892	163
.7	.00001	42148	46589	30674	99007	15384	638
.8	.00001	28621	25085	64558	58060	81817	263
.9	.00001	16381	32052	95109	72519	37515	872
46.0	0.00001	05306	17357	55381	23787	63324	449
.1	.00000	95284	96620	13365	08941	90311	942
.2	.00000	86217	40279	52610	01613	69548	955
.3	.00000	78012	73213	50302	88340	07777	956
.4	.00000	70588	83911	89917	38002	38406	937
46.5	0.00000	63871	42293	05842	23502	28846	869
.6	.00000	57793	25341	07926	11131	32265	356
.7	.00000	52293	49819	61195	00313	96943	218
.8	.00000	47317	11388	78448	78157	94017	572
.9	.00000	42814	29515	91910	04355	72624	796

TABLE VI.—*Values of e^{-x} ranging from 33 to 48 places of decimals at intervals of 0.1 from 0.0 to 50.0*—Continued.

x	$e^{-x} \times 10^{20}$					
47.0	0.38739	97628	68718	71129	31477	497
.1	.35053	38011	81874	44181	14696	776
.2	.31717	60995	95737	66419	90234	132
.3	.28699	28030	20923	62903	85594	584
.4	.25968	18268	80355	27517	22837	845
47.5	0.23496	98337	45281	70976	26987	584
.6	.21260	94976	82419	38687	22606	325
.7	.19237	70289	32882	68879	79646	073
.8	.17406	99341	49058	66327	58806	721
.9	.15750	49897	73123	74854	18083	799
48.0	0.14251	64082	74093	51062	85321	028
.1	.12895	41788	90489	43765	81148	006
.2	.11668	25662	72217	70475	74887	795
.3	.10557	87519	95563	20838	55106	156
.4	.09553	16053	55124	32778	83159	356
48.5	0.08644	05711	30360	94557	72312	023
.6	.07821	46631	95149	50544	60620	901
.7	.07077	15538	98051	27453	54748	366
.8	.06403	67500	99505	46535	26327	711
.9	.05794	28476	19450	50254	69685	961
49.0	0.05242	88566	33634	63937	17180	530
.1	.04743	95912	66955	45835	05016	148
.2	.04292	51172	74673	23310	79205	757
.3	.03884	02522	83706	09393	43055	524
.4	.03514	41135	92253	90440	00963	185
49.5	0.03179	97090	01977	49498	18153	259
.6	.02877	35665	87644	17741	59121	756
.7	.02603	53996	98849	71336	86526	680
.8	.02355	78038	41041	37407	48630	165
.9	.02131	59824	02125	48791	98821	454
50.0	0.01928	74984	79639	17783	01734	282

TABLE VII.—*Values of e^{-x} to 63 places of decimals at decimal intervals from 1×10^{-10} to 9×10^{-1}.*

x	e^{-x}
1×10^{-10}	0.99999 99999 00000 00000 49999 99999 83333 33333 37499 99999 99166 66666 668
2	.99999 99998 00000 00001 99999 99998 66666 66667 33333 33333 06666 66666 756
3	.99999 99997 00000 00004 49999 99995 50000 00003 37499 99997 97500 00001 012
4	.99999 99996 00000 00007 99999 99989 33333 33343 99999 99991 46666 66672 356
5	0.99999 99995 00000 00012 49999 99979 16666 66692 70833 33307 29166 66688 368
6	.99999 99994 00000 00017 99999 99964 00000 00053 99999 99935 20000 00064 800
7	.99999 99993 00000 00024 49999 99942 83333 33433 37499 99859 94166 66830 068
8	.99999 99992 00000 00031 99999 99914 66666 66837 33333 33060 26666 67030 755
9	.99999 99991 00000 00040 49999 99878 50000 00273 37499 99507 92500 00737 112
1×10^{-9}	0.99999 99990 00000 00049 99999 99833 33333 33749 99999 99166 66666 68055 556
2	.99999 99980 00000 00199 99999 98666 66666 73333 33333 06666 66667 55555 556
3	.99999 99970 00000 00449 99999 95500 00000 33749 99997 97500 00010 12500 000
4	.99999 99960 00000 00799 99999 89333 33334 39999 99991 46666 66723 55555 552
5	0.99999 99950 00000 01249 99999 79166 66669 27083 33307 29166 66883 68055 540
6	.99999 99940 00000 01799 99999 64000 00005 39999 99935 20000 00647 99999 944
7	.99999 99930 00000 02449 99999 42833 33343 33749 99859 94166 68300 68055 392
8	.99999 99920 00000 03199 99999 14666 66683 73333 33060 26666 70307 55555 139
9	.99999 99910 00000 04049 99998 78500 00027 33749 99507 92500 07381 12499 051
1×10^{-8}	0.99999 99900 00000 04999 99998 33333 33374 99999 99166 66666 80555 55553 571
2	.99999 99800 00000 19999 99986 66666 67333 33333 06666 66675 55555 55301 587
3	.99999 99700 00000 44999 99955 00000 03374 99997 97500 00101 24999 95660 714
4	.99999 99600 00000 79999 99893 33333 43999 99991 46666 67235 55555 23047 619
5	0.99999 99500 00001 24999 99791 66666 92708 33307 29166 68836 80554 00545 636
6	.99999 99400 00001 79999 99640 00000 53999 99935 20000 06479 99994 44571 433
7	.99999 99300 00002 44999 99428 33334 33374 99859 94166 83006 80539 21541 681
8	.99999 99200 00003 19999 99146 66668 37333 33060 26667 03075 55513 94539 724
9	.99999 99100 00004 04999 98785 00002 73374 99507 92500 73811 24905 09982 250
1×10^{-7}	0.99999 99000 00004 99999 98333 33337 49999 99166 66668 05555 55357 14285 962
2	.99999 98000 00019 99999 86666 66733 33333 06666 66755 55555 30158 73079 365
3	.99999 97000 00044 99999 55000 00337 49997 97500 01012 49995 66071 44484 375
4	.99999 96000 00079 99998 93333 34399 99991 46666 72355 55523 04762 06730 158
5	0.99999 95000 00124 99997 91666 69270 83307 29166 88368 05400 54564 46087 544
6	.99999 94000 00179 99996 40000 05399 99935 20000 64799 99444 57147 02285 686
7	.99999 93000 00244 99994 28333 43337 49859 94168 30068 03921 54180 96428 708
8	.99999 92000 00319 99991 46666 83733 33060 26670 30755 51394 54009 86412 329
9	.99999 91000 00404 99987 85000 27337 49507 92507 38112 40509 98321 04840 450
1×10^{-6}	0.99999 90000 00499 99983 33333 74999 99166 66680 55555 35714 28819 44441 689
2	.99999 80000 01999 99866 66673 33333 06666 67555 55530 15873 65079 35097 002
3	.99999 70000 04499 99550 00033 74997 97500 10124 99566 07159 12945 88616 088
4	.99999 60000 07999 98933 33439 99991 46667 23555 52304 76353 01580 07760 430
5	0.99999 50000 12499 97916 66927 08307 29168 83680 40054 57318 01781 49459 812
6	.99999 40000 17999 96400 00539 99935 20006 47999 44457 18451 42579 42873 806
7	.99999 30000 24499 94283 34333 74859 94183 00678 92154 30964 27707 40799 293
8	.99999 20000 31999 91466 68373 33060 26703 07551 39454 38435 51856 87773 848
9	.99999 10000 40499 87850 02733 74507 92573 81115 50999 28191 16555 88237 649
1×10^{-5}	0.99999 00000 49999 83333 37499 99166 66805 55535 71431 05158 45458 58134 918
2	.99998 00001 99998 66667 33333 06666 75555 53015 87936 50652 55760 14104 217
3	.99997 00004 49995 50003 37497 97501 01249 56607 30557 98147 33770 08484 782
4	.99996 00007 99989 33343 99991 46672 35552 30477 81586 57919 16021 05894 536
5	0.99995 00012 49979 16692 70807 29188 36790 05466 03727 25662 68892 76375 056
6	.99994 00017 99964 00053 99935 20064 79944 45755 94257 94302 37705 19692 856
7	.99993 00024 49942 83433 37359 94330 06642 15559 64176 99149 98717 24649 249
8	.99992 00031 99914 66837 33060 27030 75139 45812 92328 54757 97343 88835 463
9	.99991 00040 49878 50273 37007 93238 10301 00889 04490 41831 39216 07633 246
1×10^{-4}	0.99990 00049 99833 33749 99166 68055 53571 45337 27403 02579 34002 74654 599
2	.99980 00199 98666 73333 06667 55553 01593 65065 25601 41042 16856 39331 863
3	.99970 00449 95500 33747 97510 12456 60877 00350 46270 04490 77652 95014 129
4	.99960 00799 89334 39991 46723 55230 49244 37220 74750 27202 47629 64868 422

TABLE VII.—*Values of e^{-x} to 63 places of decimals at decimal intervals from 1×10^{-10} to 9×10^{-1}*—Continued.

x	e^{-x}												
5×10^{-4}..	0. 99950	01249	79169	27057	29383	66505	55322	50247	26223	02486	90844	35858	478
6........	. 99940	01799	64005	39935	20647	94446	13082	93730	94766	25908	96603	96600	982
7........	. 99930	02449	42843	33609	95800	51716	84631	76235	05215	89370	76242	29612	618
8........	. 99920	03199	14683	73060	30307	13949	55747	14313	50962	82217	88989	57251	830
9........	. 99910	04048	78527	33257	99880	17610	49663	11297	83926	91894	32872	95547	938
1×10^{-3}..	0. 99900	04998	33374	99166	80553	57167	65597	47023	55902	36008	20590	52028	511
2........	. 99800	19986	67333	06675	55301	65077	95442	67564	61972	92438	30580	58147	637
3........	. 99700	44955	03372	97601	20662	34097	56091	07417	74804	89844	71559	07658	933
4........	. 99600	79893	43991	47235	23063	86579	47756	69165	28160	35315	78639	52494	899
5........	0. 99501	24791	92682	31335	25642	46232	50418	53859	08435	97232	32954	07758	993
6........	. 99401	79640	53935	26474	44987	72245	22520	14254	35943	49101	19096	12949	836
7........	. 99302	44429	33235	10490	47970	31756	01454	93896	51151	39821	82973	32899	764
8........	. 99203	19148	37060	63033	98697	00268	87164	93433	59144	20431	49248	41054	349
9........	. 99104	03787	72883	66216	45647	74627	71266	41139	11021	68667	32984	92224	925
1×10^{-2}..	0. 99004	98337	49168	05357	39059	77180	03655	77720	79081	25383	74668	83878	745
2........	. 98019	86733	06755	30222	08141	04225	30886	62997	12400	46914	40777	25203	931
3........	. 97044	55335	48508	17693	25283	51959	19433	34867	36815	52894	36205	32113	500
4........	. 96078	94391	52323	20943	92106	91323	24588	60279	72093	71791	65716	23439	634
5........	0. 95122	94245	00714	00909	14253	19779	65216	06570	87449	34037	31345	30249	566
6........	. 94176	45335	84248	70953	71527	83271	14970	60946	88662	54183	92213	74047	235
7........	. 93239	38199	05948	22885	79726	32484	96785	43600	68377	74845	73976	05493	459
8........	. 92311	63463	86635	78291	07598	49572	38881	00435	83063	14608	64993	87892	996
9........	. 91393	11852	71228	18674	73535	46499	52061	02105	85194	82680	97766	63588	265
1×10^{-1}..	0. 90483	74180	35959	57316	42490	59446	43662	11947	05360	98040	09520	56257	317
2........	. 81873	07530	77981	85866	99355	08619	03942	43585	91256	26901	56724	78028	762
3........	. 74081	82206	81717	86606	68737	79317	81687	21822	51231	99900	63482	95310	067
4........	. 67032	00460	35639	30074	44329	25147	82607	19369	80925	21081	21998	88910	332
5........	0. 60653	06597	12633	42360	37995	34991	18045	34419	18135	48718	69556	82892	159
6........	. 54881	16360	94026	43262	84589	17232	56787	53323	11956	69062	80669	80712	117
7........	. 49658	53037	91409	51470	48000	93397	52896	17076	67165	71181	62620	54711	497
8........	. 44932	89641	17221	59143	01023	85015	56279	59342	14941	27218	44908	97989	334
9........	. 40656	96597	40599	11188	34542	39645	62598	78337	03376	17037	81677	46288	648

TABLE VIII.—*Values of* $e^{\pm\frac{n\pi}{360}}$ *to 23 places of decimals or significant figures at intervals of unity from n=0 to n=360.*

n	$e^{\frac{n\pi}{360}}$					$e^{-\frac{n\pi}{360}}$				
0	1.00000	00000	00000	00000	000	1.00000	00000	00000	00000	000
1	.00876	48344	41532	05452	851	0.99131	13203	96706	33089	086
2	.01760	64912	05851	57557	922	.98269	81339	46661	35321	293
3	.02652	56436	27899	20923	149	.97415	97847	14044	24673	777
4	.03552	29709	44284	88002	636	.96569	56224	62250	37866	736
5	1.04459	91583	45014	94512	101	0.95730	50026	04372	64019	161
6	.05375	48970	25672	72838	534	.94898	72861	54113	03484	687
7	.06299	08842	40056	40824	120	.94074	18396	77120	78035	553
8	.07230	78233	53278	26787	366	.93256	80352	42753	21810	750
9	.08170	64238	95329	35157	955	.92446	52503	76255	85664	364
10	1.09118	74016	15113	60646	067	0.91643	28680	11357	90742	001
11	.10075	14785	34955	62442	319	.90847	02764	43279	70277	354
12	.11039	93830	05586	13551	147	.90057	68692	82148	41737	686
13	.12013	18497	61609	43998	758	.89275	20454	06818	54556	387
14	.12994	96199	77457	00326	997	.88499	52089	19093	61773	205
15	1.13985	34413	23831	47486	813	0.87730	57690	98345	66958	381
16	.14984	40680	24645	42979	844	.86968	31403	56529	00825	980
17	.15992	22609	14459	16864	183	.86212	67421	93584	84944	480
18	.17008	87874	96421	95040	967	.85463	59991	53233	42929	363
19	.18034	44220	00721	07072	304	.84721	03407	79150	22453	252
20	1.19068	99454	43543	23648	548	0.83984	92015	71522	94334	292
21	.20112	61456	86552	72724	323	.83255	20209	43985	97863	261
22	.21165	38174	96890	87278	267	.82531	82431	80929	04404	425
23	.22227	37626	07701	41621	544	.81814	73173	95176	74154	744
24	.23298	67897	79186	37185	061	.81103	86974	86035	83770	873
25	1.24379	37148	60197	02755	267	0.80399	18420	97707	05373	653
26	.25469	53608	50364	78203	834	.79700	62145	78058	20215	764
27	.26569	25579	62776	54867	601	.79008	12829	37755	53050	019
28	.27678	61436	87199	49882	300	.78321	65198	09750	15963	722
29	.28797	69628	53859	95957	082	.77641	14024	09116	53148	702
30	1.29926	58676	97781	32296	996	0.76966	54124	93239	80757	377
31	.31065	37179	23685	86637	747	.76297	80363	22349	18652	629
32	.32214	13807	71465	42651	469	.75634	87646	20394	13493	606
33	.33372	97310	82225	91314	403	.74977	70925	36260	55211	011
34	.34541	96513	64910	69197	373	.74326	25196	05323	91514	228
35	1.35721	20318	63507	91048	389	0.73680	45497	11336	47638	859
36	.36910	77706	24846	88483	649	.73040	26910	48645	61087	260
37	.38110	77735	66988	71089	244	.72405	64560	84740	43636	504
38	.39321	29545	48216	30761	162	.71776	53615	23123	85388	162
39	.40542	42354	36629	14676	135	.71152	89282	66507	18112	526
40	1.41774	25461	80347	96890	877	0.70534	66813	80324	57596	608
41	.43016	88248	78334	83212	599	.69921	81500	56564	47140	580
42	.44270	40178	51833	88669	716	.69314	28675	77915	26761	562
43	.45534	90797	16438	31638	830	.68712	03712	82222	55056	820
44	.46810	49734	54788	93452	606	.68115	02025	27255	13050	927
45	1.48097	26704	89909	97123	524	0.67523	19066	55777	21703	207
46	.49395	31507	59187	63671	005	.66936	50329	60924	07083	184
47	.50704	74027	88997	09434	472	.66354	91346	51878	49532	865
48	.52025	64237	69983	52692	842	.65778	37688	19845	55425	741
49	.53358	12196	33003	02892	202	.65206	84964	04322	92403	513
50	1.54702	28051	25729	10808	292	0.64640	28821	59664	31222	960
51	.56058	22038	89930	63039	387	.64078	64946	21933	39577	206
52	.57426	04485	39427	09338	494	.63521	89060	76045	75468	076
53	.58805	85807	38727	16452	016	.62969	96925	23196	29899	492
54	.60197	76512	82356	47335	368	.62422	84336	48569	70835	975
55	1.61601	87201	74880	69865	091	0.61880	47127	89331	42525	659
56	.63018	28567	11630	04461	975	.61342	81169	02896	76423	722
57	.64447	11395	60131	25381	130	.60809	82365	35475	72070	156
58	.65888	46568	42253	35813	200	.60281	46657	90891	08375	381
59	.67342	45062	17073	42376	336	.59757	70022	99667	47848	532

TABLE VIII.—*Values of* $e^{\pm\frac{n\pi}{360}}$ *to 23 places of decimals or significant figures at intervals of unity from* $n=0$ *to* $n=360$—Continued.

n	$e^{\frac{n\pi}{360}}$					$e^{-\frac{n\pi}{360}}$				
60	1.68809	17949	64468	60061	685	0.59238	48471	88388	98366	542
61	.70288	76400	69440	84226	286	.58723	78050	49322	99127	427
62	.71781	31683	07180	71806	918	.58213	54839	10307	99458	797
63	.73286	95163	28876	79556	987	.57707	74952	04903	01162	498
64	.74805	78307	48277	12786	401	.57206	34537	42796	37068	836
65	1.76337	92682	29009	43812	068	0.56709	29776	80471	60448	935
66	.77883	49955	72666	65104	439	.56216	56884	92128	21891	844
67	.79442	61898	07664	47944	055	.55728	12109	40855	12193	974
68	.81015	40382	78877	83281	609	.55243	91730	50054	51732	561
69	.82601	97387	38062	87426	137	.54763	92060	75114	08702	308
70	1.84202	44994	35071	61169	047	0.54288	09444	75325	30485	148
71	.85816	95392	09865	96987	248	.53816	40258	86045	74297	510
72	.87445	60875	85338	35057	029	.53348	80910	91103	25117	573
73	.89088	53848	60945	74952	191	.52885	27839	95439	90737	003
74	.90745	86822	07164	56095	507	.52425	77515	97993	65607	591
75	1.92417	72417	60773	26282	571	0.51970	26439	64815	56963	376
76	.94104	23367	20970	23901	766	.51518	71142	02420	68493	176
77	.95805	52514	46334	05834	083	.51071	08184	31370	38617	250
78	.97521	72815	52633	59432	218	.50627	34157	60084	32185	175
79	.99252	97340	11495	43450	340	.50187	45682	58879	86159	993
80	2.00999	39272	49936	09324	583	0.49751	39409	34237	11586	546
81	.02761	11912	50766	60790	224	.49319	12017	03287	55859	665
82	.04538	28676	53877	16465	121	.48890	60213	68524	31010	739
83	.06331	03098	58409	46730	841	.48465	80735	92732	15419	220
84	.08139	48831	25824	63003	504	.48044	70348	74135	38029	032
85	2.09963	79646	83874	44306	209	0.47627	25845	21761	55808	661
86	.11804	09438	31483	92934	532	.47213	44046	31019	36838	163
87	.13660	52220	44553	17946	512	.46803	21800	59488	63036	447
88	.15533	22130	82686	52209	299	.46396	55984	02920	68158	185
89	.17422	33430	96857	15796	707	.45993	43499	71447	28291	620
90	2.19328	00507	38015	45655	977	0.45593	81277	65996	23676	592
91	.21250	37872	66649	18648	471	.45197	66274	54911	92236	275
92	.23189	60166	63304	02318	445	.44804	95473	50778	96776	722
93	.25145	82157	40072	75057	055	.44415	65883	87447	29355	240
94	.27119	18742	53061	64705	804	.44029	74540	97256	77852	204
95	2.29109	84950	15842	62085	369	0.43647	18505	88459	81301	115
96	.31117	95940	13899	73442	713	.43267	94865	22840	02038	712
97	.33143	67005	20078	83382	137	.42892	00730	93525	44230	900
98	.35187	13572	11049	07485	054	.42519	33240	02994	49811	129
99	.37248	51202	84785	21529	362	.42149	89554	41273	04335	996
100	2.39327	95595	79079	61992	899	0.41783	66860	64320	86718	102
101	.41425	62586	91093	00367	250	.41420	62369	72605	98238	893
102	.43541	68150	97953	01718	646	.41060	73316	89865	07674	338
103	.45676	28402	78409	85912	539	.40703	96961	42048	50783	996
104	.47829	59598	35558	17968	202	.40350	30586	36448	23819	430
105	2.50001	78136	20634	62130	019	0.39999	71498	41007	12101	097
106	.52193	00558	57900	42433	634	.39652	17027	63807	96093	901
107	.54403	43552	70618	60808	433	.39307	64527	32740	78780	699
108	.56633	23952	08135	32093	526	.38966	11373	75346	79490	198
109	.58882	58737	74075	03753	213	.38627	54965	98837	40681	099
110	2.61151	65039	55659	36560	368	0.38291	92725	70286	95517	995
111	.63440	60137	54159	31073	041	.37959	22096	96997	45396	697
112	.65749	61463	16490	93361	367	.37629	40546	07033	97887	229
113	.68078	86600	67964	42149	403	.37302	45561	29929	16861	994
114	.70428	53288	46196	68320	288	.36978	34652	77555	37864	563
115	2.72798	79420	36197	66593	929	0.36657	05352	25163	03051	267
116	.75189	83047	06640	68124	869	.36338	55212	92583	71303	459
117	.77601	82377	47327	11784	814	.36022	81809	25596	60362	966
118	.80034	95780	07856	00990	098	.35709	82736	77456	79087	012
119	.82489	41784	37509	02109	953	.35399	55611	90584	09151	846

TABLE VIII.—*Values of $e^{\pm\frac{n\pi}{360}}$ to 23 places of decimals or significant figures at intervals of unity from $n=0$ to $n=360$*—Continued.

n	$e^{\frac{n\pi}{360}}$					$e^{-\frac{n\pi}{360}}$				
120	2.84965	39082	26361	49747	413	0.35091	98071	73410	96756	574
121	.87463	06529	47630	33521	788	.34787	07774	07388	16090	274
122	.89982	63147	01269	50400	604	.34484	82396	79146	67526	631
123	.92524	28122	58824	16130	382	.34185	19638	12814	74700	914
124	.95088	20812	09554	38900	424	.33888	17216	27488	45804	511
125	2.97674	60741	07839	68042	567	0.33593	72869	24854	65602	228
126	3.00283	67606	21875	40323	374	.33301	84354	71964	85837	499
127	.02915	61276	83672	56224	289	.33012	49449	84158	82840	442
128	.05570	61796	40372	28530	512	.32725	65951	08136	52293	545
129	.08248	89384	06886	55561	611	.32441	31674	05177	12239	654
130	3.10950	64436	19876	81476	913	0.32159	44453	34503	86537	070
131	.13676	07527	93082	16277	217	.31880	02142	36793	42076	927
132	.16425	39414	74008	98402	237	.31603	02613	17828	54178	719
133	.19198	81034	01993	93191	080	.31328	43756	32292	75671	061
134	.21996	53506	67652	30931	867	.31056	23480	67705	86246	376
135	3.24818	78138	73723	98777	048	0.30786	39713	28498	99750	548
136	.27665	76422	97329	11443	926	.30518	90399	20228	08131	505
137	.30537	70040	53645	96356	078	.30253	73501	33924	41824	435
138	.33434	80862	61023	39711	728	.29990	87000	30581	27395	911
139	.36357	30952	07540	50890	318	.29730	28894	25775	24304	706
140	3.39305	42565	19026	13629	581	0.29471	97198	74421	23663	566
141	.42279	38153	28551	03522	967	.29215	89946	55659	92903	789
142	.45279	40364	47405	62602	350	.28962	05187	57876	51253	226
143	.48305	72045	37576	33084	269	.28710	40988	63849	61938	250
144	.51358	56242	85733	63770	467	.28460	95433	36029	28011	557
145	3.54438	16205	78745	14106	010	0.28213	66622	01942	79690	337
146	.57544	75386	80726	92511	704	.27968	52671	39727	42063	470
147	.60678	57444	11646	77322	765	.27725	51714	63788	72992	081
148	.63839	86243	27492	80483	585	.27484	61901	10583	61985	084
149	.67028	85859	02021	26069	947	.27245	81396	24526	81780	282
150	3.70245	80577	10097	27735	985	0.27009	08381	44019	85302	330
151	.73490	94896	12642	61314	616	.26774	41053	87601	41601	409
152	.76764	53529	43204	41037	843	.26541	77626	40218	05300	877
153	.80066	81406	96159	20188	329	.26311	16327	39614	14998	621
154	.83398	03677	16566	49446	801	.26082	55400	62840	16975	235
155	3.86758	45708	91686	38762	187	0.25855	93105	12878	11462	718
156	.90148	33093	44175	81243	803	.25631	27715	05383	19620	141
157	.93567	91646	26978	10358	430	.25408	57519	55540	70247	661
158	.97017	47409	19920	74610	667	.25187	80822	65037	06147	574
159	4.00497	26652	28036	26893	504	.24968	95943	09144	10910	763
160	4.04007	55875	81621	38818	691	0.24752	01214	23915	57768	972
161	.07548	61812	38049	63574	063	.24536	94983	93494	83007	990
162	.11120	71428	85352	84208	626	.24323	75614	37532	87283	989
163	.14724	11928	47586	97716	905	.24112	41481	98715	69025	103
164	.18359	10752	91997	98882	778	.23902	90977	30399	94932	856
165	4.22025	95584	38003	41550	903	0.23695	22504	84356	13423	360
166	.25724	94347	68005	68821	813	.23489	34482	98618	17666	319
167	.29456	35212	40053	17615	989	.23285	25343	85438	65690	883
168	.33220	46595	02365	17123	644	.23082	93533	19348	65831	400
169	.37017	57161	09737	14851	779	.22882	37510	25321	36583	071
170	4.40847	95827	41842	78299	274	0.22683	55747	67038	50727	579
171	.44711	91764	23449	34735	479	.22486	46731	35258	74371	974
172	.48609	74397	46563	26129	116	.22291	08960	36287	12320	482
173	.52541	73410	94522	70973	329	.22097	40946	80544	71968	532
174	.56508	18748	68054	39580	576	.21905	41215	71237	58671	293
175	4.60509	40617	13311	64378	935	0.21715	08304	93124	16295	306
176	.64545	69487	51911	21830	277	.21526	40765	01380	27411	572
177	.68617	36098	12986	37812	013	.21339	37159	10560	88331	709
178	.72724	71456	67273	83658	665	.21153	96062	83657	74925	548
179	.76868	06842	63252	45548	740	.20970	16064	21252	15888	956

TABLE VIII.—*Value of* $e^{\pm\frac{n\pi}{360}}$ *to 23 places of decimals or significant figures at intervals of unity from* $n=o$ *to* $n=360$—Continued.

n	$e^{\frac{n\pi}{360}}$					$e^{-\frac{n\pi}{360}}$				
180	4.81047	73809	65351	65547	304	0.20787	95763	50761	90854	696
181	.85264	04187	94247	68376	546	.20607	33773	15781	71456	866
182	.89517	30086	69266	03886	689	.20428	28717	65516	24171	015
183	.93807	83896	52908	51238	967	.20250	79233	44301	94157	308
184	.98135	98291	97523	46992	426	.20074	83968	81237	92433	351
185	5.02502	06233	94138	15608	064	0.19900	41583	79862	00996	409
186	.06906	40972	23471	97348	712	.19727	50750	07976	28001	825
187	.11349	36048	09149	85162	220	.19556	10150	87516	24785	622
188	.15831	25296	73134	98890	260	.19386	18480	84525	93994	446
189	.20352	42849	93400	42046	668	.19217	74145	99217	10355	391
190	5.24913	23138	63859	03458	976	0.19050	76763	56114	78681	754
191	.29514	00895	56571	83265	967	.18885	24161	94288	54068	539
192	.34155	11157	86254	40114	022	.18721	15380	57668	49883	602
193	.38836	89269	77101	73897	017	.18558	49169	85445	59806	685
194	.43559	70885	31951	76039	916	.18397	24291	02555	20809	382
195	5.48323	91971	03807	97136	379	0.18237	39516	10243	44604	271
196	.53129	88808	69741	99716	920	.18078	93627	76715	45721	131
197	.57977	97998	07196	82047	891	.17921	85419	27864	94992	380
198	.62868	56459	72711	77144	131	.17766	13694	38084	27848	651
199	.67802	01437	83090	49620	933	.17611	77267	21154	37438	854
200	5.72778	70502	99033	31615	443	0.17458	74962	21213	83197	126
201	.77799	01555	11255	57775	134	.17307	05614	03806	46081	903
202	.82863	32826	29113	78243	032	.17156	68067	47006	62309	877
203	.87972	02883	71761	47667	295	.17007	61177	32621	68000	005
204	.93125	50632	61857	07528	136	.16859	83808	37470	87729	933
205	5.98324	15319	21845	98509	259	0.16713	34835	24740	00589	333
206	6.03568	36533	72839	59245	528	.16568	13142	35411	17891	704
207	.08858	54213	36113	87554	966	.16424	17623	79767	07278	251
208	.14195	08645	37250	60212	885	.16281	47183	28968	98514	529
209	.19578	40470	12944	27450	523	.16140	00734	06708	06842	675
210	6.25008	90684	20498	18661	514	0.15999	77198	80929	10309	347
211	.30487	00643	50033	16278	404	.15860	75509	55626	18041	887
212	.36013	12066	39432	75439	813	.15722	94607	62709	66992	878
213	.41587	67036	92048	87908	302	.15586	33443	53943	85216	098
214	.47211	08007	97192	09721	086	.15450	90976	92954	60275	062
215	6.52883	77804	53430	93262	120	0.15316	66176	47306	51918	782
216	.58606	19626	94724	85836	308	.15183	58019	80648	88688	251
217	.64378	77054	19415	78406	301	.15051	65493	44929	88641	365
218	.70201	94047	22103	09921	242	.14920	87592	72678	44903	714
219	.76076	14952	28427	54626	483	.14791	23321	69353	17267	840
220	6.82001	84504	32789	41895	435	0.14662	71693	05757	71574	243
221	.87979	47830	39026	80471	041	.14535	31728	10522	09113	701
222	.94009	50453	04079	81546	502	.14409	02456	62649	28792	320
223	7.00092	38293	84666	97854	660	.14283	82916	84126	65298	220
224	.06228	57676	87000	18874	469	.14159	72155	32601	47001	970
225	7.12418	55332	19564	85403	134	0.14036	69226	94120	17811	747
226	.18662	78399	48992	10085	391	.13914	73194	75930	67688	869
227	.24961	74431	59050	14038	971	.13793	83129	99347	17009	761
228	.31315	91398	12782	13469	168	.13673	98111	92677	00436	697
229	.37725	77689	17818	14127	594	.13555	17227	84208	96431	763
230	7.44191	82118	94888	95642	314	0.13437	39572	95262	49016	520
231	.50714	53929	49569	92130	574	.13320	64250	33297	28843	804
232	.57294	42794	47283	00103	068	.13204	90370	85082	81107	998
233	.63931	98822	91585	69482	006	.13090	17053	09927	08276	070
234	.70627	72563	05775	58586	066	.12976	43423	32964	36073	612
235	7.77382	15006	17839	59185	504	0.12863	68615	38501	11608	165
236	.84195	77590	48777	23202	201	.12751	91770	63419	82956	263
237	.91069	12205	04327	48324	230	.12641	12037	90640	09980	896
238	.98002	71193	70129	05724	452	.12531	28573	42636	56582	583
239	8.04997	07359	10344	19219	860	.12422	40540	75013	15019	892

TABLE VIII.—*Values of* $e^{\pm\frac{n\pi}{360}}$ *to 23 places of decimals or significant figures at intervals of unity from* $n=0$ *to* $n=360$—Continued.

n	$e^{\frac{n\pi}{360}}$	$e^{-\frac{n\pi}{360}}$
240	8. 12052 73966 69776 31584 700	0. 12314 47110 70133 13364 153
241	. 19170 24748 79512 20337 923	. 12207 47461 30804 57578 317
242	. 26350 13908 66119 52166 269	. 12101 40777 74020 60131 364
243	. 33592 96124 64430 92220 258	. 11996 26252 24754 07477 529
244	. 40899 26554 33946 11833 666	. 11892 03084 09806 19143 782
245	8. 48269 60838 78883 65769 776	0. 11788 70479 51708 51579 606
246	. 55704 55106 71914 37891 883	. 11686 27651 62678 00330 151
247	. 63204 65978 81608 82193 324	. 11584 73820 38624 54497 323
248	. 70770 50572 03631 14405 896	. 11484 08212 53210 57853 367
349	. 78402 66503 95712 37936 932	. 11384 30061 51962 31368 003
250	8. 86101 71897 16436 16666 869	0. 11285 38607 46432 12303 254
251	. 93868 25383 67870 36172 912	. 11187 33097 08411 65419 737
252	9. 01702 86109 42078 24232 692	. 11090 12783 64195 22224 482
253	. 09606 13738 71543 31006 758	. 10993 76926 88893 04573 217
254	. 17578 68458 83541 99102 710	. 10898 24793 00793 89319 680
255	9. 25621 10984 58498 83788 916	0. 10803 55654 55776 71080 766
256	. 33734 02562 92359 13954 448	. 10709 68790 41770 80559 368
257	. 41918 04977 63014 15006 354	. 10616 63485 73264 16236 506
258	. 50173 80554 00814 45758 072	. 10524 39031 85859 47610 938
259	. 58501 92163 63207 32495 911	. 10432 94726 30877 48527 800
260	9. 66903 03229 13534 14816 610	0. 10342 29872 70007 19498 038
261	. 75377 77729 04024 49510 263	. 10252 43780 70002 58267 516
262	. 83926 80202 63023 50721 902	. 10163 35765 97425 38248 614
263	. 92550 75754 86489 76864 133	. 10075 05150 13433 54778 112
264	10. 01250 30061 33801 07274 906	. 09987 51260 68614 99512 953
265	10. 10026 09373 27905 84421 191	0. 09900 73430 97866 23620 359
266	. 18878 80522 59858 30543 655	. 09814 71000 15315 50760 602
267	. 27809 10926 97775 81021 711	. 09729 43313 09290 01199 605
268	. 36817 68595 00257 10415 295	. 09644 89720 37326 88724 480
269	. 45905 22131 34300 61111 832	. 09561 09578 21227 52368 126
270	10. 55072 40741 97762 18776 707	0. 09478 02248 42154 85279 102
271	. 64319 94239 46392 13375 833	. 09395 67098 35773 23400 282
272	. 73648 53048 25491 59412 138	. 09314 03500 87430 56944 153
273	. 83058 88210 06228 84196 705	. 09233 10834 27382 27974 251
274	. 92551 71389 26656 28462 535	. 09152 88482 26056 77720 981
275	11. 02127 74878 37469 39427 194	0. 09073 35833 89362 07576 123
276	. 11787 71603 52549 12522 656	. 08994 52283 54033 18023 577
277	. 21532 35130 04329 74439 212	. 08916 37230 83019 90074 479
278	. 31362 39668 04034 36878 174	. 08838 90080 60914 74082 642
279	. 41278 60078 06820 87478 044	. 08762 10242 89420 51121 503
280	11. 51281 71876 81881 21773 686	0. 08685 97132 82857 32406 240
281	. 61372 51242 87537 57770 642	. 08610 50170 63708 62544 664
282	. 71551 75022 51379 12769 988	. 08535 68781 58205 92697 753
283	. 81820 20735 55483 60465 906	. 08461 52395 91951 90025 436
284	. 92178 66581 26768 25061 373	. 08388 00448 85581 50085 363
285	12. 02627 91444 32515 08210 017	0. 08315 12380 50460 79142 032
286	. 13168 74900 81115 83997 199	. 08242 87635 84423 13630 715
287	. 23801 97224 28082 36923 751	. 08171 25664 67542 44305 263
288	. 34528 39391 87368 57954 636	. 08100 25921 57943 12880 954
289	. 45348 83090 48050 54145 018	. 08029 87865 87646 49263 274
290	12. 56264 10722 96411 68161 042	0. 07960 10961 58453 17730 730
291	. 67275 05414 43480 45175 074	. 07890 94677 37861 40714 661
292	. 78382 51018 58068 26138 346	. 07822 38486 55020 69091 440
293	. 89587 32124 05355 88321 175	. 07754 41866 96720 68172 576
294	13. 00890 34060 91076 96265 160	. 07687 04301 03414 88845 924
295	13. 12292 42907 11347 68916 485	0. 07620 25275 65278 93586 651
296	. 23794 45495 08192 11707 660	. 07554 04282 18303 07319 705
297	. 35397 29418 30813 05730 179	. 07488 40816 40418 63376 343
298	. 47101 83038 02658 89895 883	. 07423 34378 47658 15045 833
299	. 58908 95489 94337 16123 604	. 07358 84472 90348 83479 756

TABLE VIII.—*Value of* $e^{\pm\frac{n\pi}{360}}$ *to 23 places of decimals or significant figures at intervals of unity from* $n=0$ *to* $n=360$—Continued.

n	$e^{\frac{n\pi}{360}}$					$e^{-\frac{n\pi}{360}}$				
300	13. 70819	56691	02426	02113	374	0. 07294	90608	49339	12960	414
301	. 82834	57346	34235	51186	394	. 07231	52298	32258	04796	701
302	. 94954	88955	98570	53978	601	. 07168	69059	69807	01360	509
303	14. 07181	43822	02548	32482	414	. 07106	40414	12083	92024	232
304	. 19515	15055	54523	33038	599	. 07044	65887	24939	13005	305
305	14. 31956	96583	73173	21391	698	0. 06983	45008	86363	13366	939
306	. 44507	83157	02799	79841	590	. 06922	77312	82905	59665	336
307	. 57168	70356	34899	53854	180	. 06862	62337	06125	51972	697
308	. 69940	54600	36058	43239	420	. 06802	99623	49072	24242	264
309	. 82824	33152	82226	81168	582	. 06743	88718	02797	02216	549
310	14. 95821	04129	99429	92888	587	0. 06685	29170	52894	92312	725
311	15. 08931	66508	10970	75002	865	. 06627	20534	76076	75149	998
312	. 22157	20130	91181	85629	540	. 06569	62368	36770	77612	585
313	. 35498	65717	25783	85622	353	. 06512	54232	83753	97568	760
314	. 48957	04868	78908	21351	505	. 06455	95693	46812	55591	273
315	15. 62533	40077	66842	90294	364	0. 06399	86319	33431	48247	389
316	. 76228	74734	38559	81883	558	. 06344	25683	25512	77747	710
317	. 90044	13135	63083	37706	315	. 06289	13361	76122	32962	040
318	16. 03980	60492	23760	27247	885	. 06234	48935	06264	97027	672
319	. 18039	22937	19490	87927	558	. 06180	31987	01687	56990	743
320	16. 32221	07533	72983	31192	026	0. 06126	62105	09709	91134	679
321	. 46527	22283	46091	69911	820	. 06073	38880	36083	09861	317
322	. 60958	76134	62300	76276	270	. 06020	61907	41875	26199	944
323	. 75516	78990	36419	33804	977	. 05968	30784	40384	32227	413
324	. 90202	41717	11546	01993	393	. 05916	45112	94077	57888	523
325	17. 05016	76153	03370	67490	843	0. 05865	04498	11557	88910	155
326	. 19960	95116	51876	11575	491	. 05814	08548	44556	20705	151
327	. 35036	12414	80504	80046	591	. 05763	56875	84950	25362	667
328	. 50243	42852	62855	98504	099	. 05713	49095	61809	09020	719
329	. 65584	02240	96979	33333	821	. 05663	84826	38463	37113	933
330	17. 81059	07405	87331	56566	908	0. 05614	63690	09601	05185	051
331	. 96669	76197	34463	31140	320	. 05565	85311	98388	33142	596
332	18. 12417	27498	32503	91954	067	. 05517	49320	53615	61039	295
333	. 28302	81233	74512	57506	306	. 05469	55347	46868	24636	339
334	. 44327	58379	65764	66793	056	. 05422	03027	69721	89207	412
335	18. 60492	80972	45042	96590	118	0. 05374	91999	30962	20223	646
336	. 76799	72118	14003	75195	206	. 05328	21903	53828	69746	206
337	. 93249	56001	74688	70203	097	. 05281	92384	73282	57537	209
338	19. 09843	57896	75253	89920	315	. 05236	03090	33298	26082	030
339	. 26583	04174	63988	00603	375	. 05190	53670	84178	48896	838
340	19. 43469	22314	51692	24830	545	0. 05145	43779	79892	71674	436
341	. 60503	40912	82495	49996	317	. 05100	73073	75438	65999	091
342	. 77686	89693	13178	40155	162	. 05056	41212	24226	75537	216
343	. 95020	99516	01081	09241	569	. 05012	47857	75487	34785	287
344	20. 12507	02389	00669	79061	689	. 04968	92675	71700	40629	501
345	20. 30146	31476	68838	11393	295	0. 04925	75334	46047	57143	187
346	. 47940	21110	79019	70049	959	. 04882	95505	19886	34218	104
347	. 65890	06800	44189	35867	752	. 04840	52862	00246	20794	325
348	. 83997	25242	48830	65263	134	. 04798	47081	77346	53620	544
349	21. 02263	14331	89948	51294	515	. 04756	77844	22136	02642	314
350	21. 20689	13172	27206	15042	175	0. 04715	44831	83853	54279	980
351	. 39276	62086	42266	24607	222	. 04674	47729	87610	14020	846
352	. 58027	02627	07417	09125	285	. 04633	86226	31992	09911	568
353	. 76941	77587	63565	05900	027	. 04593	60011	86684	78696	681
354	. 96022	31013	07675	50090	701	. 04553	68779	90117	16507	855
355	22. 15270	08210	89744	88342	326	0. 04514	12226	47126	76165	636
356	. 34686	55762	19387	70332	009	. 04474	90050	26644	93311	316
357	. 54273	21532	82122	45426	202	. 04436	01952	59402	23741	086
358	. 74031	54684	65441	65506	596	. 04397	47637	35653	74467	767
359	. 93963	05686	94751	69532	741	. 04359	26811	02924	11187	273
360	23. 14069	26327	79269	00572	909	0. 04321	39182	63772	24977	442

TABLE IX.—*Values of $e^{\pm n\pi}$ to 25 places of decimals or significant figures for various values of* n.

n	$e^{n\pi}$							$e^{-n\pi}$				
7/6			39. 06361	33631	89410	86273	103	0. 02559	92703	67096	25596	73767
13/6			903. 95906	99632	95003	11733	87	. 00110	62447	77255	90464	07938
19/6			20918. 23899	06336	20474	74990		. 00004	78051	71384	06160	18611
5/4			50. 75401	95117	34935	60233	883	0. 01970	28729	86617	11028	26839
9/4			1174. 48316	53991	39896	15170	1	. 00085	14383	42805	15803	58525
13/4			27178. 35393	28751	52262	56105		. 00003	67939	86952	62379	65643
4/3			65. 94296	52000	64414	66050	359	0. 01516	46198	64546	56995	25407
7/3			1525. 96588	89887	50315	18599	0	. 00065	53226	43327	69247	97556
10/3			35311. 90760	51944	42270	60088		. 00002	83190	59145	16207	79080
3/2			111. 31777	84898	56226	02684	10	0. 00898	32910	21129	42788	96650
5/2			2575. 97049	65975	70550	92240	7	. 00038	82032	03926	76624	72325
7/2			59609. 74149	28721	55884	50138		. 00001	67757	81524	22578	70825
5/3			187. 91462	85023	98509	43960	74	0. 00532	15654	78800	58297	30579
8/3			4348. 47465	93769	06427	43192	0	. 00022	99656	95636	20042	96150
11/3		1	00626. 71551	40705	19800	4780		. 00000	99377	18774	69429	21058
7/4			244. 15106	28542	75029	02837	17	0. 00409	58248	89350	83589	25536
11/4			5649. 82470	14771	50409	93657	8	. 00017	69966	41991	13104	12384
15/4		1	30740. 85684	59666	27285	2389		. 00000	76487	18419	96689	60038
11/6			317. 21714	25286	95191	88997	58	0. 00315	24147	52962	28940	00659
17/6			7340. 62439	31050	68162	80074	8	. 00013	62281	93468	02216	28376
23/6		1	69867. 13281	35262	43509	0267		. 00000	58869	54017	74846	24408
2			535. 49165	55247	64736	50304	93	0. 00186	74427	31707	98881	44302
3			12391. 64780	79166	97481	50654		. 00008	06995	17570	30459	92392
4		2	86751. 31313	66532	99746	6916		. 00000	34873	42356	20899	54918
5		66	35623. 99934	11342	33266	264		0. 00000	01507	01727	53900	64611
6		1535	52935. 39544	66939	22626	2		. 00000	00065	12412	13607	99007
7		35533	21280. 84704	43596	96468			. 00000	00002	81426	84574	85553
8	8	22263	15585. 59499	52749	6691			0. 00000	00000	12161	55670	94093
9	190	27738	95292. 16129	16866	54			. 00000	00000	00525	54851	76006
10	4403	15058	60632. 02901	14005	4			. 00000	00000	00022	71101	06832

TABLE X.—*Values of sin x and cos x to 23 places of decimals at intervals of unity from 0 to 100.*

x	sin x	cos x
0	0. 00000 00000 00000 00000 000	1. 00000 00000 00000 00000 000
1	+ . 84147 09848 07896 50665 250	+0. 54030 23058 68139 71740 094
2	+ . 90929 74268 25681 69539 602	− . 41614 68365 47142 38699 757
3	+ . 14112 00080 59867 22210 074	− . 98999 24966 00445 45727 157
4	− . 75680 24953 07928 25137 264	− . 65364 36208 63611 91463 917
5	−0. 95892 42746 63138 46889 315	+0. 28366 21854 63226 26446 664
6	− . 27941 54981 98925 87281 156	+ . 96017 02866 50366 02054 565
7	+ . 65698 65987 18789 09039 700	+ . 75390 22543 43304 63814 120
8	+ . 98935 82466 23381 77780 812	− . 14550 00338 08613 52586 884
9	+ . 41211 84852 41756 56975 627	− . 91113 02618 84676 98836 829
10	−0. 54402 11108 89369 81340 475	−0. 83907 15290 76452 45225 886
11	− . 99999 02065 50703 45705 156	+ . 00442 56979 88050 78574 836
12	− . 53657 29180 00434 97166 537	+ . 84385 39587 32492 10465 396
13	+ . 42016 70368 26640 92186 896	+ . 90744 67814 50196 21385 269
14	+ . 99060 73556 94870 30787 535	+ . 13673 72182 07833 59424 893
15	+0. 65028 78401 57116 86582 974	−0. 75968 79128 58821 27384 815
16	− . 28790 33166 65065 29478 446	− . 95765 94803 23384 64189 964
17	− . 96139 74918 79556 85726 164	− . 27516 33380 51596 92222 034
18	− . 75098 72467 71676 10375 016	+ . 66031 67082 44080 14481 610
19	+ . 14987 72096 62952 32975 424	+ . 98870 46181 86669 25289 835
20	+0. 91294 52507 27627 65437 610	+0. 40808 20618 13391 98606 227
21	+ . 83665 56385 36056 03186 648	− . 54772 92602 24268 42138 427
22	− . 00885 13092 90403 87592 169	− . 99996 08263 94637 12645 417
23	− . 84622 04041 75170 63524 133	− . 53283 30203 33397 55521 576
24	− . 90557 83620 06623 84513 579	+ . 42417 90073 36996 97593 705
25	−0. 13235 17500 97773 02890 201	+0. 99120 28118 63473 59808 329
26	+ . 76255 84504 79602 73751 582	+ . 64691 93223 28640 34272 138
27	+ . 95637 59284 04503 01343 234	− . 29213 88087 33836 19337 140
28	+ . 27090 57883 07869 01998 634	− . 96260 58663 13566 60197 545
29	− . 66363 38842 12967 50215 117	− . 74805 75296 89000 35176 519
30	−0. 98803 16240 92861 78998 775	+0. 15425 14498 87584 05071 866
31	− . 40403 76453 23065 00604 877	+ . 91474 23578 04531 27896 244
32	+ . 55142 66812 41690 55066 156	+ . 83422 33605 06510 27221 553
33	+ . 99991 18601 07267 14572 808	− . 01327 67472 23059 47891 522
34	+ . 52908 26861 20023 82083 249	− . 84857 02747 84605 18659 997
35	−0. 42818 26694 96151 00440 675	−0. 90369 22050 91506 75984 730
36	− . 99177 88534 43115 73683 529	− . 12796 36896 27404 68102 833
37	− . 64353 81333 56999 46068 567	+ . 76541 40519 45343 35649 108
38	+ . 29636 85787 09385 31739 230	+ . 95507 36440 47294 85758 654
39	+ . 96379 53862 84087 75326 066	+ . 26664 29323 59937 25152 683
40	+0. 74511 31604 79348 78698 771	−0. 66693 80616 52261 84438 409
41	− . 15862 26688 04708 98710 332	− . 98733 92775 23826 45822 883
42	− . 91652 15479 15633 78589 899	− . 39998 53149 88351 29395 471
43	− . 83177 47426 28598 28820 958	+ . 55511 33015 20625 67704 483
44	+ . 01770 19251 05413 57780 795	+ . 99984 33086 47691 22006 901
45	+0. 85090 35245 34118 42486 238	+0. 52532 19888 17729 69604 746
46	+ . 90178 83476 48809 18503 329	− . 43217 79448 84778 29495 278
47	+ . 12357 31227 45224 00406 153	− . 99233 54691 50928 71827 975
48	− . 76825 46613 23666 79904 497	− . 64014 43394 69199 73131 294
49	− . 95375 26527 59471 81836 042	+ . 30059 25437 43637 08368 703
50	−0. 26237 48537 03928 78591 439	+0. 96496 60284 92113 27406 896
51	+ . 67022 91758 43374 73449 435	+ . 74215 41968 13782 53946 738
52	+ . 98662 75920 40485 29658 757	− . 16299 07807 95705 48100 333
53	+ . 39592 51501 81834 18150 339	− . 91828 27862 12118 89119 973
54	− . 55878 90488 51616 24581 787	− . 82930 98328 63150 14772 785
55	−0. 99975 51733 58619 83659 863	+0. 02212 67562 61955 73456 356
56	− . 52155 10020 86911 88018 741	+ . 85322 01077 22584 11396 968
57	+ . 43616 47552 47824 95908 053	+ . 89986 68269 69193 78650 300
58	+ . 99287 26480 84537 11816 509	+ . 11918 01354 48819 28543 584
59	+ . 63673 80071 39137 88077 123	− . 77108 02229 75845 22938 744

TABLE X.—*Values of sin x and cos x to 23 places of decimals at intervals of unity from 0 to 100*—Continued.

x	$\sin x$					$\cos x$				
60	−0. 30481	06211	02216	70562	565	−0. 95241	29804	15156	29269	382
61	− . 96611	77700	08392	94701	829	− . 25810	16359	38267	44570	121
62	− . 73918	06966	49222	86727	602	+ . 67350	71623	23586	25288	783
63	+ . 16735	57003	02806	92152	784	+ . 98589	65815	82549	69743	864
64	+ . 92002	60381	96790	68335	154	+ . 39185	72304	29550	00516	171
65	+0. 82682	86794	90103	46771	021	−0. 56245	38512	38172	03106	212
66	− . 02655	11540	23966	79446	384	− . 99964	74559	66349	96483	045
67	− . 85551	99789	75322	25899	683	− . 51776	97997	89505	06565	339
68	− . 89792	76806	89291	26040	073	+ . 44014	30224	96040	70593	105
69	− . 11478	48137	83187	22054	507	+ . 99339	03797	22271	63756	155
70	+0. 77389	06815	57889	09778	733	+0. 63331	92030	86299	83233	201
71	+ . 95105	46532	54374	63665	657	− . 30902	27281	66070	70291	749
72	+ . 25382	33627	62036	27306	903	− . 96725	05882	73882	48729	171
73	− . 67677	19568	87307	62215	498	− . 73619	27182	27315	96016	815
74	− . 98514	62604	68247	37085	189	+ . 17171	73418	30777	55609	845
75	−0. 38778	16354	09430	43773	094	+0. 92175	12697	24749	31639	230
76	+ . 56610	76368	98180	32361	028	+ . 82433	13311	07557	75991	501
77	+ . 99952	01585	80731	24386	610	− . 03097	50317	31216	45752	196
78	+ . 51397	84559	87535	21169	609	− . 85780	30932	44987	85540	835
79	− . 44411	26687	07508	36850	760	− . 89597	09467	90963	14833	703
80	−0. 99388	86539	23375	18973	081	−0. 11038	72438	39047	55811	787
81	− . 62988	79942	74453	87856	521	+ . 77668	59820	21631	15768	342
82	+ . 31322	87824	33085	15263	353	+ . 94967	76978	82543	20471	326
83	+ . 96836	44611	00185	40435	015	+ . 24954	01179	73338	12437	735
84	+ . 73319	03200	73292	16636	321	− . 68002	34955	87338	79542	720
85	−0. 17607	56199	48587	07696	212	−0. 98437	66433	94041	89491	821
86	− . 92345	84470	04059	80260	163	− . 38369	84449	49741	84477	893
87	− . 82181	78366	30822	54487	211	+ . 56975	03342	65311	92000	851
88	+ . 03539	83027	33660	68362	543	+ . 99937	32836	95124	65698	442
89	+ . 86006	94058	12453	22683	685	+ . 51017	70449	41668	89902	379
90	+0. 89399	66636	00557	89051	827	−0. 44807	36161	29170	15236	548
91	+ . 10598	75117	51156	85002	021	− . 99436	74609	28201	52610	672
92	− . 77946	60696	15804	68855	400	− . 62644	44479	10339	06880	027
93	− . 94828	21412	69947	23213	104	+ . 31742	87015	19701	64974	551
94	− . 24525	19854	67654	32522	044	+ . 96945	93666	69987	60380	439
95	+0. 68326	17147	36120	98369	958	+0. 73017	35609	94819	66479	352
96	+ . 98358	77454	34344	85760	773	− . 18043	04492	91083	95011	850
97	+ . 37960	77390	27521	69648	192	− . 92514	75365	96413	89170	475
98	− . 57338	18719	90422	88494	922	− . 81928	82452	91459	25267	566
99	− . 99920	68341	86353	69443	272	+ . 03982	08803	93138	89816	180
100	−0. 50636	56411	09758	79365	656	+0. 86231	88722	87683	93410	194

Table XI.—*Values of sin x and cos x to 23 places of decimals at intervals of 0.1 from 0.0 to 10.0.*

x	$\sin x$					$\cos x$				
0.0	+0.00000	00000	00000	00000	000	+1.00000	00000	00000	00000	000
.1	.09983	34166	46828	15230	681	0.99500	41652	78025	76609	556
.2	.19866	93307	95061	21545	941	.98006	65778	41241	63112	420
.3	.29552	02066	61339	57510	532	.95533	64891	25606	01964	231
.4	.38941	83423	08650	49166	631	.92106	09940	02885	08279	853
0.5	+0.47942	55386	04203	00027	329	+0.87758	25618	90372	71611	628
.6	.56464	24733	95035	35720	095	.82533	56149	09678	29724	095
.7	.64421	76872	37691	05367	261	.76484	21872	84488	42625	586
.8	.71735	60908	99522	76162	717	.69670	67093	47165	42092	075
.9	.78332	69096	27483	38846	138	.62160	99682	70664	45648	472
1.0	+0.84147	09848	07896	50665	250	+0.54030	23058	68139	71740	094
.1	.89120	73600	61435	33995	180	.45359	61214	25577	38777	137
.2	.93203	90859	67226	34967	013	.36235	77544	76673	57763	837
.3	.96355	81854	17192	96470	135	.26749	88286	24587	40699	798
.4	.98544	97299	88460	18065	947	.16996	71429	00240	93861	675
1.5	+0.99749	49866	04054	43094	172	+0.07073	72016	67702	91008	819
.6	.99957	36030	41505	16434	211	− .02919	95223	01288	72620	577
.7	.99166	48104	52468	61534	613	.12884	44942	95524	68408	764
.8	.97384	76308	78195	18653	237	.22720	20946	93087	05531	667
.9	.94630	00876	87414	48848	971	.32328	95668	63503	42227	883
2.0	+0.90929	74268	25681	69539	602	−0.41614	68365	47142	38699	757
.1	.86320	93666	48873	77068	076	.50484	61045	99857	45162	094
.2	.80849	64038	19590	18430	404	.58850	11172	55345	70852	414
.3	.74570	52121	76720	17738	541	.66627	60212	79824	19331	788
.4	.67546	31805	51150	92656	577	.73739	37155	41245	49960	882
2.5	+0.59847	21441	03956	49405	185	−0.80114	36155	46933	71483	350
.6	.51550	13718	21464	23525	773	.85688	87533	68947	23379	770
.7	.42737	98802	33829	93455	605	.90407	21420	17061	14798	253
.8	.33498	81501	55904	91954	385	.94222	23406	68658	15258	679
.9	.23924	93292	13982	32818	426	.97095	81651	49590	52178	111
3.0	+0.14112	00080	59867	22210	074	−0.98999	24966	00445	45727	157
.1	+ .04158	06624	33290	57919	470	.99913	51502	73279	46449	238
.2	− .05837	41434	27579	90913	722	.99829	47757	94753	08466	166
.3	.15774	56941	43248	38201	165	.98747	97699	08864	88393	659
.4	.25554	11020	26831	31924	990	.96679	81925	79461	01428	220
3.5	−0.35078	32276	89619	84812	037	−0.93645	66872	90796	33769	866
.6	.44252	04432	94852	38426	673	.89675	84163	34147	00587	029
.7	.52983	61409	08493	21321	078	.84810	00317	10408	15883	567
.8	.61185	78909	42719	07573	359	.79096	77119	14416	69999	657
.9	.68776	61591	83973	81809	089	.72593	23042	00140	12937	233
4.0	−0.75680	24953	07928	25137	264	−0.65364	36208	63611	91463	917
.1	.81827	71110	64410	50426	504	.57482	39465	33268	91153	503
.2	.87157	57724	13588	06001	858	.49026	08213	40699	57765	554
.3	.91616	59367	49454	98403	171	.40079	91720	79975	29690	676
.4	.95160	20738	89515	95403	539	.30733	28699	78419	68311	914
4.5	−0.97753	01176	65097	05538	914	−0.21079	57994	30779	70598	048
.6	.99369	10036	33464	45613	810	− .11215	25269	35054	51742	991
.7	.99992	32575	64100	88417	954	− .01238	86634	62890	73715	051
.8	.99616	46088	35840	67178	160	+ .08749	89834	39446	56932	022
.9	.98245	26126	24332	51227	638	.18651	23694	22575	40449	433
5.0	−0.95892	42746	63138	46889	315	+0.28366	21854	63226	26446	664
.1	.92581	46823	27732	29694	615	.37797	77427	12980	56332	058
.2	.88345	46557	20153	26467	308	.46851	66713	00376	95863	909
.3	.83226	74422	23901	16356	456	.55437	43361	79160	92944	495
.4	.77276	44875	55987	36235	847	.63469	28759	42634	36240	675
5.5	−0.70554	03255	70391	90623	192	+0.70866	97742	91260	00002	742
.6	.63126	66378	72321	31146	367	.77556	58785	10249	79765	581
.7	.55068	55425	97637	76122	735	.83471	27848	39159	68274	923
.8	.46460	21794	13757	21141	823	.88551	95169	41319	00416	466
.9	.37387	66648	30236	35981	485	.92747	84307	44035	74090	610

TABLE XI.—*Values of sin x and cos x to 23 places of decimals at intervals of 0.1 from 0.0 to 10.0*—Continued.

x	$\sin x$	$\cos x$
6.0	−0. 27941 54981 98925 87281 156	+0. 96017 02866 50366 02054 565
.1	− .18216 25042 72095 54002 413	.98326 84384 42584 59658 502
.2	− .08308 94028 17496 57800 058	.99654 20970 23217 47513 940
.3	+ .01681 39004 84349 89031 097	.99985 86363 83415 14228 667
.4	.11654 92048 50493 28948 042	.99318 49187 58192 65859 474
6.5	+0. 21511 99880 87815 52429 695	+0. 97658 76257 28023 49988 631
.6	.31154 13635 13378 17435 499	.95023 25919 58529 46621 974
.7	.40484 99206 16598 16163 219	.91438 31482 35319 44113 790
.8	.49411 33511 38608 32222 208	.86939 74903 49825 17244 162
.9	.57843 97643 88199 87017 378	.81572 51001 25357 07265 676
7.0	+0. 65698 65987 18789 09039 700	+0. 75390 22543 43304 63814 120
.1	.72896 90401 25876 15207 599	.68454 66664 42806 34062 180
.2	.79366 78638 49153 05246 445	.60835 13145 32254 67100 485
.3	.85043 66206 28564 51751 737	.52607 75173 81105 18891 541
.4	.89870 80958 11626 75926 950	.43854 73275 74390 64913 410
7.5	+0. 93799 99767 74738 85794 846	+0. 34663 53178 35025 81097 162
.6	.96791 96720 31486 42590 346	+ .25125 98425 82255 38005 815
.7	.98816 82338 77000 36855 239	+ .15337 38620 37864 52597 738
.8	.99854 33453 74604 96343 877	+ .05395 54205 62649 57303 257
.9	.99894 13418 39772 03630 491	− .04600 21256 39536 59449 775
8.0	+0. 98935 82466 23381 77780 812	−0. 14550 00338 08613 52586 884
.1	.96988 98108 45086 24322 432	.24354 41537 35791 46446 505
.2	.94073 05566 79772 90115 365	.33915 48609 83835 20740 049
.3	.90217 18337 56293 64000 050	.43137 68449 70620 17370 933
.4	.85459 89080 88280 66283 324	.51928 86541 16685 29914 480
8.5	+0. 79848 71126 23490 28666 691	−0. 60201 19026 84823 61534 843
.6	.73439 70978 74113 14371 716	.67872 00473 20012 70086 447
.7	.66296 92300 82182 79220 235	.74864 66455 97399 15731 879
.8	.58491 71928 91762 25353 093	.81109 30140 61655 56288 909
.9	.50102 08564 57884 98201 617	.86543 52092 41112 05963 983
9.0	+0. 41211 84852 41756 56975 627	−0. 91113 02618 84676 98836 829
.1	+ .31909 83623 49351 77079 400	.94772 16021 31112 02471 907
.2	+ .22288 99141 00246 95752 807	.97484 36214 04163 74194 145
.3	+ .12445 44235 07062 40798 941	.99222 53254 52603 40775 691
.4	+ .02477 54254 53358 12107 977	.99969 30420 35206 47217 795
9.5	−0. 07515 11204 61809 30728 348	−0. 99717 21561 96378 47289 160
.6	.17432 67812 22979 98512 410	.98468 78557 94126 91002 034
.7	.27176 06264 10943 12433 774	.96236 48798 31310 03407 036
.8	.36647 91292 51927 74816 925	.93042 62721 04753 51854 938
.9	.45753 58937 75321 04441 382	.88919 11526 25361 05463 444
10.0	−0. 54402 11108 89369 81340 475	−0. 83907 15290 76452 45225 886

TABLE XII.—*Values of sin x and cos x to 23 places of decimals at intervals of 0.001 from 0.000 to 1.600.*

x	$\sin x$					$\cos x$				
0.000	0.00000	00000	00000	00000	000	1.00000	00000	00000	00000	000
.001	.00099	99998	33333	34166	667	0.99999	95000	00041	66666	528
.002	.00199	99986	66666	93333	331	.99999	80000	00666	66657	778
.003	.00299	99955	00002	02499	957	.99999	55000	03374	99898	750
.004	.00399	99893	33341	86666	342	.99999	20000	10666	66097	778
0.005	0.00499	99791	66692	70831	783	0.99998	75000	26041	64496	529
.006	.00599	99640	00064	79994	446	.99998	20000	53999	93520	004
.007	.00699	99428	33473	39150	327	.99997	55001	00041	50326	542
.008	.00799	99146	66939	73291	723	.99996	80001	70666	30257	819
.009	.00899	98785	00492	07405	100	.99995	95002	73374	26188	857
0.010	0.00999	98333	34166	66468	254	0.99995	00004	16665	27778	026
.011	.01099	97781	68008	75446	684	.99993	95006	10039	20617	059
.012	.01199	97120	02073	59289	053	.99992	80008	63995	85281	066
.013	.01299	96338	36427	42921	659	.99991	55011	90034	96278	551
.014	.01399	95426	71148	51241	801	.99990	20016	00656	20901	438
0.015	0.01499	94375	06328	09109	944	0.99988	75021	09359	17975	106
.016	.01599	93173	42071	41340	585	.99987	20027	30643	36508	430
.017	.01699	91811	78498	72691	726	.99985	55034	80008	14243	829
.018	.01799	90280	15746	27852	832	.99983	80043	73952	76107	331
.019	.01899	88568	53967	31431	205	.99981	95054	29976	32558	650
0.020	0.01999	86666	93333	07936	649	0.99980	00066	66577	77841	270
.021	.02099	84565	34033	81764	335	.99977	95081	03255	88132	556
.022	.02199	82253	76279	77175	771	.99975	80097	60509	19593	878
.023	.02299	79722	20302	18277	769	.99973	55116	59836	06320	750
.024	.02399	76960	66354	28999	311	.99971	20138	23734	58193	002
0.025	0.02499	73959	14712	33066	217	0.99968	75162	75702	58624	967
.026	.02599	70707	65676	53973	517	.99966	20190	40237	62215	698
.027	.02699	67196	19572	14955	411	.99963	55221	42836	92299	214
.028	.02799	63414	76750	38952	746	.99960	80256	09997	38394	779
.029	.02899	59353	37589	48577	881	.99957	95294	69215	53557	207
0.030	0.02999	55002	02495	66076	853	0.99955	00337	48987	51627	216
.031	.03099	50350	71904	13288	752	.99951	95384	78809	04381	810
.032	.03199	45389	46280	11602	188	.99948	80436	89175	38584	710
.033	.03299	40108	26119	81908	762	.99945	55494	11581	32936	824
.034	.03399	34497	11951	44553	435	.99942	20556	78521	14926	773
0.035	0.03499	28546	04336	19281	702	0.99938	75625	23488	57581	460
.036	.03599	22245	03869	25183	461	.99935	20699	80976	76116	700
.037	.03699	15584	11180	80633	489	.99931	55780	86478	24487	902
.038	.03799	08553	26937	03228	414	.99927	80868	76484	91840	819
.039	.03899	01142	51841	09720	085	.99923	95963	88487	98862	358
0.040	0.03998	93341	86634	15945	255	0.99920	01066	60977	94031	457
.041	.04098	85141	32096	36751	449	.99915	96177	33444	49770	040
.042	.04198	76530	89047	85918	946	.99911	81296	46376	58494	043
.043	.04298	67500	58349	76078	755	.99907	56424	41262	28564	524
.044	.04398	58040	40905	18626	492	.99903	21561	60588	80138	853
0.045	0.04498	48140	37660	23632	066	0.99898	76708	47842	40921	992
.046	.04598	37790	49604	99745	054	.99894	21865	47508	41817	869
.047	.04698	26980	77774	54095	689	.99889	57033	05071	12480	849
.048	.04798	15701	23249	92191	340	.99884	82211	67013	76767	299
.049	.04898	03941	87159	17808	403	.99879	97401	80818	48087	272
0.050	0.04997	91692	70678	32879	487	0.99875	02603	94966	24656	287
.051	.05097	78943	75032	37375	800	.99869	97818	58936	84647	237
.052	.05197	65685	01496	29184	649	.99864	83046	23208	81242	407
.053	.05297	51906	51396	03981	925	.99859	58287	39259	37585	623
.054	.05397	37598	26109	55099	505	.99854	23542	59564	41634	531
0.055	0.05497	22750	27067	73387	446	0.99848	78812	37598	40913	005
.056	.05597	07352	55755	47070	891	.99843	24097	27834	37163	704
.057	.05696	91395	13712	61601	567	.99837	59397	85743	80900	770
.058	.05796	74868	02534	99503	794	.99831	84714	67796	65862	676
.059	.05896	57761	23875	40214	896	.99826	00048	31461	23365	235

TABLE XII.—*Values of sin x and cos x to 23 places of decimals at intervals of 0.001 from 0.000 to 1.600*—Continued.

x	*sin x*	*cos x*
0.060	0.05996 40064 79444 59919 909	0.99820 05399 35204 16554 766
.061	.06096 21768 71012 31380 500	.99814 00768 38490 34561 437
.062	.06196 02863 00408 23757 982	.99807 86156 01782 86552 769
.063	.06295 83337 69523 02430 343	.99801 61562 86542 95687 334
.064	.06395 63182 80309 28803 166	.99795 26989 55229 92968 628
0.065	0.06495 42388 34782 60114 361	0.99788 82436 71301 10999 144
.066	.06595 20944 35022 49232 601	.99782 27904 99211 77634 635
.067	.06694 98840 83173 44449 361	.99775 63395 04415 09538 592
.068	.06794 76067 81445 89264 458	.99768 88907 53362 05636 926
.069	.06894 52615 32117 22165 004	.99762 04443 13501 40472 866
0.070	0.06994 28473 37532 76397 655	0.99755 10002 53279 57462 091
.071	.07094 03632 00106 79734 071	.99748 05586 42140 62048 084
.072	.07193 78081 22323 54229 480	.99740 91195 50526 14757 726
.073	.07293 51811 06738 15974 250	.99733 66830 49875 24157 139
.074	.07393 24811 55977 74838 360	.99726 32492 12624 39707 777
0.075	0.07492 97072 72742 34208 684	0.99718 88181 12207 44522 774
.076	.07592 68584 59805 90718 980	.99711 33898 23055 48023 568
.077	.07692 39337 20017 33972 485	.99703 69644 20596 78496 785
.078	.07792 09320 56301 46257 015	.99695 95419 81256 75551 417
.079	.07891 78524 71660 02252 478	.99688 11225 82457 82476 279
0.080	0.07991 46939 69172 68730 688	0.99680 17063 02619 38497 771
.081	.08091 14555 51998 04247 389	.99672 12932 21157 70937 933
.082	.08190 81362 23374 58826 394	.99663 98834 18485 87272 823
.083	.08290 47349 86621 73635 718	.99655 74769 76013 67091 212
.084	.08390 12508 45140 80655 638	.99647 40739 76147 53953 598
0.085	0.08489 76828 02416 02338 544	0.99638 96745 02290 47151 570
.086	.08589 40298 62015 51260 514	.99630 42786 38841 93367 506
.087	.08689 02910 27592 29764 492	.99621 78864 71197 78234 626
.088	.08788 64653 02885 29594 973	.99613 04980 85750 17797 412
.089	.08888 25516 91720 31524 112	.99604 21135 69887 49872 388
0.090	0.08987 85491 98011 04969 125	0.99595 27330 11994 25309 284
.091	.09087 44568 25760 07600 919	.99586 23565 01450 99152 586
.092	.09187 02735 79059 84943 819	.99577 09841 28634 21703 483
.093	.09286 59984 62093 69966 323	.99567 86159 84916 29482 217
.094	.09386 16304 79136 82662 751	.99558 52521 62665 36090 844
0.095	0.09485 71686 34557 29625 724	0.99549 08927 55245 22976 426
.096	.09585 26119 32817 03609 347	.99539 55378 57015 30094 649
.097	.09684 79593 78472 83083 006	.99529 91875 63330 46473 881
.098	.09784 32099 76177 31775 683	.99520 18419 70541 00679 686
.099	.09883 83627 30679 98210 683	.99510 35011 75992 51179 796
0.100	0.09983 34166 46828 15230 681	0.99500 41652 78025 76609 556
.101	.10082 83707 29567 99512 975	.99490 38343 75976 65937 840
.102	.10182 32239 83945 51074 864	.99480 25085 70176 08533 469
.103	.10281 79754 15107 52769 040	.99470 01879 61949 84132 117
.104	.10381 26240 28302 69768 897	.99459 68726 53618 52703 737
0.105	0.10480 71688 28882 49043 655	0.99449 25627 48497 44220 501
.106	.10580 16088 22302 18823 209	.99438 72583 50896 48325 268
.107	.10679 59430 14121 88052 588	.99428 09595 66120 03900 596
.108	.10779 01704 10007 45835 941	.99417 36665 00466 88538 307
.109	.10878 42900 15731 60869 939	.99406 53792 61230 07909 607
0.110	0.10977 83008 37174 80866 495	0.99395 60979 56696 85035 784
.111	.11077 22018 80326 31964 714	.99384 58226 96148 49459 483
.112	.11176 59921 51285 18131 952	.99373 45535 89860 26316 578
.113	.11275 96706 56261 20553 909	.99362 22907 49101 25308 652
.114	.11375 32364 01575 97013 636	.99350 90342 86134 29576 080
0.115	0.11474 66883 93663 81259 372	0.99339 47843 14215 84471 755
.116	.11574 00256 39072 82361 097	.99327 95409 47595 86235 439
.117	.11673 32471 44465 84055 722	.99316 33043 01517 70568 768
.118	.11772 63519 16621 44080 790	.99304 60744 92218 01110 921
.119	.11871 93389 62434 93496 613	.99292 78516 36926 57814 950

TABLE XII.—*Values of sin x and cos x to 23 places of decimals at intervals of 0.001 from 0.000 to 1.600*—Continued.

x	$\sin x$					$\cos x$				
0.120	0.11971	22072	88919	35996	735	0.99280	86358	53866	25224	810
.121	.12070	49559	03206	47206	615	.99268	84272	62252	80653	067
.122	.12169	75838	12547	73970	447	.99256	72259	82294	82259	329
.123	.12269	00900	24315	33626	003	.99244	50321	35193	57029	382
.124	.12368	24735	46003	13267	407	.99232	18458	43142	88655	070
0.125	0.12467	47333	85227	68995	744	0.99219	76672	29329	05314	910
.126	.12566	68685	49729	25157	389	.99207	24964	17930	67355	462
.127	.12665	88780	47372	73569	978	.99194	63335	34118	54873	474
.128	.12765	07608	86148	72735	909	.99181	91787	04055	55198	803
.129	.12864	25160	74174	47043	273	.99169	10320	54896	50278	123
0.130	0.12963	41426	19694	85954	121	0.99156	18937	14788	03959	451
.131	.13062	56395	31083	43179	968	.99143	17638	12868	49177	481
.132	.13161	70058	16843	35844	433	.99130	06424	79267	75039	751
.133	.13260	82404	85608	43632	907	.99116	85298	45107	13813	659
.134	.13359	93425	46144	07929	171	.99103	54260	42499	27814	325
0.135	0.13459	03110	07348	30938	844	0.99090	13312	04547	96193	339
.136	.13558	11448	78252	74799	575	.99076	62454	65348	01628	375
.137	.13657	18431	68023	60677	867	.99063	01689	59985	16913	714
.138	.13756	24048	85962	67852	453	.99049	31018	24535	91451	667
.139	.13855	28290	41508	32784	107	.99035	50441	96067	37644	937
0.140	0.13954	31146	44236	48171	799	0.99021	59962	12637	17189	895
.141	.14053	32607	03861	61995	092	.99007	59580	13293	27270	829
.142	.14152	32662	30237	76542	691	.98993	49297	38073	86655	145
.143	.14251	31302	33359	47427	025	.98979	29115	28007	21689	546
.144	.14350	28517	23362	82584	791	.98964	99035	25111	52197	214
0.145	0.14449	24297	10526	41263	332	0.98950	59058	72394	77275	984
.146	.14548	18632	05272	32992	773	.98936	09187	13854	60997	551
.147	.14647	11512	18167	16543	800	.98921	49421	94478	18007	704
.148	.14746	02927	59922	98870	997	.98906	79764	60241	99027	617
.149	.14844	92868	41398	34041	627	.98892	00216	58111	76256	193
0.150	0.14943	81324	73599	22149	773	0.98877	10779	36042	28673	498
.151	.15042	68286	67680	08215	725	.98862	11454	42977	27245	283
.152	.15141	53744	34944	81070	532	.98847	02243	28849	20028	611
.153	.15240	37687	86847	72225	604	.98831	83147	44579	17178	614
.154	.15339	20107	34994	54727	267	.98816	54168	42076	75856	382
0.155	0.15438	00992	91143	41996	190	0.98801	15307	74239	85038	006
.156	.15536	80334	67205	86651	555	.98785	66566	94954	50224	794
.157	.15635	58122	75247	79319	902	.98770	07947	59094	78054	663
.158	.15734	34347	27490	47428	529	.98754	39451	22522	60814	736
.159	.15833	08998	36311	53983	354	.98738	61079	42087	60855	150
0.160	0.15931	82066	14245	96331	146	0.98722	72833	75626	94904	095
.161	.16030	53540	73987	04906	020	.98706	74715	81965	18284	099
.162	.16129	23412	28387	41960	095	.98690	66727	20914	09029	574
.163	.16227	91670	90460	00278	226	.98674	48869	53272	51905	638
.164	.16326	58306	73379	01876	705	.98658	21144	40826	22328	234
0.165	0.16425	23309	90480	96685	825	0.98641	83553	46347	70185	554
.166	.16523	86670	55265	61216	228	.98625	36098	33596	03560	791
.167	.16622	48378	81396	97208	916	.98608	78780	67316	72356	233
.168	.16721	08424	82704	30268	843	.98592	11602	13241	51818	712
.169	.16819	66798	73183	08481	981	.98575	34564	38088	25966	434
0.170	0.16918	23490	66996	01015	762	0.98558	47669	09560	70917	193
.171	.17016	78490	78473	96702	805	.98541	50917	96348	38117	998
.172	.17115	31789	22117	02607	812	.98524	44312	68126	37476	124
.173	.17213	83376	12595	42577	560	.98507	27854	95555	20391	598
.174	.17312	33241	64750	55773	865	.98490	01546	50280	62691	158
0.175	0.17410	81375	93595	95189	433	0.98472	65389	04933	47463	670
.176	.17509	27769	14318	26146	505	.98455	19384	33129	47797	052
.177	.17607	72411	42278	24778	176	.98437	63534	09469	09416	699
.178	.17706	15292	93011	76492	317	.98419	97840	09537	33225	443
.179	.17804	56403	82230	74417	975	.98402	22304	09903	57745	046

TABLE XII.—*Values of sin x and cos x to 23 places of decimals at intervals of 0.001 from 0.000 to 1.600*—Continued.

x	$\sin x$	$\cos x$
0.180	0.17902 95734 25824 17834 180	0.98384 36927 88121 41459 272
.181	.18001 33274 39859 10581 029	.98366 41713 22728 45058 522
.182	.18099 69014 40581 59452 980	.98348 36661 93246 13586 083
.183	.18198 02944 44417 72574 233	.98330 21775 80179 58485 974
.184	.18296 35054 67974 57756 116	.98311 97056 65017 39552 448
0.185	0.18394 65335 28041 20836 370	0.98293 62506 30231 46781 122
.186	.18492 93776 41589 64000 231	.98275 18126 59276 82121 799
.187	.18591 20368 25775 84083 224	.98256 63919 36591 41132 959
.188	.18689 45100 97940 70855 554	.98237 99886 47595 94537 971
.189	.18787 67964 75611 05288 013	.98219 26029 78693 69683 022
0.190	0.18885 88949 76500 57799 285	0.98200 42351 17270 31896 788
.191	.18984 08046 18510 86484 571	.98181 48852 51693 65751 875
.192	.19082 25244 19732 35325 424	.98162 45535 71313 56228 034
.193	.19180 40533 98445 32380 691	.98143 32402 66461 69777 178
.194	.19278 53905 73120 87958 485	.98124 09455 28451 35290 214
0.195	0.19376 65349 62421 92769 058	0.98104 76695 49577 24965 723
.196	.19474 74855 85204 16058 510	.98085 34125 23115 35080 479
.197	.19572 82414 60517 03723 204	.98065 81746 43322 66661 867
.198	.19670 88016 07604 76404 820	.98046 19561 05437 06062 170
.199	.19768 91650 45907 27565 917	.98026 47571 05677 05434 796
0.200	0.19866 93307 95061 21545 941	0.98006 65778 41241 63112 420
.201	.19964 92978 74900 91597 545	.97986 74185 10310 03887 090
.202	.20062 90653 05459 37903 151	.97966 72793 12041 59192 306
.203	.20160 86321 06969 25571 640	.97946 61604 46575 47187 084
.204	.20258 79972 99863 82615 083	.97926 40621 15030 52742 047
0.205	0.20356 71599 04777 97905 397	0.97906 09845 19505 07327 536
.206	.20454 61189 42549 19110 856	.97885 69278 63076 68803 784
.207	.20552 48734 34218 50612 330	.97865 18923 49802 01113 156
.208	.20650 34224 01031 51399 175	.97844 58781 84716 53874 491
.209	.20748 17648 64439 32944 665	.97823 88855 73834 41879 553
0.210	0.20845 98998 46099 57060 871	0.97803 09147 24148 24491 614
.211	.20943 78263 67877 33732 895	.97782 19658 43628 84946 201
.212	.21041 55434 51846 18932 346	.97761 20391 41225 09554 014
.213	.21139 30501 20289 12409 982	.97740 11348 26863 66806 039
.214	.21237 03453 95699 55467 398	.97718 92531 11448 86380 882
0.215	0.21334 74283 00782 28707 677	0.97697 63942 06862 38054 344
.216	.21432 42978 58454 49764 905	.97676 25583 25963 10511 247
.217	.21530 09530 91846 71012 439	.97654 77456 82586 90059 555
.218	.21627 73930 24303 77249 851	.97633 19564 91546 39246 782
.219	.21725 36166 79385 83368 434	.97611 51909 68630 75378 736
0.220	0.21822 96230 80869 31995 179	0.97589 74493 30605 48940 602
.221	.21920 54112 52747 91115 124	.97567 87317 95212 21920 392
.222	.22018 09802 19233 51671 977	.97545 90385 81168 46034 788
.223	.22115 63290 04757 25146 920	.97523 83699 08167 40857 388
.224	.22213 14566 33970 41115 484	.97501 67259 96877 71849 392
0.225	0.22310 63621 31745 44782 417	0.97479 41070 68943 28292 737
.226	.22408 10445 23176 94494 428	.97457 05133 46983 01125 708
.227	.22505 55028 33582 59230 720	.97434 59450 54590 60681 052
.228	.22602 97360 88504 16071 214	.97412 04024 16334 34326 607
.229	.22700 37433 13708 47642 363	.97389 38856 57756 84008 477
0.230	0.22797 75235 35188 39540 462	0.97366 63950 05374 83696 773
.231	.22895 10757 79163 77732 354	.97343 79306 86678 96733 940
.232	.22992 43990 72082 45933 437	.97320 84929 30133 53085 695
.233	.23089 74924 40621 22962 869	.97297 80819 65176 26494 602
.234	.23187 03549 11686 80075 884	.97274 66980 22218 11536 294
0.235	0.23284 29855 12416 78273 112	0.97251 43413 32643 00578 389
.236	.23381 53832 70180 65586 809	.97228 10121 28807 60642 091
.237	.23478 75472 12580 74343 904	.97204 67106 44041 10166 529
.238	.23575 94763 67453 18405 752	.97181 14371 12644 95675 843
.239	.23673 11697 62868 90384 520	.97157 51917 69892 68349 034

Table XII.—*Values of sin x and cos x to 23 places of decimals at intervals of 0.001 from 0.000 to 1.600*—Continued.

x	$\sin x$					$\cos x$				
0.240	0.23770	26264	27134	58836	079	0.97133	79748	52029	60492	618
.241	.23867	38453	88793	65429	334	.97109	97865	96272	61916	095
.242	.23964	48256	76627	22091	869	.97086	06272	40809	96210	262
.243	.24061	55663	19655	08131	828	.97062	04970	24800	96928	391
.244	.24158	60663	47136	67335	933	.97037	93961	88375	83670	294
0.245	0.24255	63247	88572	05043	522	0.97013	73249	72635	38069	313
.246	.24352	63406	73702	85196	546	.96989	42836	19650	79682	233
.247	.24449	61130	32513	27365	389	.96965	02723	72463	41782	166
.248	.24546	56408	95231	03750	445	.96940	52914	75084	47054	425
.249	.24643	49232	92328	36159	337	.96915	93411	72494	83195	397
0.250	0.24740	39592	54522	92959	685	0.96891	24217	10644	78414	459
.251	.24837	27478	12778	86007	332	.96866	45333	36453	76838	955
.252	.24934	12879	98307	67549	922	.96841	56762	97810	13822	250
.253	.25030	95788	42569	27105	742	.96816	58508	43570	91154	897
.254	.25127	76193	77272	88317	722	.96791	50572	23561	52178	941
0.255	0.25224	54086	34378	05782	506	0.96766	32956	88575	56805	375
.256	.25321	29456	46095	61854	486	.96741	05664	90374	56434	780
.257	.25418	02294	44888	63424	714	.96715	68698	81687	68781	180
.258	.25514	72590	63473	38674	587	.96690	22061	16211	52599	126
.259	.25611	40335	34820	33804	209	.96664	65754	48609	82314	035
0.260	0.25708	05518	92155	09735	339	0.96638	99781	34513	22555	822
.261	.25804	68131	68959	38788	820	.96613	24144	30519	02595	835
.262	.25901	28163	98972	01336	401	.96587	38845	94190	90687	131
.263	.25997	85606	16189	82426	844	.96561	43888	84058	68308	107
.264	.26094	40448	54868	68386	239	.96535	39275	59618	04309	520
0.265	0.26190	92681	49524	43392	399	0.96509	25008	81330	28964	923
.266	.26287	42295	34933	86023	278	.96483	01091	10622	07924	537
.267	.26383	89280	46135	65779	278	.96456	67525	09885	16072	584
.268	.26480	33627	18431	39579	372	.96430	24313	42476	11288	118
.269	.26576	75325	87386	48230	942	.96403	71458	72716	08109	368
0.270	0.26673	14366	88831	12873	229	0.96377	08963	65890	51301	623
.271	.26769	50740	58861	31394	301	.96350	36830	88248	89328	696
.272	.26865	84437	33839	74821	451	.96323	55063	07004	47727	972
.273	.26962	15447	50396	83684	915	.96296	63662	90334	02389	084
.274	.27058	43761	45431	64354	828	.96269	62633	07377	52736	246
0.275	0.27154	69369	56112	85351	302	0.96242	51976	28237	94814	248
.276	.27250	92262	19879	73627	557	.96215	31695	23980	94278	169
.277	.27347	12429	74443	10825	981	.96188	01792	66634	59286	807
.278	.27443	29862	57786	29507	043	.96160	62271	29189	13299	879
.279	.27539	44551	08166	09350	952	.96133	13133	85596	67778	997
0.280	0.27635	56485	64113	73331	967	0.96105	54383	10770	94792	459
.281	.27731	65656	64435	83865	270	.96077	86021	80586	99523	878
.282	.27827	72054	48215	38926	293	.96050	08052	71880	92684	682
.283	.27923	75669	54812	68142	411	.96022	20478	62449	62830	504
.284	.28019	76492	23866	28856	909	.95994	23302	31050	48581	495
0.285	0.28115	74512	95294	02165	110	0.95966	16526	57401	10746	590
.286	.28211	69722	09293	88922	591	.95938	00154	22179	04351	746
.287	.28307	62110	06345	05725	374	.95909	74188	07021	50572	193
.288	.28403	51667	27208	80861	997	.95881	38630	94525	08568	713
.289	.28499	38384	12929	50237	384	.95852	93485	68245	47227	984
0.290	0.28595	22251	04835	53268	394	0.95824	38755	12697	16807	013
.291	.28691	03258	44540	28750	981	.95795	74442	13353	20481	688
.292	.28786	81396	73943	10698	841	.95767	00549	56644	85799	478
.293	.28882	56656	35230	24153	475	.95738	17080	29961	36036	308
.294	.28978	29027	70875	80965	551	.95709	24037	21649	61457	636
0.295	0.29073	98501	23642	75547	489	0.95680	21423	21013	90483	768
.296	.29169	65067	36583	80597	155	.95651	09241	18315	60759	429
.297	.29265	28716	53042	42792	582	.95621	87494	04772	90127	632
.298	.29360	89439	16653	78457	616	.95592	56184	72560	47507	858
.299	.29456	47225	71345	69198	389	.95563	15316	14809	23678	590

TABLE XII.—*Values of sin x and cos x to 23 places of decimals at intervals of 0.001 from 0.000 to 1.600*—Continued.

x	*sin x*	*cos x*
0. 300	0. 29552 02066 61339 57510 532	0. 95533 64891 25606 01964 231
. 301	. 29647 53952 31151 42357 025	. 95504 04912 99993 28826 414
. 302	. 29743 02873 25592 74716 586	. 95474 35384 33968 84359 763
. 303	. 29838 48819 89771 53102 518	. 95444 56308 24485 52692 116
. 304	. 29933 91782 69093 19051 897	. 95414 67687 69450 92289 242
0. 305	0. 30029 31752 09261 52585 026	0. 95384 69525 67727 06164 084
. 306	. 30124 68718 56279 67635 045	. 95354 61825 19130 11990 559
. 307	. 30220 02672 56451 07447 613	. 95324 44589 24430 12121 945
. 308	. 30315 33604 56380 39950 549	. 95294 17820 85350 63513 878
. 309	. 30410 61505 02974 53093 365	. 95263 81523 04568 47552 001
0. 310	0. 30505 86364 43443 50156 564	0. 95233 35698 85713 39784 281
. 311	. 30601 08173 25301 45030 632	. 95202 80351 33367 79558 038
. 312	. 30696 26921 96367 57464 615	. 95172 15483 53066 39561 711
. 313	. 30791 42601 04767 08284 189	. 95141 41098 51295 95271 383
. 314	. 30886 55200 98932 14579 138	. 95110 57199 35494 94302 111
0. 315	0. 30981 64712 27602 84860 120	0. 95079 63789 14053 25664 080
. 316	. 31076 71125 39828 14184 658	. 95048 60870 96311 88923 617
. 317	. 31171 74430 84966 79252 234	. 95017 48447 92562 63269 094
. 318	. 31266 74619 12688 33468 402	. 94986 26523 14047 76481 749
. 319	. 31361 71680 72974 01977 833	. 94954 95099 72959 73811 467
0. 320	0. 31456 65606 16117 76666 176	0. 94923 54180 82440 86757 531
. 321	. 31551 56385 92727 11130 659	. 94892 03769 56583 01754 395
. 322	. 31646 44010 53724 15619 332	. 94860 43869 10427 28762 501
. 323	. 31741 28470 50346 51938 844	. 94828 74482 59963 69764 173
. 324	. 31836 09756 34148 28330 674	. 94796 95613 22130 87164 613
0. 325	0. 31930 87858 57000 94315 718	0. 94765 07264 14815 72098 048
. 326	. 32025 62767 71094 35507 128	. 94733 09438 56853 12639 034
. 327	. 32120 34474 28937 68391 319	. 94701 02139 68025 61918 976
. 328	. 32215 02968 83360 35077 048	. 94668 85370 69063 06147 877
. 329	. 32309 68241 87512 98012 460	. 94636 59134 81642 32541 351
0. 330	0. 32404 30283 94868 34670 020	0. 94604 23435 28386 97152 941
. 331	. 32498 89085 59222 32199 224	. 94571 78275 32866 92611 768
. 332	. 32593 44637 34694 82047 011	. 94539 23658 19598 15765 535
. 333	. 32687 96929 75730 74545 756	. 94506 59587 14042 35228 939
. 334	. 32782 45953 37100 93468 777	. 94473 86065 42606 58837 502
0. 335	0. 32876 91698 73903 10553 241	0. 94441 03096 32643 01006 864
. 336	. 32971 34156 41562 79990 386	. 94408 10683 12448 49997 577
. 337	. 33065 73316 95834 32882 957	. 94375 08829 11264 35085 413
. 338	. 33160 09170 92801 71669 766	. 94341 97537 59275 93637 243
. 339	. 33254 41708 88879 64517 288	. 94308 76811 87612 38092 499
0. 340	0. 33348 70921 40814 39678 177	0. 94275 46655 28346 22850 264
. 341	. 33442 96799 05684 79816 635	. 94242 07071 14493 11062 025
. 342	. 33537 19332 40903 16300 519	. 94208 58062 80011 41330 105
. 343	. 33631 38512 04216 23460 104	. 94174 99633 59801 94311 834
. 344	. 33725 54328 53706 12813 399	. 94141 31786 89707 59229 468
0. 345	0. 33819 66772 47791 27257 928	0. 94107 54526 06513 00285 905
. 346	. 33913 75834 45227 35228 880	. 94073 67854 47944 22986 218
. 347	. 34007 81505 05108 24823 531	. 94039 71775 52668 40365 059
. 348	. 34101 83774 86866 97891 850	. 94005 66292 60293 39119 944
. 349	. 34195 82634 50276 64093 188	. 93971 51409 11367 45650 473
0. 350	0. 34289 78074 55451 34918 963	0. 93937 27128 47378 92003 503
. 351	. 34383 70085 62847 17681 237	. 93902 93454 10755 81724 321
. 352	. 34477 58658 33263 09467 102	. 93868 50389 44865 55613 841
. 353	. 34571 43783 27841 91058 778	. 93833 97937 94014 57391 869
. 354	. 34665 25451 08071 20819 319	. 93799 36103 03447 99266 461
0. 355	0. 34759 03652 35784 28543 852	0. 93764 64888 19349 27409 412
. 356	. 34852 78377 73161 09276 237	. 93729 84296 88839 87337 915
. 357	. 34946 49617 82729 17091 064	. 93694 94332 59978 89202 418
. 358	. 35040 17363 27364 58840 891	. 93659 94998 81762 72980 716
. 359	. 35133 81604 70292 87868 632	. 93624 86299 04124 73578 312

TABLE XII.—*Values of sin x and cos x to 23 places of decimals at intervals of 0.001 from 0.000 to 1.600*—Continued.

x	sin x	cos x
0.360	0.35227 42332 75089 97684 991	0.93589 68236 77934 85835 091
.361	.35320 99538 05683 15610 866	.93554 40815 54999 29438 322
.362	.35414 53211 26351 96384 608	.93519 01038 88060 13742 042
.363	.35508 03343 01729 15734 065	.93483 57910 30795 02492 855
.364	.35601 49923 96801 63913 294	.93448 02433 37816 78462 165
0.365	0.35694 92944 76911 39203 863	0.93412 37611 64673 07984 897
.366	.35788 32396 07756 41380 647	.93376 63448 67846 05404 739
.367	.35881 68268 55391 65142 021	.93340 79948 04751 97425 922
.368	.35975 00552 86229 93504 354	.93304 87113 33740 87371 606
.369	.36068 29239 67042 91160 721	.93268 84948 14096 19348 871
0.370	0.36161 54319 64961 97803 729	0.93232 73456 06034 42320 381
.371	.36254 75783 47479 21412 373	.93196 52640 70704 74082 737
.372	.36347 93621 82448 31502 813	.93160 22505 70188 65151 560
.373	.36441 07825 38085 52343 006	.93123 83054 67499 62553 347
.374	.36534 18384 82970 56131 067	.93087 34291 26582 73524 125
0.375	0.36627 25290 86047 56137 291	0.93050 76219 12314 29114 948
.376	.36720 28534 16625 99809 733	.93014 08841 90501 47704 265
.377	.36813 28105 44381 61843 251	.92977 32163 27881 98417 211
.378	.36906 23995 39357 37211 926	.92940 46186 92123 64451 836
.379	.36999 16194 71964 34164 758	.92903 50916 51824 06312 328
0.380	0.37092 04694 12982 67184 549	0.92866 46355 76510 24949 253
.381	.37184 89484 33562 49909 881	.92829 32508 36638 24806 858
.382	.37277 70556 05224 88020 096	.92792 09378 03592 76777 471
.383	.37370 47899 99862 72083 184	.92754 76968 49686 81063 030
.384	.37463 21506 89741 70366 479	.92717 35283 48161 29943 792
0.385	0.37555 91367 47501 21610 089	0.92679 84326 73184 70454 235
.386	.37648 57472 46155 27762 945	.92642 24101 99852 66966 223
.387	.37741 19812 59093 46681 397	.92604 54613 04187 63679 438
.388	.37833 78378 60081 84790 240	.92566 75863 63138 47019 143
.389	.37926 33161 23263 89706 110	.92528 87857 54580 07941 297
0.390	0.38018 84151 23161 42823 118	0.92490 90598 57313 04145 068
.391	.38111 31339 34675 51860 671	.92452 84090 51063 22192 776
.392	.38203 74716 33087 43373 349	.92414 68337 16481 39537 314
.393	.38296 14272 94059 55222 774	.92376 43342 35142 86457 070
.394	.38388 49999 93636 29011 366	.92338 09109 89547 07898 401
0.395	0.38480 81888 08245 02477 888	0.92299 65643 63117 25225 693
.396	.38573 09928 14697 01854 707	.92261 12947 40199 97879 040
.397	.38665 34110 90188 34186 658	.92222 51025 06064 84939 589
.398	.38757 54427 12300 79611 426	.92183 79880 46904 06602 584
.399	.38849 70867 59002 83601 363	.92144 99517 49832 05558 150
0.400	0.38941 83423 08650 49166 631	0.92106 09940 02885 08279 853
.401	.39033 92084 39988 29019 595	.92067 11151 95020 86221 075
.402	.39125 96842 32150 17700 358	.92028 03157 16118 16919 248
.403	.39217 97687 64660 43663 363	.91988 85959 56976 45007 979
.404	.39309 94611 17434 61324 955	.91949 59563 09315 43137 110
0.405	0.39401 87603 70780 43071 820	0.91910 23971 65774 72800 745
.406	.39493 76656 05398 71230 202	.91870 79189 19913 45073 295
.407	.39585 61759 02384 29995 816	.91831 25219 66209 81253 568
.408	.39677 42903 43226 97324 356	.91791 62067 00060 73416 956
.409	.39769 20080 09812 36782 508	.91751 89735 17781 44875 737
0.410	0.39860 93279 84422 89359 380	0.91712 08228 16605 10547 564
.411	.39952 62493 49738 65238 251	.91672 17549 94682 37232 150
.412	.40044 27711 88838 35528 558	.91632 17704 51081 03796 202
.413	.40135 88925 85200 23958 010	.91592 08695 85785 61266 649
.414	.40227 46126 22702 98524 766	.91551 90527 99696 92832 194
0.415	0.40318 99303 85626 63109 550	0.91511 63204 94631 73753 232
.416	.40410 48449 58653 49047 645	.91471 26730 73322 31180 180
.417	.40501 93554 26869 06660 654	.91430 81109 39416 03880 251
.418	.40593 34608 75762 96747 939	.91390 26344 97475 01872 722
.419	.40684 71603 91229 82037 655	.91349 62441 52975 65972 725

TABLE XII.—*Values of sin x and cos x to 23 places of decimals at intervals of 0.001 from 0.000 to 1.600*—Continued.

x	*sin x*					*cos x*				
0.420	0.40776	04530	59570	18597	279	0.91308	89403	12308	27243	609
.421	.40867	33379	67491	47203	546	.91268	07233	82776	66357	915
.422	.40958	58142	02108	84671	703	.91227	15937	72597	72866	996
.423	.41049	78808	50946	15143	980	.91186	15518	90901	04379	332
.424	.41140	95370	01936	81337	201	.91145	05981	47728	45647	576
0.425	0.41232	07817	43424	75749	435	0.91103	87329	54033	67564	373
.426	.41323	16141	64165	31825	593	.91062	59567	21681	86066	990
.427	.41414	20333	53326	15081	889	.91021	22698	63449	20950	808
.428	.41505	20384	00488	14189	067	.90979	76727	93022	54591	701
.429	.41596	16283	95646	32014	301	.90938	21659	24998	90577	360
0.430	0.41687	08024	29210	76621	692	0.90896	57496	74885	12247	591
.431	.41777	95595	92007	52231	243	.90854	84244	59097	41143	638
.432	.41868	78989	75279	50136	257	.90813	01906	94960	95366	563
.433	.41959	58196	70687	39579	028	.90771	10488	00709	47844	729
.434	.42050	33207	70310	58584	774	.90729	09991	95484	84510	435
0.435	0.42141	04013	66648	04753	684	0.90687	00422	99336	62385	731
.436	.42231	70605	52619	26011	018	.90644	81785	33221	67577	465
.437	.42322	32974	21565	11315	146	.90602	54083	19003	73181	601
.438	.42412	91110	67248	81323	456	.90560	17320	79452	97096	848
.439	.42503	45005	83856	79016	027	.90517	71502	38245	59747	647
0.440	0.42593	94650	65999	60276	972	0.90475	16632	19963	41716	554
.441	.42684	40036	08712	84433	381	.90432	52714	50093	41286	061
.442	.42774	81153	07458	04751	750	.90389	79753	55027	31889	904
.433	.42865	17992	58123	58891	823	.90346	97753	62061	19473	892
.444	.42955	50545	57025	59317	745	.90304	06718	99394	99766	305
0.445	0.43045	78803	00908	83666	443	0.90261	06653	96132	15457	899
.446	.43136	02755	86947	65073	141	.90217	97562	82279	13291	573
.447	.43226	22395	12746	82453	917	.90174	79449	88745	01061	718
.448	.43316	37711	76342	50745	219	.90131	52319	47341	04523	319
.449	.43406	48696	76203	11100	244	.90088	16175	90780	24210	832
0.450	0.43496	55341	11230	21042	084	0.90044	71023	52676	92166	884
.451	.43586	57635	80759	44573	567	.90001	16866	67546	28580	847
.452	.43676	55571	84561	42243	681	.89957	53709	70803	98337	319
.453	.43766	49140	22842	61170	507	.89913	81556	98765	67474	569
.454	.43856	38331	96246	25020	568	.89870	00412	88646	59552	965
0.455	0.43946	23138	05853	23944	492	0.89826	10281	78561	11933	463
.456	.44036	03549	53183	04468	918	.89782	11168	07522	31966	167
.457	.44125	79557	40194	59344	542	.89738	03076	15441	53089	030
.458	.44215	51152	69287	17350	215	.89693	86010	43127	90836	721
.459	.44305	18326	43301	33053	008	.89649	59975	32287	98759	714
0.460	0.44394	81069	65519	76524	151	0.89605	24975	25525	24253	639
.461	.44484	39373	39668	23010	752	.89560	81014	66339	64298	937
.462	.44573	93228	69916	42563	218	.89516	28097	99127	21110	867
.463	.44663	42626	60878	89618	275	.89471	66229	69179	57699	918
.464	.44752	87558	17615	92537	506	.89426	95414	22683	53342	602
0.465	0.44842	28014	45634	43101	319	0.89382	15656	06720	58962	873
.466	.44931	63986	50888	85958	244	.89337	26959	69266	52423	883
.467	.45020	95465	39782	08029	479	.89292	29329	59190	93730	459
.468	.45110	22442	19166	27868	603	.89247	22770	26256	80142	134
.469	.45199	44907	96343	84976	342	.89202	07286	21120	01196	857
0.470	0.45288	62853	79068	29070	327	0.89156	82881	95328	93645	402
.471	.45377	76270	75545	09309	736	.89111	49562	01323	96296	541
.472	.45466	85149	94432	63474	735	.89066	07330	92437	04773	005
.473	.45555	89482	44843	07100	635	.89020	56193	22891	26178	292
.474	.45644	89259	36343	22566	671	.88974	96153	47800	33674	367
0.475	0.45733	84471	78955	48139	307	0.88929	27216	23168	20970	288
.476	.45822	75110	83158	66969	994	.88883	49386	05888	56721	822
.477	.45911	61167	59888	96047	279	.88837	62667	53744	38842	074
.478	.46000	42633	20540	75103	180	.88791	67065	25407	48723	197
.479	.46089	19498	76967	55473	739	.88745	62583	80438	05369	212

TABLE XII.—*Values of sin x and cos x to 23 places of decimals at intervals of 0.001 from 0.000 to 1.600*—Continued.

x	$\sin x$	$\cos x$
0.480	0.46177 91755 41482 88913 664	0.88699 49227 79284 19439 995
.481	.46266 59394 26861 16364 968	.88653 27001 83281 47206 469
.482	.46355 22406 46338 56679 522	.88606 95910 54652 44417 051
.483	.46443 80783 13613 95295 430	.88560 55958 56506 20075 401
.484	.46532 34515 42849 72867 132	.88514 07150 52837 90129 517
0.485	0.46620 83594 48672 73849 162	0.88467 49491 08528 31072 223
.486	.46709 28011 46175 15033 451	.88420 82984 89343 33453 094
.487	.46797 67757 50915 34040 104	.88374 07636 61933 55301 874
.488	.46886 02823 78918 77761 558	.88327 23450 93833 75463 416
.489	.46974 33201 46678 90760 024	.88280 30432 53462 46844 214
0.490	0.47062 58881 71158 03618 136	0.88233 28586 10121 49570 547
.491	.47150 79855 69788 21242 715	.88186 17916 33995 44058 307
.492	.47238 96114 60472 11121 556	.88138 98427 96151 23994 541
.493	.47327 07649 61583 91533 149	.88091 70125 68537 69230 763
.494	.47415 14451 91970 19709 261	.88044 33014 23984 98588 075
0.495	0.47503 16512 70950 79950 264	0.87996 87098 36204 22574 157
.496	.47591 13823 18319 71693 150	.87949 32382 79786 96012 154
.497	.47679 06374 54345 97532 118	.87901 68872 30204 70581 529
.498	.47766 94157 99774 51191 668	.87853 96571 63808 47270 917
.499	.47854 77164 75827 05452 099	.87806 15485 57828 28743 023
0.500	0.47942 55386 04203 00027 329	0.87758 25618 90372 71611 628
.501	.48030 28813 07080 29394 947	.87710 26976 40428 38630 733
.502	.48117 97437 07116 30578 414	.87662 19562 87859 50795 903
.503	.48205 61249 27448 70881 314	.87614 03383 13407 39357 847
.504	.48293 20240 91696 35573 583	.87565 78441 98689 97748 295
0.505	0.48380 74403 23960 15529 617	0.87517 44744 26201 33418 203
.506	.48468 23727 48823 94818 170	.87469 02294 79311 19588 355
.507	.48555 68204 91355 38243 967	.87420 51098 42264 46912 391
.508	.48643 07826 77106 78840 928	.87371 91160 00180 75052 318
.509	.48730 42584 32116 05316 931	.87323 22484 39053 84166 561
0.510	0.48817 72468 82907 49450 013	0.87274 45076 45751 26310 581
.511	.48904 97471 56492 73435 934	.87225 58941 08013 76750 129
.512	.48992 17583 80371 57187 006	.87176 64083 14454 85187 176
.513	.49079 32796 82532 85582 104	.87127 60507 54560 26898 565
.514	.49166 43101 91455 35667 778	.87078 48219 18687 53787 441
0.515	0.49253 48490 36108 63810 364	0.87029 27222 98065 45347 504
.516	.49340 48953 45953 92799 025	.86979 97523 84793 59540 132
.517	.49427 44482 50944 98899 617	.86930 59126 71841 83584 429
.518	.49514 35068 81528 98859 309	.86881 12036 53049 84660 240
.519	.49601 20703 68647 36861 855	.86831 56258 23126 60524 189
0.520	0.49688 01378 43736 71433 446	0.86781 91796 77649 90038 785
.521	.49774 77084 38729 62299 043	.86732 18657 13065 83614 647
.522	.49861 47812 86055 57189 109	.86682 36844 26688 33565 898
.523	.49948 13555 18641 78596 658	.86632 46363 16698 64378 779
.524	.50034 74302 69914 10484 518	.86582 47218 82144 82893 524
0.525	0.50121 30046 73797 84942 748	0.86532 39416 22941 28399 561
.526	.50207 80778 64718 68796 092	.86482 22960 39868 22644 077
.527	.50294 26489 77603 50161 411	.86431 97856 34571 19753 996
.528	.50380 67171 47881 24954 981	.86381 64109 09560 56071 436
.529	.50467 02815 11483 83349 596	.86331 21723 68210 99902 671
0.530	0.50553 33412 04846 96181 366	0.86280 70705 14761 01180 670
.531	.50639 58953 64911 01306 143	.86230 11058 54312 41041 248
.532	.50725 79431 29121 89905 473	.86179 42788 92829 81312 894
.533	.50811 94836 35431 92741 999	.86128 65901 37140 13920 311
.534	.50898 05160 22300 66364 220	.86077 80400 94932 10201 726
0.535	0.50984 10394 28695 79260 534	0.86026 86292 74755 70140 025
.536	.51070 10529 94093 97962 456	.85975 83581 86021 71507 760
.537	.51156 05558 58481 73096 946	.85924 72273 39001 18926 068
.538	.51241 95471 62356 25387 754	.85873 52372 44824 92837 581
.539	.51327 80260 46726 31605 686	.85822 23884 15482 98393 339

TABLE XII.—*Values of sin x and cos x to 23 places of decimals at intervals of 0.001 from 0.000 to 1.600*—Continued.

x	*sin x*					*cos x*				
0.540	0.51413	59916	53113	10467	728	0.85770	86813	63824	14253	797
.541	.51499	34431	23551	08484	914	.85719	41166	03555	41303	947
.542	.51585	03796	00588	85758	874	.85667	86946	49241	51282	623
.543	.51670	68002	27290	01726	969	.85616	24160	16304	35326	032
.544	.51756	27041	47234	00855	920	.85564	52812	21022	52425	567
0.545	0.51841	80905	04516	98283	861	0.85512	72907	80530	77799	957
.546	.51927	29584	43752	65410	714	.85460	84452	12819	51181	787
.547	.52012	73071	10073	15436	812	.85408	87450	36734	25018	472
.548	.52098	11356	49129	88849	675	.85356	81907	71975	12587	703
.549	.52183	44432	07094	38858	868	.85304	67829	39096	36027	442
0.550	0.52268	72289	30659	16778	838	0.85252	45220	59505	74280	498
.551	.52353	94919	67038	57359	653	.85200	14086	55464	10953	761
.552	.52439	12314	63969	64065	565	.85147	74432	50084	82092	114
.553	.52524	24465	69712	94301	297	.85095	26263	67333	23867	110
.554	.52609	31364	33053	44585	976	.85042	69585	32026	20180	431
0.555	0.52694	33002	03301	35674	635	0.84990	04402	69831	50182	218
.556	.52779	29370	30292	97627	180	.84937	30721	07267	35704	287
.557	.52864	20460	64391	54824	757	.84884	48545	71701	88608	318
.558	.52949	06264	56488	10933	415	.84831	57881	91352	58049	047
.559	.53033	86773	58002	33815	002	.84778	58734	95285	77652	517
0.560	0.53118	61979	20883	40385	187	0.84725	51110	13416	12609	452
.561	.53203	31872	97610	81418	533	.84672	35012	76506	06683	799
.562	.53287	96446	41195	26300	543	.84619	10448	16165	29136	481
.563	.53372	55691	05179	47726	585	.84565	77421	64850	21564	438
.564	.53457	09598	43639	06347	607	.84512	35938	55863	44654	991
0.565	0.53541	58160	11183	35362	572	0.84458	86004	23353	24855	579
.566	.53626	01367	62956	25057	521	.84405	27624	02313	00958	945
.567	.53710	39212	54637	07291	168	.84351	60803	28580	70603	796
.568	.53794	71686	42441	39926	969	.84297	85547	38838	36691	011
.569	.53878	98780	83121	91211	553	.84244	01861	70611	53715	445
0.570	0.53963	20487	33969	24099	446	0.84190	09751	62268	74013	376
.571	.54047	36797	52812	80524	005	.84136	09222	53020	93925	658
.572	.54131	47702	98021	65614	465	.84082	00279	82920	99876	632
.573	.54215	53195	28505	31859	028	.84027	82928	92863	14368	839
.574	.54299	53266	03714	63213	905	.83973	57175	24582	41893	605
0.575	0.54383	47906	83642	59158	222	0.83919	23024	20654	14757	543
.576	.54467	37109	28825	18694	718	.83864	80481	24493	38825	019
.577	.54551	20865	00342	24296	136	.83810	29551	80354	39176	658
.578	.54634	99165	59818	25797	231	.83755	70241	33330	05683	918
.579	.54718	72002	69423	24232	321	.83701	02555	29351	38499	807
0.580	0.54802	39367	91873	55618	270	0.83646	26499	15186	93465	789
.581	.54886	01252	90432	74682	851	.83591	42078	38442	27434	927
.582	.54969	57649	28912	38538	382	.83536	49298	47559	43511	337
.583	.55053	08548	71672	90300	563	.83481	48164	91816	36205	988
.584	.55136	53942	83624	42652	424	.83426	38683	21326	36508	907
0.585	0.55219	93823	30227	61353	309	0.83371	20858	87037	56877	861
.586	.55303	28181	77494	48692	799	.83315	94697	40732	36143	543
.587	.55386	57009	91989	26889	504	.83260	60204	35026	84331	337
.588	.55469	80299	40829	21434	637	.83205	17385	23370	27399	720
.589	.55552	98041	91685	44380	278	.83149	66245	60044	51895	332
0.590	0.55636	10229	12783	77572	254	0.83094	06791	00163	49524	800
.591	.55719	16852	72905	55827	556	.83038	39026	99672	61643	346
.592	.55802	17904	41388	50056	192	.82982	62959	15348	23660	255
.593	.55885	13375	88127	50327	409	.82926	78593	04797	09361	243
.594	.55968	03258	83575	48880	201	.82870	85934	26455	75147	786
0.595	0.56050	87544	98744	23078	004	0.82814	84988	39590	04193	468
.596	.56133	66226	05205	18307	516	.82758	75761	04294	50517	407
.597	.56216	39293	75090	30821	541	.82702	58257	81491	82974	799
.598	.56299	06739	81092	90525	792	.82646	32484	32932	29164	660
.599	.56381	68555	96468	43709	545	.82589	98446	21193	19254	799

TABLE XII.—*Values of sin x and cos x to 23 places of decimals at intervals of 0.001 from 0.000 to 1.600*—Continued.

x	$\sin x$	$\cos x$
0.600	0.56464 24733 95035 35720 095	0.82533 56149 09678 29724 095
.601	.56546 75265 51175 93580 897	.82477 05598 62617 27022 123
.602	.56629 20142 39837 08553 336	.82420 46800 45065 11146 193
.603	.56711 59356 36531 18642 028	.82363 79760 22901 59135 858
.604	.56793 92899 17336 91043 574	.82307 04483 62830 68484 934
0.605	0.56876 20762 58900 04538 687	0.82250 20976 32380 00471 116
.606	.56958 42938 38434 31827 607	.82193 29243 99900 23403 216
.607	.57040 59418 33722 21808 719	.82136 29292 34564 55786 102
.608	.57122 70194 23115 81800 299	.82079 21127 06368 09403 380
.609	.57204 75257 85537 59705 300	.82022 04753 86127 32317 893
0.610	0.57286 74601 00481 26119 098	0.81964 80178 45479 51790 075
.611	.57368 68215 48012 56380 111	.81907 47406 56882 17114 225
.612	.57450 56093 08770 12563 221	.81850 06443 93612 42372 770
.613	.57532 38225 63966 25415 904	.81792 57296 29766 49108 549
.614	.57614 14604 95387 76236 989	.81734 99969 40259 08915 198
0.615	0.57695 85222 85396 78697 975	0.81677 34469 00822 85945 685
.616	.57777 50071 16931 60606 809	.81619 60800 88007 79339 051
.617	.57859 09141 73507 45614 047	.81561 78970 79180 65565 411
.618	.57940 62426 39217 34861 330	.81503 88984 52524 40689 288
.619	.58022 09916 98732 88572 073	.81445 90847 87037 62551 318
0.620	0.58103 51605 37305 07584 296	0.81387 84566 62533 92868 400
.621	.58184 87483 40765 14825 522	.81329 70146 59641 39252 335
.622	.58266 17542 95525 36729 641	.81271 47593 59801 97147 027
.623	.58347 41775 88579 84595 681	.81213 16913 45270 91684 290
.624	.58428 60174 07505 35888 387	.81154 78111 99116 19458 331
0.625	0.58509 72729 40462 15480 540	0.81096 31195 05217 90218 953
.626	.58590 79433 76194 76836 923	.81037 76168 48267 68483 556
.627	.58671 80279 04032 83139 861	.80979 13038 13768 15067 973
.628	.58752 75257 13891 88356 252	.80920 41809 88032 28536 214
.629	.58833 64359 96274 18246 006	.80861 62489 58182 86569 178
0.630	0.58914 47579 42269 51311 811	0.80802 75083 12151 87252 371
.631	.58995 24907 43555 99690 151	.80743 79596 38679 90282 722
.632	.59075 96335 92400 89983 484	.80684 76035 27315 58094 522
.633	.59156 61856 81661 44033 509	.80625 64405 68414 96904 569
.634	.59237 21462 04785 59635 440	.80566 44713 53140 97676 566
0.635	0.59317 75143 55812 91193 198	0.80507 16964 73462 77004 837
.636	.59398 22893 29375 30315 454	.80447 81165 22155 17917 411
.637	.59478 64703 20697 86352 425	.80388 37320 92798 10598 548
.638	.59559 00565 25599 66873 364	.80328 85437 79775 93030 752
.639	.59639 30471 40494 58084 641	.80269 25521 78276 91556 338
0.640	0.59719 54413 62392 05188 355	0.80209 57578 84292 61358 611
.641	.59799 72383 88897 92681 375	.80149 81614 94617 26862 715
.642	.59879 84374 18215 24594 757	.80089 97636 06847 22056 216
.643	.59959 90376 49145 04673 426	.80030 05648 19380 30729 469
.644	.60039 90382 81087 16496 070	.79970 05657 31415 26635 842
0.645	0.60119 84385 14041 03535 151	0.79909 97669 42951 13571 848
.646	.60199 72375 48606 49156 949	.79849 81690 54786 65377 243
.647	.60279 54345 85984 56561 576	.79789 57726 68519 65855 159
.648	.60359 30288 27978 28662 868	.79729 25783 86546 48612 327
.649	.60439 00194 76993 47908 070	.79668 85868 12061 36819 444
0.650	0.60518 64057 36039 56037 252	0.79608 37985 49055 82891 760
.651	.60598 21868 08730 33782 358	.79547 82142 02318 08089 927
.652	.60677 73618 99284 80505 818	.79487 18343 77432 42041 183
.653	.60757 19302 12527 93778 646	.79426 46596 80778 62180 929
.654	.60836 58909 53891 48897 929	.79365 66907 19531 33114 757
0.655	0.60915 92433 29414 78343 652	0.79304 79281 01659 45900 987
.656	.60995 19865 45745 51174 755	.79243 83724 35925 57253 785
.657	.61074 41198 10140 52364 359	.79182 80243 31885 28666 909
.658	.61153 56423 30466 62074 073	.79121 68843 99886 65458 154
.659	.61232 65533 15201 34867 307	.79060 49532 51069 55734 550

Table XII.—*Values of sin x and cos x to 23 places of decimals at intervals of 0.001 from 0.000 to 1.600*—Continued.

x	*sin x*					*cos x*				
0. 660	0. 61311	68519	73433	78861	515	0. 78999	22314	97365	09278	382
. 661	. 61390	65375	14865	34819	272	. 78937	87197	51494	96354	080
. 662	. 61469	56091	49810	55178	137	. 78876	44186	26970	86436	061
. 663	. 61548	40660	89197	83019	186	. 78814	93287	38093	86857	558
. 664	. 61627	19075	44570	30974	165	. 78753	34506	99953	81380	523
0. 665	0. 61705	91327	28086	60071	171	0. 78691	67851	28428	68686	643
. 666	. 61784	57408	52521	58518	785	. 78629	93326	40184	00789	551
. 667	. 61863	17311	31267	20428	576	. 78568	10938	52672	21368	279
. 668	. 61941	71027	78333	24475	901	. 78506	20693	84132	04022	017
. 669	. 62020	18550	08348	12498	919	. 78444	22598	53587	90446	244
0. 670	0. 62098	59870	36559	68035	744	0. 78382	16658	80849	28530	294
. 671	. 62176	94980	78835	94799	654	. 78320	02880	86510	10376	414
. 672	. 62255	23873	51665	95092	281	. 78257	81270	91948	10240	374
. 673	. 62333	46540	72160	48154	700	. 78195	51835	19324	22393	698
. 674	. 62411	62974	58052	88456	349	. 78133	14579	91581	98907	578
0. 675	0. 62489	73167	27699	83921	682	0. 78070	69511	32446	87358	526
. 676	. 62567	77111	00082	14094	496	. 78008	16635	66425	68455	830
. 677	. 62645	74797	94805	48239	849	. 77945	55959	18805	93590	877
. 678	. 62723	66220	32101	23383	477	. 77882	87488	15655	22308	414
. 679	. 62801	51370	32827	22288	658	. 77820	11228	83820	59699	786
0. 680	0. 62879	30240	18468	51370	418	0. 77757	27187	50927	93718	239
. 681	. 62957	02822	11138	18547	018	. 77694	35370	45381	32416	339
. 682	. 63034	69108	33578	11028	644	. 77631	35783	96362	41105	566
. 683	. 63112	29091	09159	73043	207	. 77568	28434	33829	79438	156
. 684	. 63189	82762	61884	83499	207	. 77505	13327	88518	38411	247
0. 685	0. 63267	30115	16386	33585	507	0. 77441	90470	91938	77293	390
. 686	. 63344	71140	97929	04308	094	. 77378	59869	76376	60473	500
. 687	. 63422	05832	32410	43963	552	. 77315	21530	74891	94232	293
. 688	. 63499	34181	46361	45549	316	. 77251	75460	21318	63436	286
. 689	. 63576	56180	66947	24110	576	. 77188	21664	50263	68154	418
0. 690	0. 63653	71822	21967	94023	743	0. 77124	60149	97106	60197	354
. 691	. 63730	81098	39859	46216	467	. 77060	90922	97998	79579	541
. 692	. 63807	84001	49694	25323	984	. 76997	13989	89862	90904	069
. 693	. 63884	80523	81182	06781	899	. 76933	29357	10392	19670	418
. 694	. 63961	70657	64670	73855	200	. 76869	37030	98049	88505	132
0. 695	0. 64038	54395	31146	94603	464	0. 76805	37017	92068	53315	502
. 696	. 64115	31729	12236	98782	185	. 76741	29324	32449	39366	321
. 697	. 64192	02651	40207	54680	136	. 76677	13956	59961	77279	757
. 698	. 64268	67154	47966	45892	698	. 76612	90921	16142	38958	434
. 699	. 64345	25230	69063	48031	063	. 76548	60224	43294	73431	759
0. 700	0. 64421	76872	37691	05367	261	0. 76484	21872	84488	42625	586
. 701	. 64498	22071	88685	07414	902	. 76419	75872	83558	57055	252
. 702	. 64574	60821	57525	65445	583	. 76355	22230	85105	11442	075
. 703	. 64650	93113	80337	88940	870	. 76290	60953	34492	20253	368
. 704	. 64727	18940	93892	61979	783	. 76225	92046	77847	53166	023
0. 705	0. 64803	38295	35607	19561	705	0. 76161	15517	62061	70453	752
. 706	. 64879	51169	43546	23864	641	. 76096	31372	34787	58298	030
. 707	. 64955	57555	56422	40438	747	. 76031	39617	44439	64022	815
. 708	. 65031	57446	13597	14335	062	. 75966	40259	40193	31253	107
. 709	. 65107	50833	55081	46169	354	. 75901	33304	71984	34997	406
0. 710	0. 65183	37710	21536	68121	013	0. 75836	18759	90508	16654	146
. 711	. 65259	18068	54275	19866	915	. 75770	96631	47219	18942	159
. 712	. 65334	91900	95261	24450	173	. 75705	66925	94330	20755	235
. 713	. 65410	59199	87111	64083	709	. 75640	29649	84811	71940	852
. 714	. 65486	19957	73096	55888	565	. 75574	84809	72391	28003	128
0. 715	0. 65561	74166	97140	27566	883	0. 75509	32412	11552	84730	074
. 716	. 65637	21820	03821	93009	463	. 75443	72463	57536	12745	203
. 717	. 65712	62909	38376	27837	851	. 75378	04970	66335	91983	563
. 718	. 65787	97427	46694	44880	853	. 75312	29939	94701	46092	263
. 719	. 65863	25366	75324	69585	417	. 75246	47378	00135	76755	558

TABLE XII.—*Values of sin x and cos x to 23 places of decimals at intervals of 0.001 from 0.000 to 1.600*—Continued.

x	sin x					cos x				
0.720	0.65938	46719	71473	15361	800	0.75180	57291	40894	97944	549
.721	.66013	61478	83004	58862	952	.75114	59686	75987	70091	576
.722	.66088	69636	58443	15198	027	.75048	54570	65174	34189	363
.723	.66163	71185	46973	13079	967	.74982	41949	68966	45814	983
.724	.66238	66117	98439	69907	065	.74916	21830	48626	09078	707
0.725	0.66313	54426	63349	66778	441	0.74849	94219	66165	10497	806
.726	.66388	36103	92872	23443	354	.74783	59123	84344	52795	369
.727	.66463	11142	38839	73184	280	.74717	16549	66673	88624	209
.728	.66537	79534	53748	37633	666	.74650	66503	77410	54215	910
.729	.66612	41272	90759	01524	309	.74584	08992	81559	02955	103
0.730	0.66686	96350	03697	87373	259	0.74517	44023	44870	38879	013
.731	.66761	44758	47057	30099	195	.74450	71602	33841	50102	364
.732	.66835	86490	75996	51573	181	.74383	91736	15714	42167	693
.733	.66910	21539	46342	35102	739	.74317	04431	58475	71321	153
.734	.66984	49897	14589	99849	159	.74250	09695	30855	77713	862
0.735	0.67058	71556	37903	75177	973	0.74183	07534	02328	18528	866
.736	.67132	86509	74117	74942	523	.74115	97954	43109	01033	791
.737	.67206	94749	81736	71700	537	.74048	80963	24156	15559	237
.738	.67280	96269	19936	70863	650	.73981	56567	17168	68402	998
.739	.67354	91060	48565	84779	796	.73914	24772	94586	14660	158
0.740	0.67428	79116	28145	06748	388	0.73846	85587	29587	90979	142
.741	.67502	60429	19868	84968	216	.73779	39016	96092	48243	787
.742	.67576	34991	85605	96417	996	.73711	85068	68756	84181	492
.743	.67650	02796	87900	20669	485	.73644	23749	22975	75897	532
.744	.67723	63836	89971	13633	096	.73576	55065	34881	12335	582
0.745	0.67797	18104	55714	81235	936	0.73508	79023	81341	26664	537
.746	.67870	65592	49704	53032	193	.73440	95631	39960	28591	681
.747	.67944	06293	37191	55745	803	.73373	04894	89077	36602	285
.748	.68017	40199	84105	86745	313	.73305	06821	07766	10125	695
.749	.68090	67304	57056	87450	880	.73237	01416	75833	81627	975
0.750	0.68163	87600	23334	16673	324	0.73168	88688	73820	88631	184
.751	.68237	01079	50908	23885	163	.73100	68643	83000	05659	342
.752	.68310	07735	08431	22423	554	.73032	41288	85375	76111	160
.753	.68383	07559	65237	62625	080	.72964	06630	63683	44059	608
.754	.68456	00545	91345	04892	285	.72895	64676	01388	85978	367
0.755	0.68528	86686	57454	92691	917	0.72827	15431	82687	42395	268
.756	.68601	65974	34953	25484	772	.72758	58904	92503	49472	750
.757	.68674	38401	95911	31587	089	.72689	95102	16489	70515	436
.758	.68747	03962	13086	40963	419	.72621	24030	41026	27404	867
.759	.68819	62647	59922	57950	885	.72552	45696	53220	31961	494
0.760	0.68892	14451	10551	33914	776	0.72483	60107	40905	17233	969
.761	.68964	59365	39792	39835	383	.72414	67269	92639	68715	814
.762	.69036	97383	23154	38826	030	.72345	67190	97707	55489	548
.763	.69109	28497	36835	58582	200	.72276	59877	46116	61298	318
.764	.69181	52700	57724	63761	700	.72207	45336	28598	15545	123
0.765	0.69253	69985	63401	28295	794	0.72138	23574	36606	24219	693
.766	.69325	80345	32137	07631	223	.72068	94598	62317	00753	084
.767	.69397	83772	42896	10903	039	.71999	58415	98627	96800	072
.768	.69469	80259	75335	73038	195	.71930	15033	39157	32949	410
.769	.69541	69800	09807	26789	802	.71860	64457	78243	29362	010
0.770	0.69613	52386	27356	74701	988	0.71791	06696	10943	36337	129
.771	.69685	28011	09725	61005	296	.71721	41755	33033	64806	626
.772	.69756	96667	39351	43442	524	.71651	69642	41008	16757	355
.773	.69828	58347	99368	65024	972	.71581	90364	32078	15581	770
.774	.69900	13045	73609	25718	983	.71512	03928	04171	36356	807
0.775	0.69971	60753	46603	54062	747	0.71442	10340	55931	36051	117
.776	.70043	01464	03580	78713	256	.71372	09608	86716	83660	709
.777	.70114	35170	30469	99923	379	.71302	01739	96600	90273	093
.778	.70185	61865	13900	60948	949	.71231	86740	86370	39059	972
.779	.70256	81541	41203	19385	818	.71161	64618	57525	15198	564

TABLE XII.—*Values of sin x and cos x to 23 places of decimals at intervals of 0.001 from 0.000 to 1.600*—Continued.

x	$\sin x$					$\cos x$				
0.780	0.70327	94192	00410	18436	790	0.71091	35380	12277	35721	626
.781	.70398	99809	80256	58108	374	.71020	99032	53550	79296	239
.782	.70469	98387	70180	66337	280	.70950	55582	84980	15931	435
.783	.70540	89918	60324	70046	581	.70880	05038	10910	36614	737
.784	.70611	74395	41535	66131	480	.70809	47405	36395	82877	671
0.785	0.70682	51811	05365	92374	614	0.70738	82691	67199	76290	330
.786	.70753	22158	44073	98290	801	.70668	10904	09793	47885	059
.787	.70823	85430	50625	15901	193	.70597	32049	71355	67509	330
.788	.70894	41620	18692	30436	730	.70526	46135	59771	73107	880
.789	.70964	90720	42656	50970	857	.70455	53168	83632	99934	173
0.790	0.71035	32724	17607	80981	403	0.70384	53156	52236	09691	278
.791	.71105	67624	39345	88841	574	.70313	46105	75582	19602	208
.792	.71175	95414	04380	78239	979	.70242	32023	64376	31409	812
.793	.71246	16086	09933	58529	620	.70171	10917	30026	60306	275
.794	.71316	29633	53937	15005	776	.70099	82793	84643	63792	314
0.795	0.71386	36049	35036	79112	713	0.70028	47660	41039	70466	123
.796	.71456	35326	52590	98579	148	.69957	05524	12728	08742	151
.797	.71526	27458	06672	07482	391	.69885	56392	13922	35499	779
.798	.71596	12436	98066	96241	109	.69814	00271	59535	64661	971
.799	.71665	90256	28277	81536	630	.69742	37169	65179	95703	964
0.800	0.71735	60908	99522	76162	718	0.69670	67093	47165	42092	075
.801	.71805	24388	14736	58803	753	.69598	90050	22499	59652	695
.802	.71874	80686	77571	43741	255	.69527	06047	08886	74871	538
.803	.71944	29797	92397	50488	651	.69455	15091	24727	13123	218
.804	.72013	71714	64303	73354	263	.69383	17189	89116	26831	236
0.805	0.72083	06429	99098	50932	396	0.69311	12350	21844	23558	425
.806	.72152	33937	03310	35522	503	.69239	00579	43394	94027	956
.807	.72221	54228	84188	62476	322	.69166	81884	74945	40074	951
.808	.72290	67298	49704	19472	935	.69094	56273	38365	02528	784
.809	.72359	73139	08550	15721	677	.69022	23752	56214	89026	151
0.810	0.72428	71743	70142	51092	818	0.68949	84329	51747	01754	964
.811	.72497	63105	44620	85175	959	.68877	38011	48903	65129	158
.812	.72566	47217	42849	06266	069	.68804	84805	72316	53394	472
.813	.72635	24072	76416	00277	085	.68732	24719	47306	18165	280
.814	.72703	93664	57636	19583	027	.68659	57759	99881	15892	545
0.815	0.72772	55985	99550	51786	534	0.68586	83934	56737	35262	969
.816	.72841	11030	15926	88414	775	.68514	03250	45257	24529	414
.817	.72909	58790	21260	93542	651	.68441	15714	93509	18772	652
.818	.72977	99259	30776	72343	223	.68368	21335	30246	67094	544
.819	.73046	32430	60427	39565	302	.68295	20118	84907	59742	692
0.820	0.73114	58297	26895	87938	131	0.68222	12072	87613	55166	656
.821	.73182	76852	47595	56503	084	.68148	97204	69169	07005	802
.822	.73250	88089	40670	98872	320	.68075	75521	61060	91008	857
.823	.73318	92001	24998	51414	329	.68002	47030	95457	31885	232
.824	.73386	88581	20187	01366	283	.67929	11740	05207	30088	213
0.825	0.73454	77822	46578	54873	150	0.67855	69656	23839	88530	058
.826	.73522	59718	25249	04953	477	.67782	20786	85563	39229	106
.827	.73590	34261	78008	99391	793	.67708	65139	25264	69888	949
.828	.73658	01446	27404	08557	557	.67635	02720	78508	50409	750
.829	.73725	61264	96715	93150	579	.67561	33538	81536	59331	781
0.830	0.73793	13711	09962	71872	858	0.67487	57600	71267	10211	246
.831	.73860	58777	91899	89026	752	.67413	74913	85293	77928	481
.832	.73927	96458	68020	82039	434	.67339	85485	61885	24928	580
.833	.73995	26746	64557	48913	544	.67265	89323	39984	27394	537
.834	.74062	49635	08481	15603	989	.67191	86434	59207	01352	983
0.835	0.74129	65117	27503	03320	808	0.67117	76826	59842	28712	570
.836	.74196	73186	50074	95758	049	.67043	60506	82850	83235	098
.837	.74263	73836	05390	06248	576	.66969	37482	69864	56439	445
.838	.74330	67059	23383	44844	755	.66895	07761	63185	83438	385
.839	.74397	52849	34732	85324	932	.66820	71351	05786	68708	357

TABLE XII.—*Values of sin x and cos x to 23 places of decimals at intervals of 0.001 from 0.000 to 1.600*—Continued.

x	*sin x*	*cos x*
0. 840	0. 74464 31199 70859 32125 657	0. 66746 28258 41308 11792 267
. 841	. 74531 02103 63927 87199 577	66671 78491 14059 32935 396
. 842	. 74597 65554 46848 16798 923	. 66597 22056 69016 98654 482
. 843	. 74664 21545 53275 18184 539	. 66522 58962 51824 47240 065
. 844	. 74730 70070 17609 86260 385	. 66447 89216 08791 14192 152
0. 845	0. 74797 11121 74999 80133 429	0. 66373 12824 86891 57589 286
. 846	. 74863 44693 61339 89598 886	. 66298 29796 33764 83391 100
. 847	. 74929 70779 13273 01550 724	. 66223 40137 97713 70674 409
. 848	. 74995 89371 68190 66317 368	. 66148 43857 27703 96802 946
. 849	. 75062 00464 64233 63922 547	. 66073 40961 73363 62530 783
0. 850	0. 75128 04051 40292 70271 207	0. 65998 31458 84982 17039 542
. 851	. 75194 00125 36009 23260 432	. 65923 15356 13509 82909 449
. 852	. 75259 88679 91775 88815 295	. 65847 92661 10556 81024 321
. 853	. 75325 69708 48737 26849 594	. 65772 63381 28392 55410 547
. 854	. 75391 43204 48790 57151 380	. 65697 27524 19944 98010 152
0. 855	0. 75457 09161 34586 25193 237	0. 65621 85097 38799 73388 013
. 856	. 75522 67572 49528 67867 227	. 65546 36108 39199 43373 300
. 857	. 75588 18431 37776 79144 450	. 65470 80564 76042 91635 218
. 858	. 75653 61731 44244 75659 143	. 65395 18474 04884 48193 134
. 859	. 75718 97466 14602 62217 260	. 65319 49843 81933 13861 148
0. 860	0. 75784 25628 95276 97229 459	0. 65243 74681 64051 84627 203
. 861	. 75849 46213 33451 58068 441	. 65167 92995 08756 75966 794
. 862	. 75914 59212 77068 06350 566	. 65092 04791 74216 47091 357
. 863	. 75979 64620 74826 53141 684	. 65016 10079 19251 25131 418
. 864	. 76044 62430 76186 24087 122	. 64940 08865 03332 29254 574
0. 865	0. 76109 52636 31366 24465 750	0. 64864 01156 86580 94718 373
. 866	. 76174 35230 91346 04168 073	. 64787 86962 29767 96858 196
. 867	. 76239 10208 07866 22598 272	. 64711 66288 94312 75010 176
. 868	. 76303 77561 33429 13500 144	. 64635 39144 42282 56369 276
. 869	. 76368 37284 21299 49706 858	. 64559 05536 36391 79782 561
0. 870	0. 76432 89370 25505 07814 480	0. 64482 65472 40001 19477 766
. 871	. 76497 33813 00837 32779 191	. 64406 18960 17117 08727 234
. 872	. 76561 70606 02852 02438 134	. 64329 66007 32390 63447 280
. 873	. 76625 99742 87869 91953 834	. 64253 06621 51117 05733 091
. 874	. 76690 21217 12977 38182 114	. 64176 40810 39234 87329 202
0. 875	0. 76754 35022 36027 03963 458	0. 64099 68581 63325 13035 656
. 876	. 76818 41152 15638 42337 736	. 64022 89942 90610 64049 903
. 877	. 76882 39600 11198 60682 252	. 63946 04901 88955 21244 528
. 878	. 76946 30359 82862 84773 027	. 63869 13466 26862 88380 872
. 879	. 77010 13424 91555 22769 271	. 63792 15643 73477 15258 639
0. 880	0. 77073 88788 98969 29120 965	0. 63715 11441 98580 20801 550
. 881	. 77137 56445 67568 68399 506	. 63638 00868 72592 16079 131
. 882	. 77201 16388 60587 79051 337	. 63560 83931 66570 27264 710
. 883	. 77264 68611 42032 37074 497	. 63483 60638 52208 18529 695
. 884	. 77328 13107 76680 19618 049	. 63406 30997 01835 14874 218
0. 885	0. 77391 49871 30081 68504 290	0. 63328 95014 88415 24894 213
. 886	. 77454 78895 68560 53673 706	. 63251 52699 85546 63485 020
. 887	. 77518 00174 59214 36552 600	. 63174 04059 67460 74481 571
. 888	. 77581 13701 69915 33343 321	. 63096 49102 09021 53235 256
. 889	. 77644 19470 69310 78237 045	. 63018 87834 85724 69127 530
0. 890	0. 77707 17475 26823 86549 033	0. 62941 20265 73696 88020 355
. 891	. 77770 07709 12654 17776 316	. 62863 46402 49694 94643 540
. 892	. 77832 90165 97778 38577 722	. 62785 66252 91105 14919 057
. 893	. 77895 64839 53950 85676 211	. 62707 79824 75942 38222 428
. 894	. 77958 31723 53704 28683 432	. 62629 87125 82849 39581 242
0. 895	0. 78020 90811 70350 32846 443	0. 62551 88163 91096 01810 880
. 896	. 78083 42097 77980 21716 548	62473 82946 80578 37587 545
. 897	. 78145 85575 51465 39740 163	. 62395 71482 31818 11458 656
. 898	. 78208 21238 66458 14771 667	. 62317 53778 25961 61790 683
. 899	. 78270 49080 99392 20508 171	. 62239 29842 44779 22654 524

TABLE XII.—*Values of sin x and cos x to 23 places of decimals at intervals of 0.001 from 0.000 to 1.600*—Continued.

x	$sin\ x$					$cos\ x$				
0. 900	0. 78332	69096	27483	38846	138	0. 62160	99682	70664	45648	472
. 901	. 78394	81278	28730	22159	796	. 62082	63306	86633	21658	870
. 902	. 78456	85620	81914	55501	279	. 62004	20722	76323	02558	530
. 903	. 78518	82117	66602	18722	439	. 61925	71938	23992	22842	983
. 904	. 78580	70762	63143	48518	260	. 61847	16961	14519	21204	658
0. 905	0. 78642	51549	52674	00391	817	0. 61768	55799	33401	62045	040
. 906	. 78704	24472	17115	10540	713	. 61689	88460	66755	56924	921
. 907	. 78765	89524	39174	57664	940	. 61611	14953	01314	85952	792
. 908	. 78827	46700	02347	24696	094	. 61532	35284	24430	19111	466
. 909	. 78888	95992	90915	60447	888	. 61453	49462	24068	37523	020
0. 910	0. 78950	37396	89950	41187	896	0. 61374	57494	88811	54652	118
. 911	. 79011	70905	85311	32130	474	. 61295	59390	07856	37447	803
. 912	. 79072	96513	63647	48850	789	. 61216	55155	71013	27423	839
. 913	. 79134	14214	12398	18619	897	. 61137	44799	68705	61677	674
. 914	. 79195	24001	19793	41660	812	. 61058	28329	91968	93848	110
0. 915	0. 79256	25868	74854	52325	499	0. 60979	05754	32450	15011	758
. 916	. 79317	19810	67394	80192	738	. 60899	77080	82406	74518	350
. 917	. 79378	05820	88020	11086	785	. 60820	42317	34706	00764	999
. 918	. 79438	83893	28129	48016	785	. 60741	01471	82824	21909	476
. 919	. 79499	54021	79915	72036	860	. 60661	54552	20845	86522	589
0. 920	0. 79560	16200	36366	03026	828	0. 60582	01566	43462	84179	741
. 921	. 79620	70422	91262	60393	471	. 60502	42522	45973	65991	745
. 922	. 79681	16683	39183	23692	319	. 60422	77428	24282	65074	984
. 923	. 79741	54975	75501	93169	858	. 60343	06291	74899	16960	980
. 924	. 79801	85293	96389	50226	129	. 60263	29120	94936	79945	468
0. 925	0. 79862	07631	98814	17797	639	0. 60183	45923	82112	55377	043
. 926	. 79922	21983	80542	20660	537	. 60103	56708	34746	07885	466
. 927	. 79982	28343	40138	45653	978	. 60023	61482	51758	85549	703
. 928	. 80042	26704	76967	01823	638	. 59943	60254	32673	40005	791
. 929	. 80102	17061	91191	80485	294	. 59863	53031	77612	46494	584
0. 930	0. 80161	99408	83777	15208	432	0. 59783	39822	87298	23849	491
. 931	. 80221	73739	56488	41719	806	. 59703	20635	63051	54424	260
. 932	. 80281	40048	11892	57726	899	. 59622	95478	06791	03960	905
. 933	. 80340	98328	53358	82661	218	. 59542	64358	21032	41397	846
. 934	. 80400	48574	85059	17341	371	. 59462	27284	08887	58618	345
0. 935	0. 80459	90781	11969	03555	863	0. 59381	84263	74063	90139	324
. 936	. 80519	24941	39867	83565	545	. 59301	35305	20863	32740	634
. 937	. 80578	51049	75339	59525	671	. 59220	80416	54181	65034	867
. 938	. 80637	69100	25773	52827	488	. 59140	19605	79507	66977	785
. 939	. 80696	79086	99364	63359	313	. 59059	52881	02922	39319	443
0. 940	0. 80755	81004	05114	28687	022	0. 58978	80250	31098	22996	099
. 941	. 80814	74845	52830	83153	915	. 58898	01721	71298	18462	976
. 942	. 80873	60605	53130	16899	872	. 58817	17303	31375	04967	973
. 943	. 80932	38278	17436	34799	758	. 58736	27003	19770	59766	388
. 944	. 80991	07857	57982	15321	017	. 58655	30829	45514	77276	748
0. 945	0. 81049	69337	87809	69300	383	0. 58574	28790	18224	88177	827
. 946	. 81108	22713	20770	98639	669	. 58493	20893	48104	78446	913
. 947	. 81166	67977	71528	54920	560	. 58412	07147	45944	08339	436
. 948	. 81225	05125	55555	97938	351	. 58330	87560	23117	31310	012
. 949	. 81283	34150	89138	54154	591	. 58249	62139	91583	12874	994
0. 950	0. 81341	55047	89373	75068	542	0. 58168	30894	63883	49416	618
. 951	. 81399	67810	74171	95507	433	. 58086	93832	53142	86928	810
. 952	. 81457	72433	62256	91835	411	. 58005	50961	73067	39704	748
. 953	. 81515	68910	73166	40081	165	. 57924	02290	37944	08966	253
. 954	. 81573	57236	27252	73984	145	. 57842	47826	62640	01435	096
0. 955	0. 81631	37404	45683	42959	322	0. 57760	87578	62601	47846	300
. 956	. 81689	09409	50441	69980	433	. 57679	21554	53853	21403	511
. 957	. 81746	73245	64327	09381	654	. 57597	49762	52997	56176	536
. 958	. 81804	28907	10956	04577	644	. 57515	72210	77213	65441	113
. 959	. 81861	76388	14762	45701	891	. 57433	88907	44256	59961	007

Table XII.—*Values of sin x and cos x to 23 places of decimals at intervals of 0.001 from 0.000 to 1.600*—Continued.

x	*sin x*					*cos x*				
0.960	0.81919	15683	00998	27163	322	0.57351	99860	72456	66212	505
.961	.81976	46785	95734	05121	101	.57270	05078	80718	44551	395
.962	.82033	69691	25859	54877	569	.57188	04569	88520	07322	513
.963	.82090	84393	19084	28189	263	.57105	98342	15912	36911	940
.964	.82147	90886	03938	10495	962	.57023	86403	83518	03741	923
0.965	0.82204	89164	09771	78067	694	0.56941	68763	12530	84208	614
.966	.82261	79221	66757	55069	656	.56859	45428	24714	78562	699
.967	.82318	61053	05889	70544	986	.56777	16407	42403	28733	004
.968	.82375	34652	58985	15315	328	.56694	81708	88498	36093	162
.969	.82432	00014	58683	98799	136	.56612	41340	86469	79171	417
0.970	0.82488	57133	38450	05747	662	0.56529	95311	60354	31303	653
.971	.82545	06003	32571	52898	564	.56447	43629	34754	78229	727
.972	.82601	46618	76161	45547	087	.56364	86302	34839	35633	190
.973	.82657	78974	05158	34034	750	.56282	23338	86340	66624	480
.974	.82714	03063	56326	70155	495	.56199	54747	15554	99167	663
0.975	0.82770	18881	67257	63479	226	0.56116	80535	49341	43450	813
.976	.82826	26422	76369	37592	699	.56034	00712	15121	09200	110
.977	.82882	25681	22907	86257	689	.55951	15285	40876	22937	736
.978	.82938	16651	46947	29486	397	.55868	24263	55149	45183	654
.979	.82993	99327	89390	69534	022	.55785	27654	87042	87601	358
0.980	0.83049	73704	91970	46808	453	0.55702	25467	66217	30087	666
.981	.83105	39776	97248	95697	028	.55619	17710	22891	37806	645
.982	.83160	97538	48619	00310	290	.55536	04390	87840	78167	757
.983	.83216	46983	90304	50142	703	.55452	85517	92397	37748	295
.984	.83271	88107	67360	95650	254	.55369	61099	68448	39160	207
0.985	0.83327	20904	25676	03744	902	0.55286	31144	48435	57861	376
.986	.83382	45368	11970	13205	801	.55202	95660	65354	38911	453
.987	.83437	61493	73796	90007	262	.55119	54656	52753	13672	322
.988	.83492	69275	59543	82563	379	.55036	08140	44732	16453	272
.989	.83547	68708	18432	76889	279	.54952	56120	75943	01100	969
0.990	0.83602	59786	00520	51678	926	0.54868	98605	81587	57534	313
.991	.83657	42503	56699	33299	444	.54785	35603	97417	28224	252
.992	.83712	16855	38697	50701	883	.54701	67123	59732	24618	647
.993	.83766	82835	99079	90248	385	.54617	93173	05380	43512	268
.994	.83821	40439	91248	50455	694	.54534	13760	71756	83362	006
0.995	0.83875	89661	69442	96654	953	0.54450	28894	96802	60547	375
.996	.83930	30495	88741	15567	733	.54366	38584	19004	25576	412
.997	.83984	62937	05059	69798	245	.54282	42836	77392	79237	026
.998	.84038	86979	75154	52241	668	.54198	41661	11542	88693	907
.999	.84093	02618	56621	40408	555	.54114	35065	61572	03531	067
1.000	0.84147	09848	07896	50665	250	0.54030	23058	68139	71740	094
.001	.84201	08662	88256	92390	268	.53946	05648	72446	55654	214
.002	.84254	99057	57821	22046	578	.53861	82844	16233	47828	237
.003	.84308	81026	77549	97169	747	.53777	54653	41780	86864	465
.004	.84362	54565	09246	30271	873	.53693	21084	91907	73184	669
1.005	0.84416	19667	15556	42661	273	0.53608	82147	09970	84748	188
.006	.84469	76327	59970	18177	851	.53524	37848	39863	92716	262
.007	.84523	24541	06821	56844	116	.53439	88197	26016	77062	668
.008	.84576	64302	21289	28431	774	.53355	33202	13394	42130	747
.009	.84629	95605	69397	25943	853	.53270	72871	47496	32136	904
1.010	0.84683	18446	18015	19012	310	0.53186	07213	74355	46620	673
.011	.84736	32818	34859	07211	051	.53101	36237	40537	55841	426
.012	.84789	38716	88491	73284	331	.53016	59950	93140	16121	808
.013	.84842	36136	48323	36290	466	.52931	78362	79791	85137	984
.014	.84895	25071	84612	04660	810	.52846	91481	48651	37156	798
1.015	0.84948	05517	68464	29173	940	0.52761	99315	48406	78219	896
.016	.85000	77468	71835	55845	003	.52677	01873	28274	61274	932
.017	.85053	40919	67530	78730	164	.52591	99163	37999	01253	921
.018	.85105	95865	29204	92646	111	.52506	91194	27850	90098	832
.019	.85158	42300	31363	45804	549	.52421	77974	48627	11734	503

Table XII.—*Values of sin x and cos x to 23 places of decimals at intervals of 0.001 from 0.000 to 1.600*—Continued.

x	$\sin x$	$\cos x$
1.020	0.85210 80219 49362 92361 655	0.52336 59512 51649 56988 961
.021	.85263 09617 59411 44882 415	.52251 35816 88764 38461 245
.022	.85315 30489 38569 26719 808	.52166 06896 12341 05336 792
.023	.85367 42829 64749 24308 778	.52080 72758 75271 58150 502
.024	.85419 46633 16717 39374 945	.51995 33413 30969 63497 542
1.025	0.85471 41894 74093 41057 997	0.51909 88868 33369 68691 985
.026	.85523 28609 17351 17949 715	.51824 39132 36926 16373 373
.027	.85575 06771 27819 30046 586	.51738 84213 96612 59061 276
.028	.85626 76375 87681 60616 931	.51653 24121 67920 73657 956
.029	.85678 37417 79977 67982 525	.51567 58864 06859 75899 186
1.030	0.85729 89891 88603 37214 627	0.51481 88449 69955 34753 350
.031	.85781 33792 98311 31744 398	.51396 12887 14248 86768 878
.032	.85832 69115 94711 44887 626	.51310 32184 97296 50370 116
.033	.85883 95855 64271 51283 734	.51224 46351 77168 40101 715
.034	.85935 14006 94317 58248 998	.51138 55396 12447 80821 625
1.035	0.85986 23564 73034 57043 938	0.51052 59326 62230 21842 776
.036	.86037 24523 89466 74054 819	.50966 58151 86122 51023 535
.037	.86088 16879 33518 21889 224	.50880 51880 44242 08807 028
.038	.86139 00625 95953 50385 634	.50794 40520 97216 02209 404
.039	.86189 75758 68397 97536 975	.50708 24082 06180 18757 138
1.040	0.86240 42272 43338 40328 079	0.50622 02572 32778 40373 447
.041	.86291 00162 14123 45486 997	.50535 76000 39161 57213 919
.042	.86341 49422 74964 20150 131	.50449 44374 87986 81451 427
.043	.86391 90049 20934 62441 124	.50363 07704 42416 61010 426
.044	.86442 22036 47972 11963 456	.50276 65997 66117 93250 711
1.045	0.86492 45379 52878 00206 699	0.50190 19263 23261 38600 728
.046	.86542 60073 33318 00866 385	.50103 67509 78520 34140 520
.047	.86592 66112 87822 80077 424	.50017 10745 97070 07134 396
.048	.86642 63493 15788 46561 037	.49930 48980 44586 88513 415
.049	.86692 52209 17477 01685 140	.49843 82221 87247 26307 756
1.050	0.86742 32255 94016 89438 141	0.49757 10478 91726 99029 085
.051	.86792 03628 47403 46316 092	.49670 33760 25200 29002 975
.052	.86841 66321 80499 51123 146	.49583 52074 55338 95651 499
.053	.86891 20330 97035 74685 276	.49496 65430 50311 48726 051
.054	.86940 65651 01611 29477 198	.49409 73836 78782 21490 510
1.055	0.86990 02276 99694 19162 460	0.49322 77302 09910 43854 806
.056	.87039 30203 97621 88046 624	.49235 75835 13349 55459 008
.057	.87088 49427 02601 70443 529	.49148 69444 59246 18707 979
.058	.87137 59941 22711 39954 543	.49061 58139 18239 31756 732
.059	.87186 61741 66899 58660 794	.48974 41927 61459 41446 534
1.060	0.87235 54823 44986 26228 295	0.48887 20818 60527 56191 864
.061	.87284 39181 67663 28925 947	.48799 94820 87554 58818 317
.062	.87333 14811 46494 88556 345	.48712 63943 15140 19351 528
.063	.87381 81707 93918 11299 356	.48625 28194 16372 07757 202
.064	.87430 39866 23243 36468 402	.48537 87582 64825 06632 362
1.065	0.87478 89281 48654 85179 424	0.48450 42117 34560 23847 867
.066	.87527 29948 85211 08932 453	.48362 91807 00124 05142 311
.067	.87575 61863 48845 38105 753	.48275 36660 36547 46667 387
.068	.87623 85020 56366 30362 492	.48187 76686 19345 07484 800
.069	.87671 99415 25458 18969 874	.48100 11893 24514 22014 811
1.070	0.87720 05042 74681 61030 706	0.48012 42290 28534 12436 509
.071	.87768 01898 23473 85627 336	.47924 67886 08365 01039 904
.072	.87815 89976 92149 41877 919	.47836 88689 41447 22529 904
.073	.87863 69274 01900 46904 963	.47749 04709 05700 36282 289
.074	.87911 39784 74797 33716 111	.47661 15953 79522 38551 762
1.075	0.87959 01504 33788 98997 101	0.47573 22432 41788 74632 160
.076	.88006 54428 02703 50816 869	.47485 24153 71851 50968 911
.077	.88053 98551 06248 56244 731	.47397 21126 49538 47223 840
.078	.88101 33868 70011 88879 619	.47309 13359 55152 28292 396
.079	.88148 60376 20461 76291 297	.47221 00861 69469 56273 392

Table XII.—*Values of sin x and cos x to 23 places of decimals at intervals of 0.001 from 0.000 to 1.600*—Continued.

x	$\sin x$	$\cos x$
1.080	0.88195 78068 84947 47373 533	0.47132 83641 73740 02391 353
.081	.88242 86941 91699 79609 169	.47044 61708 49685 58871 547
.082	.88289 86990 69831 46247 031	.46956 35070 79499 50767 810
.083	.88336 78210 49337 63390 660	.46868 03737 45845 47743 217
.084	.88383 60596 61096 36998 790	.46779 67717 31856 75803 727
1.085	0.88430 34144 36869 09797 534	0.46691 27019 21135 28984 862
.086	.88476 98849 09301 08104 243	.46602 81651 97750 80991 522
.087	.88523 54706 11921 88562 972	.46514 31624 46239 96791 014
.088	.88570 01710 79145 84791 522	.46425 76945 51605 44159 401
.089	.88616 39858 46272 53940 000	.46337 17623 99315 05181 235
1.090	0.88662 69144 49487 23160 860	0.46248 53668 75300 87702 790
.091	.88708 89564 25861 35990 371	.46159 85088 65958 36738 852
.092	.88755 01113 13352 98641 470	.46071 11892 58145 45833 190
.093	.88801 03786 50807 26207 951	.45982 34089 39181 68372 764
.094	.88846 97579 77956 88779 948	.45893 51687 96847 28855 783
1.095	0.88892 82488 35422 57470 660	0.45804 64697 19382 34113 686
.096	.88938 58507 64713 50354 274	.45715 73125 95485 84487 142
.097	.88984 25633 08227 78315 047	.45626 76983 14314 84956 158
.098	.89029 83860 09252 90807 488	.45537 76277 65483 56224 382
.099	.89075 33184 11966 21527 609	.45448 71018 39062 45757 688
1.100	0.89120 73600 61435 33995 180	0.45359 61214 25577 38777 137
.101	.89166 05105 03618 67046 971	.45270 46874 16008 69206 400
.102	.89211 27692 85365 80240 901	.45181 28007 01790 30573 730
.103	.89256 41359 54417 99171 080	.45092 04621 74808 86868 576
.104	.89301 46100 59408 60693 678	.45002 76727 27402 83352 928
1.105	0.89346 41911 49863 58063 585	0.44913 44332 52361 57327 478
.106	.89391 28787 76201 85981 812	.44824 07446 42924 48852 689
.107	.89436 06724 89735 85553 594	.44734 66077 92780 11424 866
.108	.89480 75718 42671 89157 146	.44645 20235 96065 22607 305
.109	.89525 35763 88110 65223 027	.44555 69929 47363 94616 628
1.110	0.89569 86856 80047 62924 063	0.44466 15167 41706 84864 374
.111	.89614 28992 73373 56775 801	.44376 55958 74570 06453 951
.112	.89658 62167 23874 91147 427	.44286 92312 41874 38633 030
.113	.89702 86375 88234 24683 120	.44197 24237 39984 37201 474
.114	.89747 01614 24030 74633 785	.44107 51742 65707 44874 890
1.115	0.89791 07877 89740 61099 138	0.44017 74837 16293 01603 891
.116	.89835 05162 44737 51180 079	.43927 93529 89431 54849 166
.117	.89878 93463 49293 03041 321	.43838 07829 83253 69812 438
.118	.89922 72776 64577 09884 230	.43748 17745 96329 39623 410
.119	.89966 43097 52658 43829 826	.43658 23287 27666 95482 777
1.120	0.90010 04421 76504 99711 910	0.43568 24462 76712 16761 399
.121	.90053 56744 99984 38780 263	.43478 21281 43347 41055 736
.122	.90097 00062 87864 32313 880	.43388 13752 27890 74199 612
.123	.90140 34371 05813 05144 201	.43298 01884 31095 00232 420
.124	.90183 59665 20399 79088 276	.43207 85686 54146 91323 845
1.125	0.90226 75940 99095 16291 842	0.43117 65167 98666 17655 197
.126	.90269 83194 10271 62482 258	.43027 40337 66704 57257 452
.127	.90312 81420 23203 90131 256	.42937 11204 60745 05806 078
.128	.90355 70615 08069 41527 464	.42846 77777 83700 86372 749
.129	.90398 50774 35948 71758 658	.42756 40066 38914 59134 030
1.130	0.90441 21893 78825 91603 708	0.42665 98079 30157 31037 122
.131	.90483 83969 09589 10334 160	.42575 51825 61627 65422 763
.132	.90526 36996 02030 78425 425	.42485 01314 37950 91605 376
.133	.90568 80970 30848 30177 523	.42394 46554 64178 14410 340
.134	.90611 15887 71644 26245 348	.42303 87555 45785 23669 902
1.135	0.90653 41744 00926 96078 401	0.42213 24325 88672 03673 585
.136	.90695 58534 96110 80269 960	.42122 56874 99161 42580 219
.137	.90737 66256 35516 72815 632	.42031 85211 83998 41784 656
.138	.90779 64903 98372 63281 260	.41941 09345 50349 25243 478
.139	.90821 54473 64813 78880 126	.41850 29285 05800 48758 379

TABLE XII.—*Values of sin x and cos x to 23 places of decimals at intervals of 0.001 from 0.000 to 1.600*—Continued.

x	$\sin x$	$\cos x$
1.140	0.90863 34961 15883 26459 422	0.41759 45039 58358 09217 519
.141	.90905 06362 33532 34395 940	.41668 56618 16446 53794 933
.142	.90946 68673 00620 94400 939	.41577 64029 88907 89108 094
.143	.90988 21889 00918 03234 153	.41486 67283 85000 90333 707
.144	.91029 66006 19102 04326 885	.41395 66389 14400 10281 852
1.145	0.91071 01020 40761 29314 164	0.41304 61354 87194 88428 529
.146	.91112 26927 52394 39475 912	.41213 52190 13888 59906 732
.147	.91153 43723 41410 67087 073	.41122 38904 05397 64456 120
.148	.91194 51403 96130 56676 684	.41031 21505 73050 55331 381
.149	.91235 49965 05786 06195 821	.40940 00004 28587 08169 395
1.150	0.91276 39402 60521 08094 403	0.40848 74408 84157 29815 258
.151	.91317 19712 51391 90306 792	.40757 44728 52320 67107 284
.152	.91357 90890 70367 57146 165	.40666 10972 46045 15621 071
.153	.91398 52933 10330 30107 602	.40574 73149 78706 28372 706
.154	.91439 05835 65075 88579 865	.40483 31269 64086 24481 224
1.155	0.91479 49594 29314 10465 816	0.40391 85341 16372 97790 397
.156	.91519 84204 98669 12711 431	.40300 35373 50159 25449 945
.157	.91560 09663 69679 91743 383	.40208 81375 80441 76456 266
.158	.91600 25966 39800 63815 143	.40117 23357 22620 20152 779
.159	.91640 33109 07401 05261 556	.40025 61326 92496 34689 958
1.160	0.91680 31087 71766 92661 866	0.39933 95294 06273 15445 164
.161	.91720 19898 33100 42911 136	.39842 25267 80553 83402 355
.162	.91759 99536 92520 53200 023	.39750 51257 32340 93491 775
.163	.91799 69999 52063 40902 883	.39658 73271 79035 42889 706
.164	.91839 31282 14682 83374 147	.39566 91320 38435 79278 377
1.165	0.91878 83380 84250 57652 941	0.39475 05412 28737 09066 125
.166	.91918 26291 65556 80075 906	.39383 15556 68530 05567 898
.167	.91957 60010 64310 45798 178	.39291 21762 76800 17146 187
.168	.91996 84533 87139 68222 492	.39199 24039 72926 75312 486
.169	.92035 99857 41592 18336 360	.39107 22396 76682 02789 366
1.170	0.92075 05977 36135 63957 301	0.39015 16843 08230 21533 266
.171	.92114 02889 80158 08886 071	.38923 07387 88126 60718 072
.172	.92152 90590 83968 31967 851	.38830 94040 37316 64679 599
.173	.92191 69076 58796 26061 369	.38738 76809 77135 00821 054
.174	.92230 38343 16793 36915 902	.38646 55705 29304 67479 575
1.175	0.92268 98386 71033 01956 127	0.38554 30736 15936 01753 942
.176	.92307 49203 35510 88974 783	.38462 01911 59525 87293 547
.177	.92345 90789 25145 34733 097	.38369 69240 82956 62048 718
.178	.92384 23140 55777 83468 944	.38277 32733 09495 25982 487
.179	.92422 46253 44173 25312 701	.38184 92397 62792 48743 902
1.180	0.92460 60124 08020 34610 754	0.38092 48243 66881 77302 960
.181	.92498 64748 65932 08156 619	.38000 00280 46178 43547 271
.182	.92536 60123 37446 03329 642	.37907 48517 25478 71840 534
.183	.92574 46244 43024 76141 242	.37814 92963 29958 86542 917
.184	.92612 23108 04056 19188 645	.37722 33627 85174 19493 444
1.185	0.92649 90710 42853 99516 095	0.37629 70520 17058 17454 471
.186	.92687 49047 82657 96383 480	.37537 03649 51921 49518 342
.187	.92724 98116 47634 38942 352	.37444 33025 16451 14476 334
.188	.92762 37912 62876 43819 290	.37351 58656 37709 48149 962
.189	.92799 68432 54404 52606 588	.37258 80552 43133 30684 752
1.190	0.92836 89672 49166 69260 202	0.37165 98722 60532 93806 568
.191	.92874 01628 75038 97404 950	.37073 13176 18091 28040 589
.192	.92911 04297 60825 77546 899	.36980 23922 44362 89893 026
.193	.92947 97675 36260 24192 928	.36887 30970 68273 08995 672
.194	.92984 81758 32004 62877 403	.36794 34330 19116 95213 382
1.195	0.93021 56542 79650 67095 956	0.36701 34010 26558 45714 570
.196	.93058 22025 11719 95146 303	.36608 30020 20629 52004 819
.197	.93094 78201 61664 26876 083	.36515 22369 31729 06923 698
.198	.93131 25068 63866 00337 679	.36422 11066 90622 11604 876
.199	.93167 62622 53638 48349 974	.36328 96122 28438 82399 631

TABLE XII.—*Values of sin x and cos x to 23 places of decimals at intervals of 0.001 from 0.000 to 1.600*—Continued.

x	$\sin x$					$\cos x$				
1.200	0.93203	90859	67226	34967	013	0.36235	77544	76673	57763	837
.201	.93240	09776	41805	91853	542	.36142	55343	67184	05108	539
.202	.93276	19369	15485	54567	367	.36049	29528	32190	27614	189
.203	.93312	19634	27305	98748	519	.35956	00108	04273	71008	651
.204	.93348	10568	17240	76215	175	.35862	67092	16376	30309	065
1.205	0.93383	92167	26196	50966	302	0.35769	30490	01799	56527	660
.206	.93419	64427	96013	35090	992	.35675	90310	94203	63341	607
.207	.93455	27346	69465	24584	444	.35582	46564	27606	33727	018
.208	.93490	80919	90260	35070	567	.35488	99259	36382	26557	166
.209	.93526	25144	03041	37431	162	.35395	48405	55261	83165	039
1.210	0.93561	60015	53385	93341	646	0.35301	94012	19330	33870	301
.211	.93596	85530	87806	90713	291	.35208	36088	64027	04470	775
.212	.93632	01686	53752	79041	926	.35114	74644	25144	22698	521
.213	.93667	08478	99608	04663	095	.35021	09688	38826	24640	616
.214	.93702	05904	74693	45913	598	.34927	41230	41568	61124	730
1.215	0.93736	93960	29266	48199	416	0.34833	69279	70217	04069	578
.216	.93771	72642	14521	58969	959	.34739	93845	61966	52800	358
.217	.93806	41946	82590	62598	617	.34646	14937	54360	40329	260
.218	.93841	01870	86543	15169	574	.34552	32564	85289	39601	140
.219	.93875	52410	80386	79170	848	.34458	46736	92990	69704	455
1.220	0.93909	93563	19067	58093	524	0.34364	57463	16047	02047	552
.221	.93944	25324	58470	30937	151	.34270	64752	93385	66500	405
.222	.93978	47691	55418	86621	257	.34176	68615	64277	57501	890
.223	.94012	60660	67676	58302	957	.34082	69060	68336	40132	702
.224	.94046	64228	53946	57600	622	.33988	66097	45517	56153	996
1.225	0.94080	58391	73872	08723	559	0.33894	59735	36117	30011	855
.226	.94114	43146	88036	82507	685	.33800	49983	80771	74807	668
.227	.94148	18490	57965	30357	157	.33706	36852	20455	98234	533
.228	.94181	84419	46123	18091	912	.33612	20349	96483	08479	750
.229	.94215	40930	15917	59701	104	.33518	00486	50503	20093	523
1.230	0.94248	88019	31697	51002	382	0.33423	77271	24502	59823	955
.231	.94282	25683	58754	03206	998	.33329	50713	60802	72418	427
.232	.94315	53919	63320	76390	684	.33235	20823	02059	26391	462
.233	.94348	72724	12574	12870	299	.33140	87608	91261	19759	164
.234	.94381	82093	74633	70486	175	.33046	51080	71729	85740	328
1.235	0.94414	82025	18562	55790	164	0.32952	11247	87117	98424	316
.236	.94447	72515	14367	57139	322	.32857	68119	81408	78405	786
.237	.94480	53560	32999	77695	223	.32763	21705	98914	98386	387
.238	.94513	25157	46354	68328	851	.32668	72015	84277	88743	487
.239	.94545	87303	27272	60431	046	.32574	19058	82466	43066	054
1.240	0.94578	39994	49538	98628	471	0.32479	62844	38776	23657	769
.241	.94610	83227	87884	73405	063	.32385	03381	98828	67007	475
.242	.94643	17000	17986	53628	942	.32290	40681	08569	89227	042
.243	.94675	41308	16467	18984	738	.32195	74751	14269	91456	764
.244	.94707	56148	60895	92311	309	.32101	05601	62521	65238	364
1.245	0.94739	61518	29788	71844	815	0.32006	33242	00239	97855	712
.246	.94771	57414	02608	63367	118	.31911	57681	74660	77643	341
.247	.94803	43832	59766	12259	472	.31816	78930	33339	99262	871
.248	.94835	20770	82619	35461	479	.31721	96997	24152	68947	423
.249	.94866	88225	53474	53335	262	.31627	11891	95292	09714	116
1.250	0.94898	46193	55586	21434	849	0.31532	23623	95268	66544	754
.251	.94929	94671	73157	62180	713	.31437	32202	72909	11534	791
.252	.94961	33656	91340	96439	444	.31342	37637	77355	49010	665
.253	.94992	63145	96237	75008	528	.31247	39938	58064	20615	601
.254	.95023	83135	74899	10006	196	.31152	39114	64805	10363	979
1.255	0.95054	93623	15326	06166	303	0.31057	35175	47660	49664	355
.256	.95085	94605	06469	92038	225	.30962	28130	57024	22311	242
.257	.95116	86078	38232	51091	729	.30867	17989	43600	69445	729
.258	.95147	68040	01466	52726	783	.30772	04761	58403	94485	052
.259	.95178	40486	87975	83188	287	.30676	88456	52756	68021	196

TABLE XII.—*Values of sin x and cos x to 23 places of decimals at intervals of 0.001 from 0.000 to 1.600*—Continued.

x	$\sin x$	$\cos x$
1.260	0.95209 03415 90515 76385 682	0.30581 69083 78289 32688 634
.261	.95239 56824 02793 44617 416	.30486 46652 86939 08001 291
.262	.95270 00708 19468 09200 227	.30391 21173 30948 95158 833
.263	.95300 35065 36151 31003 222	.30295 92654 62866 81822 373
.264	.95330 59892 49407 40886 709	.30200 61106 35544 46859 693
1.265	0.95360 75186 56753 70045 767	0.30105 26538 02136 65060 070
.266	.95390 80944 56660 80258 512	.30009 88959 16100 11818 814
.267	.95420 77163 48552 94039 032	.29914 48379 31192 67791 595
.268	.95450 63840 32808 24694 963	.29819 04808 01472 23518 675
.269	.95480 40972 10759 06289 671	.29723 58254 81295 84019 121
1.270	0.95510 08555 84692 23509 018	0.29628 08729 25318 73355 114
.271	.95539 66588 57849 41432 673	.29532 56240 88493 39166 425
.272	.95569 15067 34427 35209 944	.29437 00799 26068 57175 182
.273	.95598 53989 19578 19640 104	.29341 42413 93588 35661 000
.274	.95627 83351 19409 78657 170	.29245 81094 46891 19906 579
1.275	0.95657 03150 40985 94719 118	0.29150 16850 42108 96613 869
.276	.95686 13383 92326 78101 497	.29054 49691 35665 98290 890
.277	.95715 14048 82408 96095 419	.28958 79626 84278 07609 308
.278	.95744 05142 21166 02109 886	.28863 06666 44951 61732 860
.279	.95772 86661 19488 64678 437	.28767 30819 74982 56616 726
1.280	0.95801 58602 89224 96370 075	0.28671 52096 31955 51277 939
.281	.95830 20964 43180 82604 453	.28575 70505 73742 72036 934
.282	.95858 73742 95120 10371 286	.28479 86057 58503 16730 332
.283	.95887 16935 59764 96853 962	.28383 98761 44681 58895 050
.284	.95915 50539 52796 17957 320	.28288 08626 91007 51923 831
1.285	0.95943 74551 90853 36739 577	0.28192 15663 56494 33192 303
.286	.95971 88969 91535 31748 357	.28096 19881 00438 28157 651
.287	.95999 93790 73400 25260 814	.28000 21288 82417 54428 993
.288	.96027 89011 55966 11427 805	.27904 19896 62291 25809 577
.289	.96055 74629 59710 84322 094	.27808 15714 00198 56310 871
1.290	0.96083 50642 06072 65890 556	0.27712 08750 56557 64138 661
.291	.96111 17046 17450 33810 354	.27615 99015 92064 75651 234
.292	.96138 73839 17203 49249 056	.27519 86519 67693 29289 769
.293	.96166 21018 29652 84528 675	.27423 71271 44692 79480 997
.294	.96193 58580 80080 50693 590	.27327 53280 84588 00512 263
1.295	0.96220 86523 94730 24982 339	0.27231 32557 49177 90379 053
.296	.96248 04845 00807 78203 231	.27135 09111 00534 74605 108
.297	.96275 13541 26481 02013 782	.27038 82951 01003 10035 206
.298	.96302 12610 00880 36103 915	.26942 54087 13198 88600 711
.299	.96329 02048 54098 95282 920	.26846 22529 00008 41057 992
1.300	0.96355 81854 17192 96470 135	0.26749 88286 24587 40699 798
.301	.96382 52024 22181 85589 331	.26653 51368 50360 07039 695
.302	.96409 12556 02048 64366 761	.26557 11785 41018 09469 650
.303	.96435 63446 90740 17032 855	.26460 69546 60519 70890 877
.304	.96462 04694 23167 36927 537	.26364 24661 73088 71318 016
1.305	0.96488 36295 35205 53009 126	0.26267 77140 43213 51456 761
.306	.96514 58247 63694 56266 806	.26171 26992 35646 16255 031
.307	.96540 70548 46439 26036 635	.26074 74227 15401 38427 774
.308	.96566 73195 22209 56221 061	.25978 18854 47755 61955 494
.309	.96592 66185 30740 81411 924	.25881 60883 98246 05556 626
1.310	0.96618 49516 12734 02916 926	0.25785 00325 32669 66133 818
.311	.96644 23185 09856 14689 520	.25688 37188 17082 22194 242
.312	.96669 87189 64740 29162 218	.25591 71482 17797 37244 030
.313	.96695 41527 20986 02983 276	.25495 03217 01385 63156 911
.314	.96720 86195 23159 62656 736	.25398 32402 34673 43517 173
1.315	0.96746 21191 16794 30085 794	0.25301 59047 84742 16937 022
.316	.96771 46512 48390 48019 478	.25204 83163 18927 20348 457
.317	.96796 62156 65416 05402 607	.25108 04758 04816 92269 738
.318	.96821 68121 16306 62628 991	.25011 23842 10251 76046 556
.319	.96846 64403 50465 76697 879	.24914 40425 03323 23067 996

TABLE XII.—*Values of sin x and cos x to 23 places of decimals at intervals of 0.001 from 0.000 to 1.600*—Continued.

x	*sin x*	*cos x*
1.320	0.96871 51001 18265 26273 590	0.24817 54516 52372 95957 398
.321	.96896 27911 71045 36648 340	.24720 66126 25991 71738 199
.322	.96920 95132 61115 04608 211	.24623 75263 93018 44974 865
.323	.96945 52661 41752 23202 252	.24526 81939 22539 30889 004
.324	.96970 00495 67204 06414 685	.24429 86161 83886 68450 760
1.325	0.96994 38632 92687 13740 188	0.24332 87941 46638 23445 582
.326	.97018 67070 74387 74662 236	.24235 87287 80615 91516 463
.327	.97042 85806 69462 13034 465	.24138 84210 55885 01181 759
.328	.97066 94838 36036 71365 051	.24041 78719 42753 16828 662
.329	.97090 94163 33208 35004 060	.23944 70824 11769 41682 448
1.330	0.97114 83779 21044 56233 768	0.23847 60534 33723 20751 578
.331	.97138 63683 60583 78261 900	.23750 47859 79643 43748 768
.332	.97162 33874 13835 59117 786	.23653 32810 20797 47988 097
.333	.97185 94348 43780 95451 405	.23556 15395 28690 21258 288
.334	.97209 45104 14372 46235 282	.23458 95624 75063 04672 221
1.335	0.97232 86138 90534 56369 230	0.23361 73508 31892 95492 805
.336	.97256 17450 38163 80187 900	.23264 49055 71391 49935 286
.337	.97279 39036 24129 04871 129	.23167 22276 66003 85946 099
.338	.97302 50894 16271 73757 046	.23069 93180 88407 85958 358
.339	.97325 53021 83406 09557 931	.22972 61778 11512 99624 085
1.340	0.97348 45416 95319 37478 787	0.22875 28078 08459 46523 264
.341	.97371 28077 22772 08238 616	.22777 92090 52617. 18849 831
.342	.97394 01000 37498 20994 365	.22680 53825 17584 84074 691
.343	.97416 64184 12205 46167 522	.22583 13291 77188 87585 859
.344	.97439 17626 20575 48173 349	.22485 70500 05482 55305 819
1.345	0.97461 61324 37264 08052 713	0.22388 25459 76744 96286 212
.346	.97483 95276 37901 46006 501	.22290 78180 65480 05279 929
.347	.97506 19479 99092 43832 603	.22193 28672 46415 65290 729
.348	.97528 33932 98416 67265 423	.22095 76944 94502 50100 463
.349	.97550 38633 14428 88217 916	.21998 23007 84913 26774 007
1.350	0.97572 33578 26659 06926 111	0.21900 66870 93041 58142 002
.351	.97594 18766 15612 73996 110	.21803 08543 94501 05261 504
.352	.97615 94194 62771 12353 536	.21705 48036 65124 29854 627
.353	.97637 59861 50591 39095 407	.21607 85358 80961 96725 291
.354	.97659 15764 62506 87244 418	.21510 20520 18281 76154 163
1.355	0.97680 61901 82927 27405 609	0.21412 53530 53567 46271 899
.356	.97701 98270 97238 89325 386	.21314 84399 63517 95410 772
.357	.97723 24869 91804 83352 894	.21217 13137 25046 24434 790
.358	.97744 41696 53965 21803 706	.21119 39753 15278 49048 406
.359	.97765 48748 72037 40225 805	.21021 64257 11553 02083 908
1.360	0.97786 46024 35316 18567 849	0.20923 86658 91419 35767 598
.361	.97807 33521 34074 02249 690	.20826 06968 32637 23964 842
.362	.97828 11237 59561 23135 125	.20728 25195 13175 64404 112
.363	.97848 79171 04006 20406 864	.20630 41349 11211 80880 089
.364	.97869 37319 60615 61343 685	.20532 55440 05130 25435 952
1.365	0.97889 85681 23574 61999 774	0.20434 67477 73521 80524 932
.366	.97910 24253 88047 07786 196	.20336 77471 95182 61151 240
.367	.97930 53035 50175 73954 516	.20238 85432 49113 16990 457
.368	.97950 72024 07082 45982 521	.20140 91369 14517 34489 495
.369	.97970 81217 56868 39862 027	.20042 95291 70801 38946 217
1.370	0.97990 80613 98614 22288 769	0.19944 97209 97572 96568 820
.371	.98010 70211 32380 30754 328	.19846 97133 74640 16515 079
.372	.98030 50007 59206 93540 094	.19748 95072 82010 52911 545
.373	.98050 20000 81114 49613 233	.19650 91036 99890 06852 798
.374	.98069 80189 01103 68424 652	.19552 85036 08682 28380 853
1.375	0.98089 30570 23155 69608 920	0.19454 77079 88987 18444 822
.376	.98108 71142 52232 42586 155	.19356 67178 21600 30840 918
.377	.98128 01903 94276 66065 826	.19258 55340 87511 74132 912
.378	.98147 22852 56212 27452 479	.19160 41577 67905 13553 129
.379	.98166 33986 45944 42153 343	.19062 25898 44156 72884 094

TABLE XII.—*Values of sin x and cos x to 23 places of decimals at intervals of 0.001 from 0.000 to 1.600*—Continued.

x	$\sin x$	$\cos x$
1.380	0.98185 35303 72359 72787 813	0.18964 08312 97834 36320 915
.381	.98204 26802 45326 48298 791	.18865 88831 10696 50314 508
.382	.98223 08480 75694 82965 850	.18767 67462 64691 25395 757
.383	.98241 80336 75296 95320 221	.18669 44217 41955 37980 715
.384	.98260 42368 56947 26961 571	.18571 19105 24813 32156 930
1.385	0.98278 94574 34442 61276 561	0.18472 92135 95776 21451 016
.386	.98297 36952 22562 42059 162	.18374 63319 37540 90577 542
.387	.98315 69500 37068 92032 708	.18276 32665 32988 97169 360
.388	.98333 92216 94707 31273 673	.18178 00183 65185 73489 451
.389	.98352 05100 13205 95537 148	.18079 65884 17379 28124 404
1.390	0.98370 08148 11276 54484 004	0.17981 29776 72999 47659 616
.391	.98388 01359 08614 29809 722	.17882 91871 15656 98336 311
.392	.98405 84731 25898 13274 870	.17784 52177 29142 27690 484
.393	.98423 58262 84790 84637 207	.17686 10704 97424 66173 860
.394	.98441 21952 07939 29485 405	.17587 67464 04651 28756 976
1.395	0.98458 75797 18974 56974 360	0.17489 22464 35146 16514 467
.396	.98476 19796 42512 17462 083	.17390 75715 73409 18192 681
.397	.98493 53948 04152 20048 145	.17292 27228 04115 11759 690
.398	.98510 78250 30479 50013 670	.17193 77011 12112 65937 830
.399	.98527 92701 49063 86162 846	.17095 25074 82423 41718 833
1.400	0.98544 97299 88460 18065 947	0.16996 71429 00240 93861 675
.401	.98561 92043 78208 63203 840	.16898 16083 50929 72373 233
.402	.98578 76931 48834 84013 966	.16799 59048 20024 23971 842
.403	.98595 51961 31850 04837 776	.16701 00332 93227 93533 854
.404	.98612 17131 59751 28769 609	.16602 39947 56412 25523 303
1.405	0.98628 72440 66021 54406 982	0.16503 77901 95615 65404 770
.406	.98645 17886 85129 92502 294	.16405 14205 97042 61039 544
.407	.98661 53468 52531 82515 912	.16306 48869 47062 64065 184
.408	.98677 79184 04669 09070 631	.16207 81902 32209 31258 571
.409	.98693 95031 78970 18307 486	.16109 13314 39179 25882 568
1.410	0.98710 01010 13850 34142 909	0.16010 43115 54831 19016 356
.411	.98725 97117 48711 74427 198	.15911 71315 66184 90869 577
.412	.98741 83352 23943 67004 304	.15812 97924 60420 32080 359
.413	.98757 59712 80922 65672 895	.15714 22952 24876 44997 336
.414	.98773 26197 62012 66048 706	.15615 46408 47050 44945 751
1.415	0.98788 82805 10565 21328 142	0.15516 68303 14596 61477 752
.416	.98804 29533 70919 57953 120	.15417 88646 15325 39606 967
.417	.98819 66381 88402 91177 144	.15319 07447 37202 41027 471
.418	.98834 93348 09330 40532 586	.15220 24716 68347 45317 231
.419	.98850 10430 81005 45199 170	.15121 40463 97033 51126 135
1.420	0.98865 17628 51719 79273 627	0.15022 54699 11685 77348 698
.421	.98880 14939 70753 66940 521	.14923 67432 00880 64281 559
.422	.98895 02362 88375 97544 222	.14824 78672 53344 74765 840
.423	.98909 79896 55844 40562 021	.14725 88430 57953 95314 499
.424	.98924 47539 25405 60478 351	.14626 96716 03732 37224 747
1.425	0.98939 05289 50295 31560 129	0.14528 03538 79851 37675 648
.426	.98953 53145 84738 52533 174	.14429 08908 75628 60810 986
.427	.98967 91106 83949 61159 714	.14330 12835 80526 98807 514
.428	.98982 19171 04132 48716 941	.14231 15329 84153 72928 666
.429	.98996 37337 02480 74376 619	.14132 16400 76259 34563 848
1.430	0.99010 45603 37177 79485 729	0.14033 16058 46736 66253 390
.431	.99024 43968 67397 01748 121	.13934 14312 85619 82699 275
.432	.99038 32431 53301 89307 176	.13835 11173 83083 31761 733
.433	.99052 10990 56046 14729 460	.13736 06651 29440 95441 799
.434	.99065 79644 37773 88889 346	.13637 00755 15144 90849 940
1.435	0.99079 38391 61619 74754 605	0.13537 93495 30784 71160 849
.436	.99092 87230 91709 01072 941	.13438 84881 67086 26554 495
.437	.99106 26160 93157 75959 459	.13339 74924 14910 85143 546
.438	.99119 55180 32073 00385 060	.13240 63632 65254 13887 244
.439	.99132 74287 75552 81565 735	.13141 51017 09245 19491 852

TABLE XII.—*Values of sin x and cos x to 23 places of decimals at intervals of 0.001 from 0.000 to 1.600*—Continued.

x	$\sin x$	$\cos x$
1.440	0.99145 83481 91686 46252 760	0.13042 37087 38145 49297 752
.441	.99158 82761 49554 53923 766	.12943 21853 43347 92153 306
.442	.99171 72125 19229 09874 676	.12844 05325 16375 79275 576
.443	.99184 51571 71773 78212 505	.12744 87512 48881 85098 002
.444	.99197 21099 79243 94748 990	.12645 68425 32647 28105 135
1.445	0.99209 80708 14686 79795 055	0.12546 48073 59580 71654 525
.446	.99222 30395 52141 50856 088	.12447 26467 21717 24785 871
.447	.99234 70160 66639 35228 024	.12348 03616 11217 43017 513
.448	.99247 00002 34203 82494 216	.12248 79530 20366 29130 391
.449	.99259 19919 31850 76923 086	.12149 54219 41572 33939 548
1.450	0.99271 29910 37588 49766 535	0.12050 27693 67366 57053 287
.451	.99283 29974 30417 91459 118	.11950 99962 90401 47620 080
.452	.99295 20109 90332 63717 946	.11851 71037 03450 05063 327
.453	.99307 00315 98319 11543 325	.11752 40925 99404 79804 068
.454	.99318 70591 36356 75120 114	.11653 09639 71276 73971 735
1.455	0.99330 30934 87418 01619 777	0.11553 77188 12194 42103 061
.456	.99341 81345 35468 56903 143	.11454 43581 15402 91829 237
.457	.99353 21821 65467 37123 830	.11355 08828 74262 84551 407
.458	.99364 52362 63366 80232 355	.11255 72940 82249 36104 618
.459	.99375 72967 16112 77380 893	.11156 35927 32951 17410 313
1.460	0.99386 83634 11644 84228 683	0.11056 97798 20069 55117 465
.461	.99397 84362 38896 32148 075	.10957 58563 37417 32232 463
.462	.99408 75150 87794 39331 194	.10858 18232 78917 88737 835
.463	.99419 55998 49260 21797 223	.10758 76816 38604 22199 915
.464	.99430 26904 15209 04300 286	.10659 34324 10617 88365 556
1.465	0.99440 87866 78550 31137 923	0.10559 90765 89208 01747 983
.466	.99451 38885 33187 76860 141	.10460 46151 68730 36201 884
.467	.99461 79958 74019 56879 043	.10361 00491 43646 25487 846
.468	.99472 11085 96938 37979 012	.10261 53795 08521 63826 230
.469	.99482 32265 98831 48727 437	.10162 06072 58026 06440 584
1.470	0.99492 43497 77580 89785 993	0.10062 57333 86931 70090 698
.471	.99502 44780 32063 44122 430	.09963 07588 90112 33595 391
.472	.99512 36112 62150 87122 898	.09863 56847 62542 38345 147
.473	.99522 17493 68709 96604 762	.09764 05119 99295 88804 678
.474	.99531 88922 53602 62729 932	.09664 52415 95545 53005 525
1.475	0.99541 50398 19685 97818 664	0.09564 98745 46561 63028 806
.476	.99551 01919 70812 46063 854	.09465 44118 47711 15478 186
.477	.99560 43486 11829 93145 787	.09365 88544 94456 71943 189
.478	.99569 75096 48581 75747 356	.09266 32034 82355 59452 948
.479	.99578 96749 87906 90969 720	.09166 74598 07058 70920 484
1.480	0.99588 08445 37640 05648 408	0.09067 16244 64309 65577 623
481	.99597 10182 06611 65569 851	.08967 56984 49943 69400 641
.482	.99606 01959 04648 04588 337	.08867 96827 59886 75526 752
.483	.99614 83775 42571 53643 374	.08768 35783 90154 44661 519
.484	.99623 55630 32200 49677 461	.08668 73863 36851 05477 303
1.485	0.99632 17522 86349 44454 246	0.08569 11075 96168 55002 845
.486	.99640 69452 18829 13277 079	.08469 47431 64385 59004 070
.487	.99649 11417 44446 63607 933	.08369 82940 37866 52356 240
.488	.99657 43417 79005 43586 693	.08270 17612 13060 39407 518
.489	.99665 65452 39305 50450 815	.08170 51456 86499 94334 076
1.490	0.99673 77520 43143 38855 320	0.08070 84484 54800 61486 832
.491	.99681 79621 09312 29093 143	.07971 16705 14659 55729 907
.492	.99689 71753 57602 15215 811	.07871 48128 62854 62770 926
.493	.99697 53917 08799 73054 448	.07771 78764 96243 39483 234
.494	.99705 26110 84688 68141 099	.07672 08624 11762 14220 152
1.495	0.99712 88334 08049 63530 364	0.07572 37716 06424 87121 354
.496	.99720 40586 02660 27521 334	.07472 66050 77322 30411 478
.497	.99727 82865 93295 41279 821	.07372 93638 21620 88691 060
.498	.99735 15173 05727 06360 877	.07273 20488 36561 79219 898
.499	.99742 37506 66724 52131 595	.07173 46611 19459 92192 943

Table XII.—*Values of sin x and cos x to 23 places of decimals at intervals of 0.001 from 0.000 to 1.600*—Continued.

x	sin x	cos x
1.500	0.99749 49866 04054 43094 172	0.07073 72016 67702 91008 819
.501	.99756 52250 46480 86109 251	.06973 96714 78750 12531 065
.502	.99763 44659 23765 37519 509	.06874 20715 50131 67342 208
.503	.99770 27001 66667 10173 501	.06774 44028 79447 39990 761
.504	.99776 99547 06942 80349 750	.06674 66664 64365 89231 245
1.505	0.99783 62024 77346 94581 063	0.06574 88633 02623 48257 343
.506	.99790 14524 11631 76379 092	.06475 09943 92023 24928 268
.507	.99796 57044 44547 32859 104	.06375 30607 30434 01988 470
.508	.99802 89585 11841 61264 976	.06275 50633 15789 37280 758
.509	.99809 12145 50260 55394 397	.06175 70031 46086 63952 953
1.510	0.99815 24724 97548 11924 274	0.06075 88812 19385 90658 160
.511	.99821 27322 92446 36636 332	.05976 06985 33809 01748 769
.512	.99827 19938 74695 50542 912	.05876 24560 87538 57464 281
.513	.99833 02571 85033 95912 947	.05776 41548 78816 94113 053
.514	.99838 75221 65198 42198 118	.05676 57959 05945 24248 072
1.515	0.99844 37887 57923 91859 188	0.05576 73801 67282 36836 851
.516	.99849 90569 06943 86092 495	.05476 89086 61243 97425 545
.517	.99855 33265 56990 10456 612	.05377 03823 86301 48297 399
.518	.99860 65976 53793 00399 163	.05277 18023 40981 08625 609
.519	.99865 88701 44081 46683 784	.05177 31695 23862 74620 716
1.520	0.99871 01439 75583 00717 231	0.05077 44849 33579 19672 613
.521	.99876 04190 97023 79776 634	.04977 57495 68814 94487 284
.522	.99880 96954 58128 72136 872	.04877 69644 28305 27218 360
.523	.99885 79730 09621 42098 089	.04777 81305 10835 23593 598
.524	.99890 52517 03224 34913 328	.04677 92488 15238 67036 388
1.525	0.99895 15314 91658 81616 285	0.04578 03203 40397 18782 371
.526	.99899 68123 28645 03749 180	.04478 13460 85239 17991 291
.527	.99904 10941 68902 17990 729	.04378 23270 48738 81854 166
.528	.99908 43769 68148 40684 234	.04278 32642 29915 05695 871
.529	.99912 66606 83100 92265 762	.04178 41586 27830 63073 262
1.530	0.99916 79452 71476 01592 427	0.04078 50112 41591 05868 899
.531	.99920 82306 91989 10170 755	.03978 58230 70343 64380 513
.532	.99924 75169 04354 76285 152	.03878 65951 13276 47406 277
.533	.99928 58038 69286 79026 436	.03778 73283 69617 42326 008
.534	.99932 30915 48498 22220 463	.03678 80238 38633 15178 390
1.535	0.99935 93799 04701 38256 819	0.03578 86825 19628 10734 312
.536	.99939 46689 01607 91817 592	.03478 93054 11943 52566 435
.537	.99942 89585 03928 83506 202	.03378 98935 14956 43115 073
.538	.99946 22486 77374 53376 306	.03279 04478 28078 63750 505
.539	.99949 45393 88654 84360 752	.03179 09693 50755 74831 796
1.540	0.99952 58306 05479 05600 596	0.03079 14590 82466 15762 248
.541	.99955 61222 96555 95674 180	.02979 19180 22720 05041 568
.542	.99958 54144 31593 85726 242	.02879 23471 71058 40314 858
.543	.99961 37069 81300 62497 095	.02779 27475 27051 98418 526
.544	.99964 09999 17383 71251 832	.02679 31200 90300 35423 217
1.545	0.99966 72932 12550 18609 586	0.02579 34658 60430 86673 867
.546	.99969 25868 40506 75272 821	.02479 37858 37097 66826 971
.547	.99971 68807 75959 78656 660	.02379 40810 19980 69885 184
.548	.99974 01749 94615 35418 249	.02279 43524 08784 69229 328
.549	.99976 24694 73179 23886 150	.02179 46010 03238 17647 934
1.550	0.99978 37641 89356 96389 761	0.02079 48278 03092 47364 391
.551	.99980 40591 21853 81488 767	.01979 50338 08120 70061 827
.552	.99982 33542 50374 86102 606	.01879 52200 18116 76905 802
.553	.99984 16495 55624 97539 966	.01779 53874 32894 38564 929
.554	.99985 89450 19308 85428 298	.01679 55370 52286 05229 507
1.555	0.99987 52406 24131 03543 342	0.01579 56698 76142 06628 284
.556	.99989 05363 53795 91538 676	.01479 57869 04329 52043 433
.557	.99990 48321 93007 76575 277	.01379 58891 36731 30323 849
.558	.99991 81281 27470 74851 093	.01279 59775 73245 09896 874
.559	.99993 04241 43888 93030 623	.01179 60532 13782 38778 533

TABLE XII.—*Values of sin x and cos x to 23 places of decimals at intervals of 0.001 from 0.000 to 1.600*—Continued.

x	$sin\ x$					$cos\ x$				
1.560	0.99994	17202	29966	29574	517	0.01079	61170	58267	44582	392
.561	.99995	20163	74406	75969	172	.00979	61701	06636	34527	146
.562	.99996	13125	66914	17856	344	.00879	62133	58835	95443	014
.563	.99996	96087	98192	36062	758	.00779	62478	14822	93777	062
.564	.99997	69050	59945	07529	731	.00679	62744	74562	75597	546
1.565	0.99998	32013	44876	06142	794	0.00579	62943	38028	66597	372
.566	.99998	84976	46689	03461	318	.00479	63084	05200	72096	784
.567	.99999	27939	60087	69348	142	.00379	63176	76064	77045	359
.568	.99999	60902	80775	72499	201	.00279	63231	50611	46023	436
.569	.99999	83866	05456	80873	162	.00179	63258	28835	23243	059
1.570	0.99999	96829	31834	62021	053	+0.00079	63267	10733	32548	541
.571	.99999	99792	58612	83315	895	−0.00020	36732	03695	22583	254
.572	.99999	92755	85495	12082	337	.00120	36729	14450	59042	804
.573	.99999	75719	13185	15626	285	.00220	36714	21533	14087	901
.574	.99999	48682	43386	61164	539	.00320	36677	24944	45343	613
1.575	0.99999	11645	78803	15654	423	−0.00420	36608	24688	30802	109
.576	.99998	64609	23138	45523	419	.00520	36497	20771	68822	280
.577	.99998	07572	81096	16298	798	.00620	36334	13205	78129	029
.578	.99997	40536	58379	92137	261	.00720	36109	02006	97812	142
.579	.99996	63500	61693	35254	568	.00820	35811	87197	87324	647
1.580	0.99995	76464	98740	05255	179	−0.00920	35432	68808	26480	539
.581	.99994	79429	78223	58361	895	.01020	34961	46876	15451	796
.582	.99993	72395	09847	46545	499	.01120	34388	21448	74764	568
.583	.99992	55361	04315	16554	408	.01220	33702	92583	45294	454
.584	.99991	28327	73330	08844	324	.01320	32895	60348	88260	743
1.585	0.99989	91295	29595	56407	893	−0.01420	31956	24825	85219	553
.586	.99988	44263	86814	83504	374	.01520	30874	86108	38055	737
.587	.99986	87233	59691	04289	313	.01620	29641	44304	68973	475
.588	.99985	20204	63927	21344	232	.01720	28245	99538	20485	440
.589	.99983	43177	16226	24106	322	.01820	26678	51948	55400	452
1.590	0.99981	56151	34290	87198	158	−0.01920	24929	01692	56809	503
.591	.99979	59127	36823	68657	422	.02020	22987	48945	28070	065
.592	.99977	52105	43527	08066	646	.02120	20843	93900	92788	583
.593	.99975	35085	75103	24582	972	.02220	18488	36773	94801	039
.594	.99973	08068	53254	14867	933	.02320	15910	77799	98151	502
1.595	0.99970	71054	00681	50917	259	−0.02420	13101	17236	87068	552
.596	.99968	24042	41086	77790	702	.02520	10049	55365	65939	492
.597	.99965	67033	99171	11241	891	.02620	06745	92491	59282	234
.598	.99963	00029	00635	35248	219	.02720	03180	28945	11714	764
.599	.99960	23027	72179	99440	759	.02819	99342	65082	87922	093
1.600	0.99957	36030	41505	16434	211	−0.02919	95223	01288	72620	577

TABLE XIII.—*Values of sin x and cos x to 25 places of decimals at decimal intervals from 1×10^{-10} to 9×10^{-4}.*

x	$\sin x$	$\cos x$
1×10^{-10}	0.00000 00001 00000 00000 00000	0.99999 99999 99999 99999 50000
2	.00000 00002 00000 00000 00000	.99999 99999 99999 99998 00000
3	.00000 00003 00000 00000 00000	.99999 99999 99999 99995 50000
4	.00000 00004 00000 00000 00000	.99999 99999 99999 99992 00000
5	.00000 00005 00000 00000 00000	.99999 99999 99999 99987 50000
6	.00000 00006 00000 00000 00000	.99999 99999 99999 99982 00000
7	.00000 00007 00000 00000 00000	.99999 99999 99999 99975 50000
8	.00000 00008 00000 00000 00000	.99999 99999 99999 99968 00000
9	.00000 00009 00000 00000 00000	.99999 99999 99999 99959 50000
1×10^{-9}	0.00000 00010 00000 00000 00000	0.99999 99999 99999 99950 00000
2	.00000 00020 00000 00000 00000	.99999 99999 99999 99800 00000
3	.00000 00030 00000 00000 00000	.99999 99999 99999 99550 00000
4	.00000 00040 00000 00000 00000	.99999 99999 99999 99200 00000
5	.00000 00050 00000 00000 00000	.99999 99999 99999 98750 00000
6	.00000 00060 00000 00000 00000	.99999 99999 99999 98200 00000
7	.00000 00069 99999 99999 99999	.99999 99999 99999 97550 00000
8	.00000 00079 99999 99999 99999	.99999 99999 99999 96800 00000
9	.00000 00089 99999 99999 99999	.99999 99999 99999 95950 00000
1×10^{-8}	0.00000 00099 99999 99999 99998	0.99999 99999 99999 95000 00000
2	.00000 00199 99999 99999 99987	.99999 99999 99999 80000 00000
3	.00000 00299 99999 99999 99955	.99999 99999 99999 55000 00000
4	.00000 00399 99999 99999 99893	.99999 99999 99999 20000 00000
5	.00000 00499 99999 99999 99792	.99999 99999 99998 75000 00000
6	.00000 00599 99999 99999 99640	.99999 99999 99998 20000 00000
7	.00000 00699 99999 99999 99428	.99999 99999 99997 55000 00000
8	.00000 00799 99999 99999 99147	.99999 99999 99996 80000 00000
9	.00000 00899 99999 99999 98785	.99999 99999 99995 95000 00000
1×10^{-7}	0.00000 00999 99999 99999 98333	0.99999 99999 99995 00000 00000
2	.00000 01999 99999 99999 86667	.99999 99999 99980 00000 00000
3	.00000 02999 99999 99999 55000	.99999 99999 99955 00000 00000
4	.00000 03999 99999 99998 93333	.99999 99999 99920 00000 00000
5	.00000 04999 99999 99997 91667	.99999 99999 99875 00000 00000
6	.00000 05999 99999 99996 40000	.99999 99999 99820 00000 00000
7	.00000 06999 99999 99994 28333	.99999 99999 99755 00000 00000
8	.00000 07999 99999 99991 46667	.99999 99999 99680 00000 00000
9	.00000 08999 99999 99987 85000	.99999 99999 99595 00000 00000
1×10^{-6}	0.00000 09999 99999 99983 33333	0.99999 99999 99500 00000 00000
2	.00000 19999 99999 99866 66667	.99999 99999 98000 00000 00007
3	.00000 29999 99999 99550 00000	.99999 99999 95500 00000 00034
4	.00000 39999 99999 98933 33333	.99999 99999 92000 00000 00107
5	.00000 49999 99999 97916 66667	.99999 99999 87500 00000 00260
6	.00000 59999 99999 96400 00000	.99999 99999 82000 00000 00540
7	.00000 69999 99999 94283 33333	.99999 99999 75500 00000 01000
8	.00000 79999 99999 91466 66667	.99999 99999 68000 00000 01707
9	.00000 89999 99999 87850 00000	.99999 99999 59500 00000 02734
1×10^{-5}	0.00000 99999 99999 83333 33333	0.99999 99999 50000 00000 04167
2	.00001 99999 99998 66666 66667	.99999 99998 00000 00000 66667
3	.00002 99999 99995 50000 00002	.99999 99995 50000 00003 37500
4	.00003 99999 99989 33333 33342	.99999 99992 00000 00010 66667
5	.00004 99999 99979 16666 66693	.99999 99987 50000 00026 04167
6	.00005 99999 99964 00000 00065	.99999 99982 00000 00054 00000
7	.00006 99999 99942 83333 33473	.99999 99975 50000 00100 04167
8	.00007 99999 99914 66666 66940	.99999 99968 00000 00170 66667
9	.00008 99999 99878 50000 00492	.99999 99959 50000 00273 37500
1×10^{-4}	0.00009 99999 99833 33333 34167	0.99999 99950 00000 00416 66667
2	.00019 99999 98666 66666 93333	.99999 99800 00000 06666 66666
3	.00029 99999 95500 00002 02500	.99999 99550 00000 33749 99990
4	.00039 99999 89333 33341 86667	.99999 99200 00001 06666 66610
5	.00049 99999 79166 66692 70833	.99999 98750 00002 60416 66450
6	.00059 99999 64000 00064 80000	.99999 98200 00005 39999 99352
7	.00069 99999 42833 33473 39167	.99999 97550 00010 00416 65033
8	.00079 99999 14666 66939 73333	.99999 96800 00017 06666 63026
9	.00089 99998 78500 00492 07499	.99999 95950 00027 33749 92619

Table XIV.—*Miscellaneous values of e^x, e^{-x}, $\sin x$ and $\cos x$ to a great number of decimals, including Boorman's value of e.*

$e^{+0.1}=1.$	10517	09180	75647	62481	17078	26490	24666	82245	47194	73751
	87187	92863	28944	09679	66747	65430	29891	43318	97074	86536
	32917	12048	54012	44536						
$e^{-0.1}=0.$	90483	74180	35959	57316	42490	59446	43662	11947	05360	98040
	09520	56257	31705	57799	65344	24836	10125	03446	03609	04572
	38478	74531	46483	18498						
$\sin 0.1=0.$	09983	34166	46828	15230	68141	98410	62202	69899	15388	01798
	22599	92766	86156	16517	44283	29242	76096	62443	80406	30362
	67832	50318	09359	89035						
$\cos 0.1=0.$	99500	41652	78025	76609	55619	87803	87029	48385	76225	41508
	40359	59352	74468	52659	10218	24046	65296	63618	52826	29279
	10723	68588	08368	71860						
$e^{+1.0}=2.$	71828	18284	59045	23536	02874	71352	60249	77572	47093	69995
	95749	66967	62772	40766	30353	54759	45713	82178	52516	64274
	27466	39193	20030	59921	81741	35966	29043	57290	03342	95260
	59563	07381	32328	62794	34907	63233	82988	07531	95251	01901
	15738	34187	93070	21540	89126	94937	99405	34631	93819	87250
	90567	36251	50082	37715	27509	03586	67692	05047	15575	85094
	92906	45748	86005	84299	93465	94757	59371	00435	26480	0
$e^{-1.0}=0.$	36787	94411	71442	32159	55237	70161	46086	74458	11131	03176
	78345	07836	80169	74614	95744	89980	33571	47274	34591	96437
	46627									
$\sin 1.0=0.$	84147	09848	07896	50665	25023	21630	29899	96225	63060	79837
	10656	72751	70999	19104	04391	23966	89486	39743	54305	26958
	54349									
$\cos 1.0=0.$	54030	23058	68139	71740	09366	07442	97660	37323	10420	61792
	22276	70097	25538	11003	94774	47176	45179	51856	08718	30893
	43572									

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